

ZERO MEASURE CANTOR SPECTRUM FOR APERIODIC METRIC TILING GRAPHS

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ABSTRACT. We study a family of aperiodic metric graphs generated by one-dimensional tilings. We prove that, generically, the spectrum of the associated Kirchhoff Laplacian has Lebesgue measure zero and, apart from a discrete set of isolated eigenvalues, is a generalized Cantor set. For a large subclass of these graphs, we further show that isolated eigenvalues are generically absent. This extends several well-known results from the setting of discrete ergodic Schrödinger operators on $\mathbb{Z}$ to a large family of metric graphs. Earlier results of this kind for metric graphs typically assume equilateral edges and exploit symmetries that reduce the analysis to a Sturm-Liouville operator on the real line. By contrast, the graph families considered here need not have equilateral edges or the symmetries needed for such a reduction and therefore require a different analytical approach.

1. INTRODUCTION AND MAIN RESULTS

This paper studies zero measure Cantor spectrum for aperiodic metric graphs whose geometry is generated by one-dimensional symbolic dynamics.

The spectral theory of aperiodic Schrödinger operators has been a central topic in mathematical physics since the 1970s. A widely studied class of operators is discrete ergodic Schrödinger operators on $\mathbb{Z}$, where even simple aperiodic models display non-trivial spectral properties. One interesting feature shared by many aperiodic models is having Cantor spectrum of zero Lebesgue measure and purely singular continuous spectral measures. This occurs, for instance, for Sturmian Hamiltonians [?, ?] and other classes of low-complexity discrete Schrödinger operators [?, ?, ?, ?, ?]. Analogous results are also known for ergodic continuum Schrödinger operators on $\mathbb{R}$ [?, ?].

The goal of this work is to study analogous results on networks, where the aperiodicity manifests through the geometric structure of the underlying space itself, rather than via a potential. The objects we study here are continuum Schrödinger operators on metric (quantum) graphs inspired by one-dimensional tilings, which we call *tiling graphs*, that provide a natural generalization of the usual models on $\mathbb{R}$. An important work in this direction is [?], where zero measure spectrum was proven for equilateral metric antitrees. Other related works include results on generic singular continuous spectrum for radially symmetric metric trees [?, ?, ?], as well as Anderson localization for random radial tree models [?] and for quantum graphs generated by subshifts of finite type [?]. A common feature to these works is the presence of symmetries, which effectively reduce the graph operator to a Sturm-Liouville operator on $\mathbb{R}$. Our goal is to go beyond that, by studying aperiodic graphs that do not necessarily possess an immediate symmetry

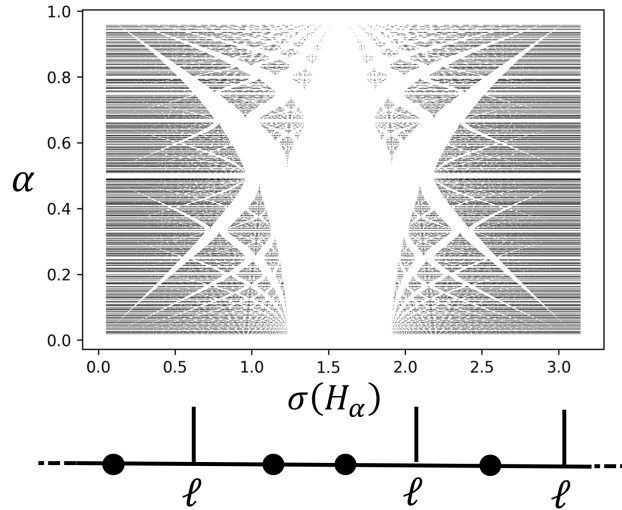


FIGURE 1.1. The “Kohmoto butterfly” for Sturmian comb graphs – parts of $\sigma(H_\alpha)$ as a function of α . The horizontal axis represents $k \in \mathbb{R}$ such that $k^2 \in \sigma(H_\alpha)$.

which allows for such a reduction. This could lay the foundation for further spectral analysis of graphs with a more complicated geometric structure.

The graphs we study in this work are constructed as one-dimensional tilings which are generated by minimal, aperiodic dynamical systems, satisfying Boshernitzan’s condition. Under an explicit and Baire-generic assumption on the tiles forming the graph, we prove that the spectrum of these graphs is of zero Lebesgue measure (Theorem 1.6), and in particular has no absolutely continuous part. After removing a possible discrete set of isolated eigenvalues, the remaining spectrum is shown to form a generalized Cantor set.

For a large class of tiling graphs, called decorated $\mathbb{Z}$ -graphs, the result can be strengthened. For a Baire-generic choice of the edge lengths, the exceptional isolated eigenvalues are absent, and the spectrum itself is a zero measure generalized Cantor set (Theorem 1.9). In this setting, the exceptional energies are associated with compactly supported eigenfunctions, which can be excluded generically by perturbing the edge lengths. Figure 1.1 illustrates these results for Sturmian comb graphs (see Example 1.4 and Section 5). These graphs depend on the Sturmian frequency, α . Whenever this frequency is irrational, the spectrum of the graph is a zero measure Cantor set, giving rise to an intricate fractal pattern which resembles the Kohmoto butterfly [?, ?].

A key ingredient in the proof is a Borg-Marchenko type theorem for tiling graphs (Theorem 2.1), showing that the underlying aperiodic tiling can be reconstructed from the Weyl-Titchmarsh m -function of the graph. Our proof is less standard than usual one-dimensional Borg-Marchenko results, and relies on ideas from quantum chaos on graphs. Namely, we use wave propagation along the graph to derive a periodic-orbit expansion of the m -function, from which the successive tiles can be recovered. This

inverse-spectral result relates the graph geometry to the more standard tools for ergodic one-dimensional operators, namely Kotani theory.

The paper is structured as follows: after presenting the necessary background and definitions in the next subsections, we state our main results. Section 2 proves the Borg-Marchenko theorem and develops the periodic-orbit expansion of the m -function. Section 3 introduces the transfer matrix and Lyapunov exponent and uses those to give a dynamical characterization of the spectrum. Section 4 combines this with Kotani theory and an oracle theorem to prove Theorems 1.6 and 1.9. Lastly, Section 5 discusses the particular family of Sturmian comb graphs.

1.1. Quantum graphs. A *metric graph* is a pair $\Gamma = (G, \vec{\ell})$, where $G = (\mathcal{V}, \mathcal{E})$ is a combinatorial graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$, and $\vec{\ell} \in \mathbb{R}_+^{|\mathcal{E}|}$ is a vector assigning a positive length to each edge in $\mathcal{E}$. Each edge $e \in \mathcal{E}$ is identified with the interval $[0, \ell_e]$, which equips Γ with the natural structure of a metric space.

For a vertex v , we write $e \sim v$ to indicate that the edge e is incident to v . The degree of a vertex v , denoted by $\deg(v)$, is defined as the number of edges incident to v , with loop edges (edges that connect v to itself) counted twice.

A *quantum graph* is a metric graph Γ equipped with a self-adjoint differential operator H acting on the Sobolev space $H^2(\Gamma) := \oplus_{e \in \mathcal{E}} H^2(0, \ell_e)$. In this work, the operator acts as the Laplacian on each edge $H = -\frac{d^2}{dx^2}$, together with the *Neumann-Kirchhoff* boundary conditions imposed at each vertex:

$$(1.1) \quad f|_e(v) = f|_{e'}(v), \quad \forall e, e' \sim v,$$

$$(1.2) \quad \sum_{e \sim v} f'|_e(v) = 0,$$

where the derivatives are taken in the outwards pointing direction from the vertex. These vertex conditions are also known as Kirchhoff, continuity-Kirchhoff, or standard vertex conditions. For an extensive introduction to quantum graphs, see [?, ?, ?, ?, ?, ?].

In this work, Γ will be an infinite, connected metric graph, where the vertex degrees $\deg(v)$ are uniformly bounded from above, and the edge lengths ℓ_e are uniformly bounded from above and also uniformly bounded from below by some $\ell_0 > 0$. Under these conditions, the associated Kirchhoff Laplacian is self-adjoint and non-negative, see [?, thm 1.4.19].

1.2. Dynamics. The aperiodic metric graphs we consider in this work are defined through one-dimensional dynamical systems that govern their geometric structure. We now introduce the relevant definitions.

Let $\mathcal{A}$ be a finite set, which we call an alphabet. We consider the space of bi-infinite sequences $\mathcal{A}^{\mathbb{Z}}$, equipped with the product topology, as induced by the following metric:

$$(1.3) \quad d(\omega, \omega') = \sum_{n \in \mathbb{Z}} \frac{1 - \delta_{\omega(n), \omega'(n)}}{2^{|n|}},$$

where $\delta_{i,j}$ is the Kronecker delta. The space $\mathcal{A}^{\mathbb{Z}}$ is naturally equipped with the *shift* map:

$$(1.4) \quad S : \mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}},$$

$$(1.5) \quad S\omega(n) = \omega(n+1).$$

Definition 1.1. A *subshift* is a closed, shift-invariant subset $\Omega \subset \mathcal{A}^{\mathbb{Z}}$. A subshift is called *minimal* if it contains no proper nontrivial subshifts, and is called *aperiodic* if it contains no periodic elements, i.e., an element ω such that $S^p\omega = \omega$ for some $p \neq 0$.

Given a finite subword $W = W(1) \dots W(k)$ of an element in Ω , we define the corresponding *cylinder set* by

$$(1.6) \quad V_W = \{\omega \in \Omega : \omega|_{[0,k-1]} = W\}.$$

If μ is a S -invariant probability measure on Ω , we define the quantity

$$(1.7) \quad \eta_\mu(n) = \min \{\mu(V_W) : |W| = n\}.$$

A minimal subshift Ω is said to satisfy *Boshernitzan's condition* if there exists an ergodic S -invariant probability measure μ on Ω such that

$$(1.8) \quad \limsup_{n \rightarrow \infty} n \cdot \eta_\mu(n) > 0.$$

This condition, introduced by Boshernitzan in [?, ?], is a quantitative recurrence condition for an aperiodic subshift. It holds for a wide variety of subshifts, as described in [?]. For minimal subshifts, Boshernitzan's condition implies uniformity for locally constant cocycles, as discussed in Section 3. This is the main role of this condition in our analysis.

Example 1.2. Let $\alpha \in (0, 1) \setminus \mathbb{Q}$. For $\theta \in [0, 1]$, we define the Sturmian sequence $(\omega_{\alpha, \theta}(n))_{n \in \mathbb{Z}}$ on the alphabet $\mathcal{A} = \{0, 1\}$ by

$$(1.9) \quad \omega_{\alpha, \theta}(n) = \chi_{(1-\alpha, 1]}(n\alpha + \theta \bmod 1).$$

The Sturmian subshift is then defined as

$$(1.10) \quad \Omega_\alpha = \overline{\{\omega_{\alpha, \theta} : \theta \in [0, 1]\}}.$$

Ω_α is a minimal, aperiodic subshift over the alphabet $\{0, 1\}$, and it satisfies Boshernitzan's condition. The most famous example is the Fibonacci subshift, obtained for $\alpha = \frac{\sqrt{5}-1}{2}$ (see [?]).

1.3. Tiling graphs. We now introduce the class of *aperiodic tiling graphs*, which are the center of this paper. To define those, we fix a minimal, aperiodic subshift Ω over a finite alphabet $\mathcal{A}$, satisfying Boshernitzan's condition.

Let $(T_a)_{a \in \mathcal{A}}$ be a family of compact metric graphs, which we call the *tiles*. For each T_a , we select two distinct vertices $v_1^a, v_2^a \in \mathcal{V}(T_a)$. Given $\omega \in \Omega$, construct an infinite metric graph Γ_ω in the following way. Form a chain of graphs $(\Gamma_\omega^{(n)})_{n \in \mathbb{Z}}$, such that

$$(1.11) \quad \Gamma_\omega^{(n)} = T_{\omega(n)},$$

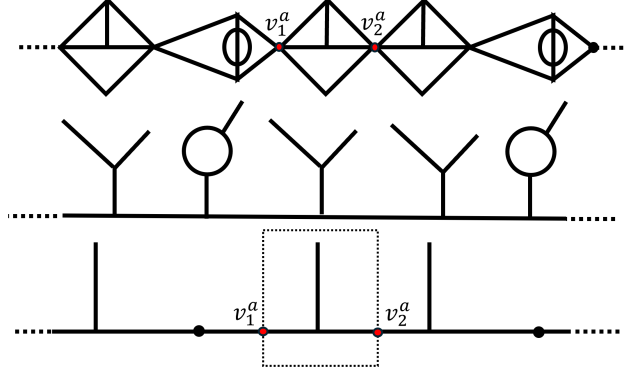


FIGURE 1.2. Three examples of tiling graphs, where the bottom two graphs are decorated $\mathbb{Z}$ -graphs. For the top and bottom graph the attachment vertices v_1^a and v_2^a of the central tile are marked in red.

and the graphs $\Gamma_\omega^{(n)}, \Gamma_\omega^{(n+1)}$ are connected to each other by identifying the vertices $v_2^{\omega(n)} \in \mathcal{V}(\Gamma_\omega^{(n)})$ and $v_1^{\omega(n+1)} \in \mathcal{V}(\Gamma_\omega^{(n+1)})$ (see Figure 1.2). This results in an infinite graph which we denote by Γ_ω . Associate to each such graph an origin $o(\Gamma_\omega) \in \Gamma_\omega$, which is taken to be the vertex $v_1^{\omega(0)} \in \mathcal{V}(\Gamma_\omega^{(0)})$.

One important family of tiling graphs on which we also focus is decorated $\mathbb{Z}$ -graphs:

Definition 1.3 (Decorated $\mathbb{Z}$ -graph). Let $\{(\mathcal{D}_a, v_a)\}_{a \in \mathcal{A}}$ be a set of compact graphs (called *decorations*) with marked vertices $v_a \in \mathcal{D}_a$ (called the *base vertices*). Given a subshift Ω over the alphabet $\mathcal{A}$, we can naturally construct a tiling graph Γ_ω by taking the line graph over $\mathbb{Z}$, and for each $a \in \mathcal{A}$, attaching a copy of $\mathcal{D}_a$ through the base vertex v_a at all integers n such that $\omega(n) = a$. We call such a Γ_ω a decorated $\mathbb{Z}$ -graph. It is indeed a particular case of a tiling graph, where the tiles $(T_a)_{a \in \mathcal{A}}$ are the graphs $(\mathcal{D}_a)_{a \in \mathcal{A}}$, with two dangling edges of size $1/2$ attached at the base $v_a \in \mathcal{D}_a$. Decorated $\mathbb{Z}$ -graphs are also studied in [?, ?], where we develop a suitable gap labelling theory for them and solve the corresponding dry ten Martini problem.

Example 1.4 (Sturmian comb). A particular family of decorated $\mathbb{Z}$ -graphs is obtained by taking the Sturmian subshift Ω_α over the alphabet $\{0, 1\}$ from Example 1.2, and choosing the first decoration to be a dangling edge of length ℓ , and the second decoration to be a single vertex (see bottom of Figure 1.2). We call such graphs Sturmian combs and demonstrate in Section 5 how our results apply for this family of graphs.

The family of graphs $\Gamma_\Omega := (\Gamma_\omega)_{\omega \in \Omega}$ is equipped with a naturally induced shift,

$$(1.12) \quad S : \Gamma_\Omega \rightarrow \Gamma_\Omega,$$

$$(1.13) \quad S\Gamma_\omega = \Gamma_{S\omega},$$

where we allowed ourselves to abuse the shift notation S . Equipping the tiling graphs $\Gamma_\omega \in \Gamma_\Omega$ with the Kirchhoff Laplacian H_ω , one can consider the family $H_\Omega := (H_\omega)_{\omega \in \Omega}$

as a dynamical system of operators. Denoting by $S^\sharp : C(\Gamma_{S\omega}) \rightarrow C(\Gamma_\omega)$ the pullback of S , it is easy to check that the family H_Ω is covariant, i.e.,

$$(1.14) \quad H_{S\omega} = (S^\sharp)^{-1} H_\omega S^\sharp, \quad \forall \omega \in \Omega,$$

which means that the operators H_ω and $H_{S\omega}$ are unitarily equivalent. As we shall see in Theorem 1.6, minimality of Ω implies that $\sigma(H_\omega)$ is in fact independent of $\omega \in \Omega$, and so we can simply denote it by $\sigma(H_\Omega)$.

Throughout the paper, we make the following technical assumption about the tiles:

Assumption 1.5. *There is at most one tile in $(T_a)_{a \in \mathcal{A}}$ in which v_a^1 and v_a^2 are connected by a single edge. In particular, at most one tile is an interval. Furthermore, no tile has degree-2 vertices, except for possibly the boundary vertices v_1^a and v_2^a .*

The following assumptions are about the tile edge lengths:

- (1) *Assume that for each $a \in \mathcal{A}$, the quantity*

$$(1.15) \quad \ell_{\min}^a := \min \{ \ell_e : e \sim v_a^1 \}$$

is realized by a unique edge, and that $\ell_{\min}^a \neq \ell_{\min}^{a'}$ for all $a \neq a'$.

- (2) *Denote*

$$\ell_{\text{left}}^a := \begin{cases} \ell_{\min}^a & \deg(v_a^1) = 1 \\ 0 & \text{else} \end{cases}.$$

If one of the tiles is an interval, denote it by $T_{\bar{a}}$ and its length by $\ell_{\bar{a}}$. Assume that for all $a', a'' \in \mathcal{A} \setminus \{\bar{a}\}$, the equality $\ell_{\text{left}}^{a'} \equiv \ell_{\text{left}}^{a''} \pmod{\ell_{\bar{a}}}$ implies that $\ell_{\text{left}}^{a'} = \ell_{\text{left}}^{a''}$.

If no tile is an interval, but there is a tile in which the boundary vertices are connected by a single edge, denote it by $T_{\bar{a}}$ and the edge length connecting v_a^1 and v_a^2 by $\ell_{\bar{a}}$. Assume that for all $a', a'' \in \mathcal{A} \setminus \{\bar{a}\}$, $\ell_{\text{left}}^{a'} - \ell_{\text{left}}^{a''} \neq \ell_{\bar{a}}$.

Remark. The assumption of excluding vertices of degree 2 is a common one when considering the Kirchhoff Laplacian. Indeed, a degree-2 vertex with Kirchhoff conditions may be removed by replacing its two incident edges with a single edge whose length is the sum of their lengths. This does not affect the Laplacian, up to unitary equivalence. All the assumptions on the edge lengths included in Assumption 1.5 may be replaced by the simple (yet stronger) assumption that there are no nontrivial integer linear relations among the tile edge lengths. We prefer not to use this stronger assumption, as it excludes some graph families, such as the decorated graphs, from the forthcoming results.

Theorem 1.6. *Let Ω be a minimal aperiodic subshift satisfying Boshernitzan's condition. Let $(\Gamma_\omega)_{\omega \in \Omega}$ be an associated family of metric tiling graphs, with tiles satisfying Assumption 1.5. Then:*

1. *There exists a closed set of zero Lebesgue measure $\sigma(H_\Omega) \subset \mathbb{R}$ such that $\sigma(H_\omega) = \sigma(H_\Omega)$ for all $\omega \in \Omega$. In particular, the absolutely continuous spectrum $\sigma_{ac}(H_\omega)$ is empty for all $\omega \in \Omega$.*

2. *There exists a (possibly empty) set of isolated eigenvalues $F \subset \mathbb{R}$ such that $\sigma(H_\Omega) \setminus F$ is a generalized Cantor set.*

Remark 1.7. The fact that $\sigma(H_\omega)$ is independent of ω is standard for minimal dynamical systems, and does not require Assumption 1.5 or Boshernitzan's condition.

By a *generalized Cantor set*, we mean a closed, nowhere-dense subset of $\mathbb{R}$ with no isolated points (relaxing the usual condition of compactness). A precise classification of the set F will be given in Section 4 (see Remark 4.6).

Consider the set of all edge lengths $\mathbb{R}_+^n$ and recall that a subset $A \subset \mathbb{R}_+^n$ is called Baire-generic if it contains a countable intersection of dense open subsets. Noting that the additional condition on the tile edge lengths in Assumption 1.5 is satisfied for a Baire-generic and Lebesgue almost-sure choice of the lengths, we immediately deduce the following:

Corollary 1.8. *For a Baire-generic and Lebesgue almost-sure choice of the tile edge lengths, the resulting tiling graph satisfies that $\sigma(H_\Omega)$ has Lebesgue measure zero and, apart from a discrete set of isolated eigenvalues, forms a generalized Cantor set.*

We further show that when restricting to the important class of decorated $\mathbb{Z}$ -graphs (see Definition 1.3), the last statement can be strengthened.

Theorem 1.9. *Let (Ω, S) be a minimal aperiodic subshift satisfying Boshernitzan's condition, and let $(\Gamma_\omega)_{\omega \in \Omega}$ be an associated family of decorated $\mathbb{Z}$ -graphs. Assume that the decorations do not contain loop edges (i.e., an edge from a vertex to itself). Then, for a Baire-generic choice of the decoration edge lengths, $\sigma(H_\Omega) \subset \mathbb{R}$ is a generalized Cantor set of zero Lebesgue measure.*

As we shall see in Subsection 4.2, $\sigma(H_\omega)$ may not be a Cantor set if H_ω admits infinitely degenerate eigenvalues with compactly supported eigenfunctions, a case that may happen for particular (non-generic) choices of the edge lengths, or in the presence of loop edges.

We mention that the questions we consider here may also be considered for more general Schrödinger operators than the Laplacian, namely of the form $H_\omega = -\frac{d^2}{dx^2} + V_\omega(x)$ where $V_\omega(x)$ is sufficiently “compatible” with the tiling structure of Γ_ω , similar to the continuum Schrödinger operators studied in [?]. We do not pursue this extension here.

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2. THE m -FUNCTION AND A BORG-MARCHENKO RESULT

In this section, we introduce a main tool for the proof of Theorem 1.6, which is the m -function associated with a quantum graph (also known as the Weyl-Titchmarsh function, or Dirichlet-to-Neumann map). We then prove that the m -function of a tiling graph Γ_ω can be used to recover the sequence of tiles $\omega(n)$. Such an inverse spectral geometric result is commonly known as a Borg-Marchenko type result, see [?, ?, ?] for some relevant results in the context of quantum graphs, or [?] for the original paper by Marchenko.

2.1. The m -function and the Borg-Marchenko result. We now introduce the m -functions associated with Γ_ω . For this, we start by partitioning the infinite graph Γ_ω into two half-infinite graphs $\Gamma_\omega^\pm$, which are obtained by cutting Γ_ω at the origin $o(\Gamma_\omega)$.

Let $z \in \mathbb{C}_+ := \{z \in \mathbb{C} : \text{Im}(z) > 0\}$. On each half-graph $\Gamma_\omega^\pm$, consider the differential equation

$$(2.1) \quad -\frac{d^2}{dx^2} f_\pm = z f_\pm,$$

with the Kirchhoff condition imposed at all vertices except for $o(\Gamma_\omega^\pm)$, where no boundary condition is imposed. Since $z \notin \mathbb{R}$, this equation has a unique solution $f_\omega^\pm(\cdot; z) \in L^2(\Gamma_\omega^\pm)$ (the proof is similar to that of [?, lem. 9.7, thm. 9.1]). The $m_\omega^\pm$ functions are then defined as follows:

$$(2.2) \quad m_\omega^\pm : \mathbb{C}_+ \rightarrow \mathbb{C},$$

$$(2.3) \quad m_\omega^\pm(z) = \frac{\sum_{e \sim o} \left. \frac{df_\omega^\pm}{dx} \right|_e(o; z)}{f_\omega^\pm(o; z)},$$

where we abbreviate notation by using o instead of $o(\Gamma_\omega^\pm)$, the edges $e \sim o$ are the ones incident to o , and the derivatives are taken in the outward pointing direction from o (the $\pm$ sign affects the chosen edges and the derivative directions). By standard theory (see [?, thm. 18.2]), $m_\omega^\pm(z)$ are Herglotz functions, i.e., they are analytic and their image is contained in $\mathbb{C}_+$.

Our goal in this section is to prove the following Theorem:

Theorem 2.1 (Borg-Marchenko). *Assume that the individual tiles $(T_a)_{a \in \mathcal{A}}$ are given and satisfy the conditions of Theorem 1.6. Then for any $\omega \in \mathcal{A}^\mathbb{Z}$, the pair of corresponding m -functions $m_\omega^\pm(z)$ uniquely determines ω .*

This result essentially states that one can reconstruct the graphs $\Gamma_\omega^\pm$ from the functions $m_\omega^\pm(z)$, assuming that the basic building blocks (tiles) are known. The proof relies on a periodic orbit expansion for $m_\omega^\pm(z)$, which is developed next. Starting from here and in the following subsections, we conveniently focus on the half-graph Γ_ω^+ (and the associated m -function $m_\omega^+(z)$), noting that Γ_ω^- and $m_\omega^-(z)$ may be treated similarly. We also keep ω fixed, and thus from now on and until the end of the section, we abbreviate and simply write o instead of $o(\Gamma_\omega^+)$.

In order to state the periodic orbit expansion for $m_\omega^+(z)$, we first introduce a definition of a *tailed* periodic orbit (see also Figure 2.1).

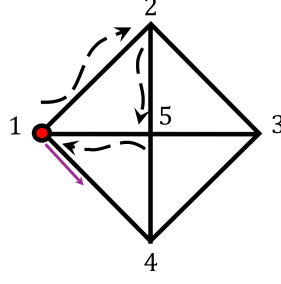


FIGURE 2.1. A square kite graph with vertex set $\{1, 2, 3, 4, 5\}$, together with a TPO $p = ((1, 2), (2, 5), (5, 1), (1, 4))$ around the marked vertex 1 (the terminal direction is marked in purple).

Definition 2.2.

- (1) Let Γ be a metric graph. A *tailed periodic orbit* (TPO) p around a vertex $v \in \Gamma$ is a finite sequence of successively connected directed edges in Γ , $(v_1, v_2), (v_2, v_3), \dots, (v_n, v_{n+1})$, where $v_1 = v_n = v$ (see Figure 2.1). We think of such TPOs as periodic orbits around v , along with a chosen terminal direction (v_n, v_{n+1}) (thought of as the “tail” of the orbit). In this work, we mostly consider TPOs around the origin of $\Gamma_\omega^\pm$ (so $v = o$).
- (2) For a TPO p , we denote its total length by $\ell_p := \sum_{i=1}^{n-1} |(v_i, v_{i+1})|$, where $|\cdot|$ denotes the edge length (Note that we do not include the tail (v_n, v_{n+1}) in the sum). We denote its topological length by $|p| := n - 1$, i.e., the number of directed edges in the sequence p , excluding the tail (v_n, v_{n+1}) .
- (3) Let Γ_ω^+ be a half-infinite tiling graph, with origin vertex o . The set of TPOs around the origin for Γ_ω^+ is denoted by $\mathcal{P}_\omega^+$.

Theorem 2.3. *The m -function $m_\omega^+(z)$ has the following expansion in terms of the tailed periodic orbits of Γ_ω^+ :*

$$(2.4) \quad m_\omega^+(z) = -\sqrt{-z} \sum_{p \in \mathcal{P}_\omega^+} A_p e^{-\sqrt{-z} \ell_p}.$$

The coefficients A_p appearing in the theorem are explicitly given in the proof. Theorem 2.3 is fully proven in the following subsection. A reader who is less interested in the proof details may proceed directly to Subsection 2.3, where the periodic orbit expansion (2.4) is applied in the proof of the Borg-Marchenko Theorem 2.1 to recover the sequence ω .

We remark that in many cases (see e.g. [?, thm 9.22] and [?, thm 3.5]), Borg-Marchenko results on the line rely on first proving asymptotic estimates for $m(z)$ as the spectral parameter z tends to infinity. Theorem 2.3 not only proves the Borg-Marchenko Theorem 2.1 directly (with no need for such asymptotics), but can also be used to provide such asymptotics for $m^\pm(z)$ as $z \rightarrow \infty$, as in the alternative proofs.

2.2. The dynamical Dirichlet-to-Neumann map and the proof of the TPO expansion. We start by introducing the relevant definitions related to the dynamical

Dirichlet-to-Neumann map (also known as the dynamical response operator), and refer to [?, sec. 19] for a more thorough background. We then use it to prove the TPO expansion for the m -function stated in Theorem 2.3.

Consider the dynamical wave equation on Γ_ω^+ :

$$(2.5) \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2},$$

$$(2.6) \quad u(x, 0) = 0,$$

$$(2.7) \quad u_t(x, 0) = 0,$$

with the Kirchhoff condition imposed at all vertices except for o , which is subject to the boundary control:

$$(2.8) \quad u(o, t) = b(t),$$

with $b \in L^2_{\text{loc}}(0, \infty)$. The *dynamical Dirichlet-to-Neumann (DDTN) map* Λ_ω^+ is the mapping:

$$(2.9) \quad b(\cdot) \xrightarrow{\Lambda_\omega^+} \partial_n u(o, \cdot) := \sum_{e \sim o} \frac{\partial u|_e(o, \cdot)}{\partial x},$$

where the edges $e \sim o$ are the ones incident to o in Γ_ω^+ , and the derivatives are taken in the outward pointing direction from o . By [?, eq. (19.12)], the DDTN map and the m -function are in one-to-one correspondence, given by:

$$(2.10) \quad m_\omega^+(-s^2) = \frac{\widehat{(\Lambda_\omega^+ b)}(s)}{\widehat{b}(s)},$$

where $\widehat{(\cdot)}$ is the Laplace transform:

$$(2.11) \quad \widehat{f}(s) = \int_0^\infty f(t) e^{-st} dt.$$

Our next goal is to obtain an expansion for Λ_ω^+ (and hence for m_ω^+) in terms of tailed periodic orbits (TPOs). Before doing so, we present the scattering matrix of a quantum graph (which was first introduced in [?, ?]). Let Γ be a quantum graph and let $v \in \mathcal{V}(\Gamma)$. On each edge e incident to v , one can write the solutions to the eigenvalue equation $-\frac{d^2}{dx^2}g = k^2g$ in the form

$$(2.12) \quad g|_e(x) = a_e e^{ikx} + a_{\hat{e}} e^{ik(\ell_e - x)},$$

with $a_e, a_{\hat{e}} \in \mathbb{C}$. Here, we think of e as a directed edge pointing towards a vertex v , and denote its reversal by $\hat{e}$. Note that for g to also satisfy the Kirchhoff vertex condition at v , the vectors $\vec{a}_{\text{in}} := (a_{e_1}, \dots, a_{e_{\deg(v)}})$, $\vec{a}_{\text{out}} := (a_{\hat{e}_1}, \dots, a_{\hat{e}_{\deg(v)}})$ must satisfy a particular linear relation. The *scattering matrix* associated with v is the unique unitary matrix $S_v(k) \in U(\deg(v))$ that expresses this linear relation:

$$(2.13) \quad \vec{a}_{\text{out}} = S_v(k) e^{ik\vec{L}} \vec{a}_{\text{in}},$$

where $\vec{L} = \text{diag}(\ell_{e_1}, \dots, \ell_{e_{\deg(v)}})$ is the diagonal matrix of edge lengths.

The scattering matrix generally depends on the imposed vertex conditions. Given two directed edges e, e' , we write that $e \rightarrow e'$ at v if the vertex v is both the end vertex of e and the starting vertex of e' . One can show that for the Kirchhoff condition, $S_v(k) =: S_v$ is independent of k and is given by (cf. [?, eq. (3.6)]),

$$(2.14) \quad [S_v]_{e',e} = \begin{cases} \frac{2}{\deg(v)} - 1, & e' = \hat{e}, \\ \frac{2}{\deg(v)}, & e \rightarrow e' \text{ at } v \text{ and } e' \neq \hat{e}, \\ 0, & \text{Otherwise.} \end{cases}$$

The scattering matrix is a common tool in the theory of quantum graphs, not only for spectral analysis, but also for describing wave propagation, which we indeed employ in the following proof.

Proof of Theorem 2.3. We show that the DDTN map can be represented as a convolution with a generalized kernel:

$$(2.15) \quad \Lambda_\omega^+ b(t) = (k_\omega^+ * b)(t) := \int_0^\infty k_\omega^+(t-s) b(s) ds.$$

Combining this with (2.10) and using that the Laplace transform is multiplicative with respect to convolution gives the simple relation between the kernel $k_\omega^+(t)$ and the m -function:

$$(2.16) \quad m_\omega^+(-s^2) = \frac{\widehat{(k_\omega^+ * b)}(s)}{\widehat{b}(s)} = \widehat{k_\omega^+}(s).$$

Hence, to prove the theorem we show (2.15) and develop a TPO expansion for the generalized kernel, $k_\omega^+(t)$. We do both simultaneously.

The key idea is that the solution $u(x, t)$ to the dynamical wave equation (2.5) consists of waves propagating along the edges of the graph, with different amplitudes corresponding to different TPOs. We systematically construct the expansion of $k_\omega^+(s)$ by analyzing the solution to the dynamical wave equation on Γ_ω^+ , keeping track of how it scatters at each vertex. Due to the finite propagation speed of the wave equation (which equals to 1 by (2.5)), the solution at time t is always supported on the geodesic ball of radius t around the origin, which allows us to focus on a bounded neighborhood of the origin.

Step 1: Short-time behavior of the wave equation.

Recall that the d'Alembert solution to the dynamical wave equation is given as a superposition of an ingoing and outgoing wave on each edge:

$$(2.17) \quad u|_e(x, t) = F_e(t-x) + G_e(t+x).$$

Through direct substitution into (2.5), (2.6), (2.7), (2.8), one can verify that for $0 < t \leq \min_{e \sim o} \{|e|\}$, the solution to the wave equation is given by

$$(2.18) \quad u|_e(x, t) = \begin{cases} b(t-x), & e \sim o, \\ 0, & \text{otherwise.} \end{cases}$$

In other words, at short times, the wave only propagates along edges that are incident to the origin o , and the associated d'Alembert solution is then given by an outgoing wave with profile equal to that of the boundary control $b(t)$. This means that at short times, the DDTN satisfies:

$$(2.19) \quad \Lambda_{\omega}^+ b(t) = \partial_n u(o, t) = - \sum_{e \sim o} b'(t),$$

and so we may choose the representing kernel to be

$$(2.20) \quad k_{\omega}^+(t) = -\deg(o) \cdot \delta'(t) + \dots,$$

where the additional terms vanish for $t < \min_{e \sim o} \{|e|\}$, and the equality above is in the sense of distributions. Here, δ' is a delta prime distribution, defined through

$$(2.21) \quad (\delta' * f) := f'(0).$$

Step 2: First scattering event.

Let v be the vertex which is the nearest to the origin and denote $L := |(o, v)|$. Then, once the wave $u|_e(x, t)$ reaches v at time $t = L$, it scatters to all neighboring edges $e' \sim v$ (see Figure 2.2), where the associated wave amplitudes at each edge are determined by the Kirchhoff condition at v . It is simple to verify using Equations (2.12), (2.13) that for a plane wave solution to the wave equation of the form

$$(2.22) \quad w|_e(x, t) = e^{i(t-kx)},$$

the wave amplitude at e following this event is determined via the scattering matrix element $[S_v]_{\hat{e}, e}$, i.e.,

$$(2.23) \quad w_e(x, t) = e^{i(t-kx)} + [S_v]_{\hat{e}, e} e^{i((t-L)-(L-kx))}.$$

Then, straightforward application of Fourier theory (writing $u|_e(x, t)$ as an infinite superposition of plane waves) shows that similarly, our d'Alembert solution at the edge e after the first scattering event is given by

$$(2.24) \quad u|_e(x, t) = b(t - x) + [S_v]_{\hat{e}, e} b((t - L) - (L - x)),$$

see also [?, sec. 19.5] for additional details. Mainly, the scattering process at this vertex introduces a secondary wave that backscatters into its initial edge e with amplitude $[S_v]_{\hat{e}, e}$, but with a reversed direction (see Figure 2.2).

At time $t = 2L$, the secondary wave reaches the origin o , and similarly to before, scatters in all directions $e' \sim o$ with amplitude $[S_o]_{e', \hat{e}}$ due to the Kirchhoff condition at the origin. At short times after this event¹, the normal derivative of $u(x, t)$ at the origin is given by:

$$(2.25) \quad \partial_n u(o, t) = - \sum_{e' \sim o} \left(b'(t) + [S_o]_{e', \hat{e}} [S_v]_{\hat{e}, e} b'(t - 2L) \right).$$

¹i.e. times with respect to which there is no additional scattering event which makes another wave reach the origin.

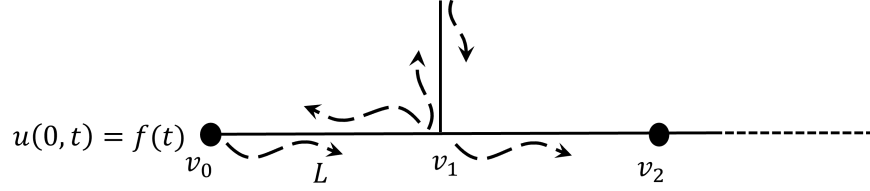


FIGURE 2.2. Wave scattering on a comb graph at short times. Initially, the solution propagates along the directed edge $e = (v_0, v_1)$, until it reaches the vertex v_1 at time $t = L$. The solution then scatters in three directions, and in particular backscatters into the directed edge (v_1, v_0) with wave amplitude $[S_{v_1}]_{\hat{e}, e} = \frac{2}{\deg(v_1)} - 1 = -\frac{1}{3}$.

Thus, for $t \in [0, 2L]$, we may choose the kernel to be

$$(2.26) \quad k_{\omega}^{+}(t) = -\deg(o) \delta'(t) - \left(\sum_{e' \sim o} [S_o]_{e', \hat{e}} [S_v]_{\hat{e}, e} \right) \delta'(t - 2L).$$

Note that the terms $\delta'(t)$ and $\delta'(t - 2L)$ correspond exactly to the shortest TPOs around the origin: $\delta'(t)$ corresponds to the trivial TPO of length 0 and $\delta'(t - 2L)$ correspond to all TPOs consisting of going back and forth along e and an arbitrary tail, $e' \sim o$.

Step 3: Scattering in longer times.

One can continue expanding the solution $u(x, t)$ in the same fashion. As time progresses beyond $t = 2L$, the d'Alembert solution $u(x, t)$ continues to propagate and undergoes additional scattering events at vertices further away from the origin. On edges adjacent to o , $u(x, t)$ will always consist of a superposition of delayed waves that scattered across some TPO around the origin. The term corresponding to a TPO p will be of the form $A_p b(t - \ell_p)$, where the delay time ℓ_p is equal to the metric length of the TPO p , and the amplitude A_p is the product of scattering coefficients along the TPO:

$$(2.27) \quad A_p := \prod_{j=1}^{|p|} [S_{v_{j+1}}]_{(v_{j+1}, v_{j+2}), (v_j, v_{j+1})}.$$

On the level of the DDTN kernel $k_{\omega}^{+}(s)$, this gives an infinite TPO expansion:

$$(2.28) \quad \partial_n u(o, t) = - \sum_{p \in \mathcal{P}_{\omega}^{+}} A_p b'(t - \ell_p),$$

$$(2.29) \quad \Rightarrow \quad k_{\omega}^{+}(t) = - \sum_{p \in \mathcal{P}_{\omega}^{+}} A_p \delta'(t - \ell_p).$$

Equation (2.4) can now be obtained by substituting (2.29) into (2.16) and using the identity $\widehat{\delta'(t - \ell)}(s) = se^{-\ell s}$, which finally gives:

$$(2.30) \quad m_{\omega}^{+}(z) = -\sqrt{-z} \sum_{p \in \mathcal{P}_{\omega}^{+}} A_p e^{-\sqrt{-z} \ell_p},$$

finishing the proof. $\square$

2.3. Proving the Borg-Marchenko result.

Proof of the Theorem 2.1 (Borg-Marchenko). Let $(\omega(n))_{n=0}^\infty \in \mathcal{A}^\mathbb{N}$. We use the TPO expansion (2.4) of $m_\omega^+(z)$ in order to inductively recover each element of $(\omega(n))_{n=0}^\infty$. Hence, the graph which is considered below is Γ_ω^+ . In exactly the same manner $m_\omega^-(z)$ may be used to recover $(\omega(n))_{n=-1}^{-\infty}$.

Base case, $n = 0$:

By Assumption 1.5, there is at most one $\bar{a} \in \mathcal{A}$ in which $v_{\bar{a}}^1$ and $v_{\bar{a}}^2$ are connected by a single edge (such a tile might be an interval). Consider the TPO expansion of $m_\omega^+(z)$. First, note that if $\omega(0) = a \neq \bar{a}$, then setting

$$(2.31) \quad \ell_{\min}^a := \min \{ \ell_e : e \sim v_a^1 \},$$

the term $-A_a \sqrt{-z} e^{-\sqrt{-z} 2\ell_{\min}^a}$ appears in this TPO expansion with $A_a \neq 0$. Indeed, in this case, there are exactly $\deg(o)$ TPOs of length $2\ell_{\min}^a$. These are the TPOs that traverse the (unique) edge $e_{\min} \sim o$ of length $\ell_{\min}^a$, backscatter and return to the origin, and then scatter to some other edge $e' \sim o$. Since we assume that $a \neq \bar{a}$, we get that there exists a vertex $v_{\min}$ in T_a such that $e_{\min} = \{o, v_{\min}\}$, then direct computation using (2.14) and the second term in (2.26) shows that the total scattering amplitude contribution of these shortest TPOs is given by

$$(2.32) \quad \begin{aligned} C_a &= \sum_{e' \sim o} [S_o]_{e', \hat{e}} [S_{v_{\min}}]_{\hat{e}, e} \\ &= \left(\frac{2}{\deg(v_{\min})} - 1 \right) \left(\frac{2}{\deg(o)} - 1 \right) + \sum_{e_{\min} \neq e' \sim o} \left(\frac{2}{\deg(v_{\min})} - 1 \right) \cdot \frac{2}{\deg(o)} \\ &= \left(\frac{2}{\deg(v_{\min})} - 1 \right) \left(\frac{2}{\deg(o)} - 1 + \frac{2(\deg(o) - 1)}{\deg(o)} \right) \\ &= \frac{2}{\deg(v_{\min})} - 1 \neq 0, \end{aligned}$$

where the last inequality follows since $v_{\min}$ is included in T_a (by $a \neq \bar{a}$) and Assumption 1.5 asserts that $\deg(v_{\min}) \geq 3$.

Now, iterate over the list $(\ell_{\min}^a)_{a \in \mathcal{A} \setminus \{\bar{a}\}}$ in increasing order, keeping in mind that no two elements in this list are equal, by Assumption 1.5. For each $\ell_{\min}^a$, examine the corresponding term $-A_a \sqrt{-z} e^{-\sqrt{-z} 2\ell_{\min}^a}$ in the TPO expansion (2.4) of $m_\omega^+(z)$ (with the prefactor A_a being possibly zero). By the discussion above, for each $a \in \mathcal{A} \setminus \{\bar{a}\}$ such that $A_a = 0$, we can immediately deduce that $\omega(0) \neq a$. Consider the minimal $\ell_{\min}^a$ with $a \in \mathcal{A} \setminus \{\bar{a}\}$ such that $A_a \neq 0$. Then there exists a TPO of length $2\ell_{\min}^a$ around o , and hence the unique edge of length $\ell_{\min}^a$ must lie in the first tile, implying that $\omega(0) = a$. If, on the other hand, $A_a = 0$ for all $a \in \mathcal{A} \setminus \{\bar{a}\}$, then the only remaining possibility is $\omega(0) = \bar{a}$. We note that by Assumption 1.5 it is not possible that there is an arrangement of a few tiles (more than one) which contains a TPO of just two scattering events, such that this TPO's length equals $2\ell_{\min}^a$ for some $a \in \mathcal{A}$. The impossibility of such arrangements validates that the process described above indeed

yields the correct $\omega(0)$.

Induction step:

Suppose that $\omega(0), \dots, \omega(n-1)$ are known, and we wish to recover $\omega(n)$. Let p be a TPO of minimal length that travels from the origin (which is also $v_{\omega(0)}^1$) to the right boundary vertex $v_{\omega(n-1)}^2$, and then returns to the origin along the reversal of the same path (there might be several such TPOs, either differing only by their tail, or by other edges). We now write an expansion of all TPOs of length ℓ_p which are contained in the first known n tiles. In this expansion there is an unknown parameter which is the degree of the right boundary vertex of the TPO. The degree of that vertex within Γ_ω^+ equals $\deg(v_{\omega(n-1)}^2) + \deg(v_{\omega(n)}^1)$, but we do not know it until we recover what is $\omega(n)$. Comparing this TPO expansion to the corresponding term $-A_p \sqrt{-z} e^{-\sqrt{-z} \ell_p}$ allows to recover the value of $\deg(v_{\omega(n)}^1)$. Doing so, we may already restrict the possible choices for $\omega(n)$ to be in a subset $\mathcal{A}_0 \subset \mathcal{A}$.

Recall that we take $\bar{a}$ to be such that the corresponding tile $T_{\bar{a}}$ has $v_{\bar{a}}^1$ and $v_{\bar{a}}^2$ connected by a single edge; there is at most one such tile (by Assumption 1.5) and there might not be such at all. As in the base case, to recover $\omega(n)$ we iterate over the list $(\ell_{\min}^a)_{a \in \mathcal{A}_0 \setminus \{\bar{a}\}}$ in increasing order. For each $a \in \mathcal{A}_0 \setminus \{\bar{a}\}$, focus on the term $-A_a \sqrt{-z} e^{-\sqrt{-z}(\ell_p + 2\ell_{\min}^a)}$ in the TPO expansion (2.4) of $m_\omega^+(z)$. Since we already know the first n tiles, we write an expansion of all TPOs of length $\ell_p + 2\ell_{\min}^a$ which are contained in the first known n tiles (if such exist). If these TPOs pass via $v_{\omega(n-1)}^2$ (and go back into $T_{\omega(n-1)}$ rather than entering $T_{\omega(n)}$) we know the corresponding scattering amplitude since the value $\deg(v_{\omega(n-1)}^2) + \deg(v_{\omega(n)}^1)$ is already known, as explained above. Hence, we may explicitly compute the overall contribution of all TPOs of length $\ell_p + 2\ell_{\min}^a$ contained in the first known n tiles to the TPO expansion of $m_\omega^+(z)$. This yields a term of the form $-\tilde{A}_a \sqrt{-z} e^{-\sqrt{-z}(\ell_p + 2\ell_{\min}^a)}$ (and again, $\tilde{A}_a$ may be 0).

Similar to before, the first $a \in \mathcal{A}_0 \setminus \{\bar{a}\}$ for which $A_a \neq \tilde{A}_a$, implies that there are additional TPOs of length $\ell_p + 2\ell_{\min}^a$ which pass through the $n+1$ tile, $T_{\omega(n)}$. By the choice of ℓ_p and $\ell_{\min}^a$, such TPOs must consist of the mentioned shortest path from the origin to $v_{\omega(n-1)}^2$, which go forth and back the unique edge $e_{\min}$ of length $\ell_{\min}^a$ in $T_{\omega(n)}$ adjacent to $v_{\omega(n)}^1$, and then return to the origin through the initial path (reversed). Since no two elements in the list $(\ell_{\min}^a)_{a \in \mathcal{A}_0 \setminus \{\bar{a}\}}$ are equal, we deduce that $\omega(n) = a$. If instead $A_a = \tilde{A}_a$ for all $a \in \mathcal{A}_0 \setminus \{\bar{a}\}$, then $\omega(n) = \bar{a}$. This completes the reconstruction of the sequence ω . $\square$

Remark 2.4. We emphasize that Theorem 2.1 shows that one can reconstruct Γ_ω from $m_\omega^\pm(z)$ assuming that the possible tiles $(T_a)_{a \in \mathcal{A}}$ are already known. It is natural to ask when one can recover Γ_ω from $m_\omega^\pm(z)$ without apriori knowing the possible tiles. This is closely tied to recovering the individual (compact) tiles from their respective m -functions. This question has been addressed in works such as [?, ?, ?]. This can, for instance, be done assuming that the tiles $(T_a)_{a \in \mathcal{A}}$ are trees, but in general counterexamples exist.

The following is an immediate corollary of the TPO expansion for $m_\omega^\pm(z)$ and the Borg-Marchenko theorem:

Corollary 2.5. $\omega_n \rightarrow \omega$ in the product topology on $\mathcal{A}^{\mathbb{Z}}$ if and only if $m_{\omega_n}^{\pm}(z) \rightarrow m_{\omega}^{\pm}(z)$ locally uniformly. In other words, the mapping $\omega \mapsto m_{\omega}^{\pm}(z)$ is bijective and continuous.

This result will be used in the proof of the oracle theorem (Theorem 4.3) to pass from identities of m -functions to continuity of the corresponding restriction maps.

The proof of the Borg-Marchenko result above was tailored for tiling graphs which satisfy the edge length condition of Assumption 1.5 (which is a generic assumption). There are many other families of tiles which do not satisfy this assumption. Nevertheless, a similar Borg-Marchenko result may be proven for those as long as the TPOs of the different tiles are “sufficiently distinguishable”. One class of graphs whose tiles do not satisfy the edge length condition of Assumption 1.5 is the class of decorated $\mathbb{Z}$ -graphs from Definition 1.3. If we adjust the edge length assumption to this class of tiles, we may get the following Borg-Marchenko result.

Assumption 2.6. Let Γ_{ω} be a decorated $\mathbb{Z}$ -graph. Suppose that the decorations $(\mathcal{D}_a)_{a \in \mathcal{A}}$ satisfy that at most one decoration is trivial (i.e., a single vertex), and that the decorations have no vertices of degree 2, except for possibly the base vertex u_a .

Proposition 2.7. Let Γ_{ω} be a decorated $\mathbb{Z}$ -graph as in Assumption 2.6. Suppose that for each $a \in \mathcal{A}$, the quantity

$$(2.33) \quad \ell_{\min}^a := \min \{ \ell_e : e \sim v_a \}$$

is realized by a unique edge, and that $\ell_{\min}^a \neq \ell_{\min}^{a'}$ for all $a \neq a'$. Then for each $\omega \in \Omega$, the pair of m -functions $m_{\omega}^{\pm}(z)$ uniquely determines the sequence ω .

Corollary 2.8. For a Baire-generic and Lebesgue almost-sure choice of the decoration edge lengths, $m_{\omega}^{\pm}(z)$ uniquely determine ω for decorated $\mathbb{Z}$ -graphs.

The proof of Proposition 2.7 is completely analogous to that of Theorem 2.1, and appears in the appendix of the PhD thesis [?, appendix A].

3. DYNAMICAL CHARACTERIZATION OF $\sigma(H_{\Omega})$

In this section, we present several important spectral and dynamical properties of tiling graphs that will be used in the proof of Theorem 1.6.

3.1. Transfer matrices and the cocycle. A basic tool which is required for our analysis is transfer matrices. We give a brief review of those, and refer to [?] for a more thorough overview.

As always, we fix the minimal subshift (Ω, S) , with an associated family of metric tiling graphs $(\Gamma_{\omega})_{\omega \in \Omega}$. Consider the following ODE on Γ_{ω} :

$$(3.1) \quad -\frac{d^2}{dx^2} \phi_{\omega} = E \phi_{\omega} \quad (E \in \mathbb{R}),$$

with the Kirchhoff condition imposed at all vertices of Γ_{ω} . For a solution ϕ_{ω} to (3.1), we denote

$$(3.2) \quad \Phi_{\omega} := \left(\begin{array}{c} \phi_{\omega}(o) \\ \sum_{e \in \Gamma_{\omega}^+ : e \sim o} \partial \phi_{\omega}|_e(o) \end{array} \right) \in \mathbb{C}^2,$$

where the sum is taken over edges incident to the origin from the right, and the derivatives are taken in the positive direction. Standard ODE theory then shows that for all but a discrete set of $E \in \mathbb{R}$ (described after Definition 3.2), there exists a unique matrix $\mathcal{M}(\omega, E) \in SL(2, \mathbb{C})$ such that for any solution ϕ_ω to (3.1),

$$(3.3) \quad \Phi_{S\omega} = \mathcal{M}(\omega, E) \Phi_\omega.$$

The matrix $\mathcal{M}(\omega, E)$ is known as the one-step transfer matrix (or monodromy matrix). One can similarly define the n -step transfer matrix by

$$(3.4) \quad \mathcal{M}_n(\omega, E) := \begin{cases} \mathcal{M}(S^{n-1}\omega, E) \cdot \dots \cdot \mathcal{M}(\omega, E), & n \geq 1, \\ I_2, & n = 0, \\ \mathcal{M}(S^n\omega, E)^{-1} \cdot \dots \cdot \mathcal{M}(S^{-1}\omega, E)^{-1}, & n \leq -1. \end{cases}$$

The transfer matrices allow to define a dynamical system on $SL(2, \mathbb{C})$ (known as a cocycle) through

$$(3.5) \quad (\omega, n) \mapsto \mathcal{M}_n(\omega, E).$$

Remark 3.1. Since neighboring tiles are attached to one another through a single vertex, our cocycle consists of 2×2 matrices. If one allows for more than a single attachment vertex, the cocycle will consist of larger matrices, which is much more technically challenging. We do not pursue this direction here, see for instance [?].

There is a discrete set of values $E \in \mathbb{R}$ such that there exists a nontrivial solution to (3.1) with initial condition $\Phi_\omega = \vec{0}$ for some $\omega \in \Omega$ (see e.g., [?, thm. 2.1 and cor. 2.4]). Hence, the cocycle definition via (3.3) fails for this set of energies (which are sometimes called resonances).

Definition 3.2. Denote by $\mathcal{B} \subset \mathbb{R}$ the set of energies $E \in \mathbb{R}$ for which the cocycle $\mathcal{M}(\cdot, E)$ is not well-defined.

In general, the set $\mathcal{B}$ may intersect $\sigma(H_\Omega)$ nontrivially – for instance, this is the case for the aperiodic antitrees studied in [?]. At such spectral points, the analysis of $\sigma(H_\Omega)$ through the transfer matrix applied below breaks down. However, since the set $\mathcal{B}$ is discrete, it is enough to analyze $\sigma(H_\Omega)$ on $\mathbb{R} \setminus \mathcal{B}$ in order to prove that $\sigma(H_\Omega)$ is of Lebesgue measure zero. In Subsection 4.2, we show that generically $\mathcal{B} \cap \sigma(H_\Omega) = \emptyset$ for decorated $\mathbb{Z}$ -graphs.

3.2. Characterization of the spectrum via the Lyapunov exponent. In this subsection, we provide a dynamical characterization of the spectrum, by showing that $\sigma(H_\omega)$ is equal (up to intersection with $\mathcal{B}$) to the zero set of the Lyapunov exponent associated with the cocycle. Such a result has been proven in the ergodic Schrödinger operator setting for many systems, including Sturmian Hamiltonians and the Almost-Mathieu operator (cf. [?]).

Given $E \in \mathbb{R} \setminus \mathcal{B}$ and our cocycle $\mathcal{M}(\omega, E)$, Kingman's ergodic theorem ensures that the quantity

$$(3.6) \quad L_\omega(E) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \|\mathcal{M}_n(\omega, E)\|,$$

is identical for almost every $\omega \in \Omega$. We denote this almost-sure value by $L_\Omega(E)$, and call $L_\Omega(E)$ the *Lyapunov exponent* of the cocycle.

Theorem 3.3. *There exists a closed subset $\sigma(H_\Omega) \subset \mathbb{R}$ such that $\sigma(H_\omega) = \sigma(H_\Omega)$ for all $\omega \in \Omega$, and furthermore,*

$$(3.7) \quad \sigma(H_\Omega) \setminus \mathcal{B} = \mathcal{Z} := \{E \in \mathbb{R} \setminus \mathcal{B} : L_\Omega(E) = 0\}.$$

The proof of Theorem 3.3 appears at the end of this subsection. To prove it, we first need to present several lemmas. These lemmas are adaptations of existing techniques to quantum graphs. Since in this subsection we are mostly adapting many of the techniques presented in previous works to the quantum graph setting, some of the results will not be proven completely, and the proofs will only be outlined, with reference to the relevant works.

Definition 3.4. Define

$$(3.8) \quad \mathcal{UH} := \{E \in \mathbb{R} \setminus \mathcal{B} : \exists \gamma > 1, C > 0; \|\mathcal{M}_n(\omega, E)\| \geq C\gamma^{|n|}, \forall \omega \in \Omega, \forall n \in \mathbb{Z}\},$$

$$(3.9) \quad \mathcal{NUH} := \{E \in \mathbb{R} \setminus \mathcal{B} : L_\Omega(E) > 0\} \setminus \mathcal{UH}.$$

It is easy to see that $\mathbb{R} \setminus \mathcal{B} = \mathcal{UH} \sqcup \mathcal{NUH} \sqcup \mathcal{Z}$, and this partition into uniformly hyperbolic, non-uniformly hyperbolic and non-hyperbolic energies forms the basis to our spectral analysis.

Lemma 3.5 ([?, thm. 1.2]). *Let $E \in \mathcal{UH}$. Then there is no nontrivial solution ϕ_ω of (3.1) such that the vector Φ_n in (3.2) is subexponentially bounded in both directions in Γ_ω , i.e. there is no nontrivial solution such that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ with*

$$(3.10) \quad \|\Phi_n\| \leq C_\varepsilon e^{\varepsilon|n|} \quad \forall n \in \mathbb{Z}.$$

Recall that a maximal spectral measure ρ_ω for H_ω is a Borel measure on $\mathbb{R}$ such that, for every Borel set $A \subset \mathbb{R}$,

$$(3.11) \quad \rho_\omega(A) = 0 \iff \chi_A(H_\omega) = 0,$$

where $\chi_A(H_\omega)$ denotes the spectral projection of H_ω associated with A . We say that a property holds for spectrally almost every E if it holds for ρ_ω -almost every E , and call A spectrally null if $\rho_\omega(A) = 0$.

Lemma 3.6. *For spectrally almost every $E \in \sigma(H_\omega)$, there is a nontrivial solution ϕ_ω of (3.1) such that for all $\beta > 0$,*

$$(3.12) \quad e^{-\beta|x|} \phi_\omega(x) \in L^2(\Gamma_\omega),$$

where $|x|$ denotes the geodesic distance of x from the origin o . In particular, the vectors Φ_n in (3.2) are subexponentially bounded as in (3.10).

Proof of Lemma 3.6. By [?, thm. 1.1], for the result to hold, it suffices to verify two conditions:

1. For all $\alpha > 0$ and $x_0 \in \Gamma_\omega$,

$$(3.13) \quad \lim_{R \rightarrow \infty} e^{-\alpha R} |B_R^\omega(x_0)| = 0,$$

where $B_R^\omega(x_0)$ is the geodesic ball of radius R around $x_0 \in \Gamma_\omega$ and $|\cdot|$ denotes the Lebesgue measure.

2. For all $\omega \in \Omega$ and $t > 0$, the heat semigroup e^{-tH_ω} is ultracontractive, i.e.,

$$(3.14) \quad e^{-tH_\omega} : L^2(\Gamma_\omega) \rightarrow L^\infty(\Gamma_\omega).$$

The first condition holds since the Lebesgue measure of geodesic balls in Γ_ω grows piecewise linearly with R . The fact that the second condition holds is shown in [?, sec. 4.2]. $\square$

Lemma 3.7. *Suppose that ϕ_ω is a subexponentially bounded solution of (3.1) on Γ_ω . Meaning, for all $\kappa > 0$ there is $C > 0$ such that*

$$(3.15) \quad |\phi_\omega(x)| \leq Ce^{\kappa|x|}, \quad \forall x \in \Gamma_\omega.$$

Then $E \in \sigma(H_\omega)$.

The proof is based on using the subexponential boundedness to construct a Weyl sequence for E . The full details are given in [?, lem. 4.13].

Proof of Theorem 3.3. The fact that $\sigma(H_\omega)$ is independent of ω follows from standard arguments for minimal dynamical systems, see for instance [?, cor. 4.5], [?, prop. 2.1], [?, thm. 5] and references therein for similar proofs. We wish to prove the equality (3.7). Recall that $\mathbb{R} \setminus \mathcal{B} = \mathcal{UH} \sqcup \mathcal{N}\mathcal{UH} \sqcup \mathcal{Z}$, and further explore this decomposition as follows. Since the dynamical system (Ω, S) satisfies Boshernitzan's condition, we conclude by [?, thm. 1] that the Lyapunov exponent $L_\Omega(E)$ in (3.6) converges uniformly in $\omega \in \Omega$. This implies that $\mathcal{N}\mathcal{UH} = \emptyset$ (see [?, app. A]), and $\mathbb{R} \setminus \mathcal{B} = \mathcal{UH} \sqcup \mathcal{Z}$. We thus only need to prove that $\sigma(H_\Omega) \setminus \mathcal{B} = (\mathbb{R} \setminus \mathcal{B}) \setminus (\mathcal{UH})$, and do it by showing both inclusions. Fix arbitrary $\omega \in \Omega$ and assume that $E_0 \in \mathcal{UH}$. Since $\mathcal{UH}$ is open, we can choose some open interval $I_{E_0} \subset \mathcal{UH}$ around E_0 . For each $E \in I_{E_0}$, Lemma 3.5 rules out a nontrivial solution of (3.1) on Γ_ω whose vectors Φ_n are subexponentially bounded in both directions. On the other hand, by Lemma 3.6, such a solution exists for spectrally almost every $E \in \sigma(H_\omega)$. Hence I_{E_0} is spectrally null, that is $\chi_{I_{E_0}}(H_\omega) = 0$. Since I_{E_0} is open and $\sigma(H_\omega) = \text{supp}(\rho_\omega)$, it follows that $I_{E_0} \cap \sigma(H_\omega) = \emptyset$. In particular $E_0 \notin \sigma(H_\omega)$ and we get that $\sigma(H_\Omega) \setminus \mathcal{B} = \sigma(H_\omega) \setminus \mathcal{B} \subset (\mathbb{R} \setminus \mathcal{B}) \setminus (\mathcal{UH})$.

To prove the other inclusion, Lemma 3.7 tells us that it suffices to show that for each $E \in \mathcal{Z}$, the associated solution of (3.1) has subexponential growth. But this is true by the definition of $\mathcal{Z}$ and the uniform convergence of the Lyapunov exponent. $\square$

4. PROOF OF THEOREMS 1.6 AND 1.9

4.1. The oracle theorem and proof of Theorem 1.6. We are now ready to prove Theorem 1.6 (we in fact prove Theorem 4.5, which immediately implies Theorem 1.6). For this, we will apply the following fundamental result by Kotani [?]:

Theorem 4.1. *For almost every $\omega \in \Omega$ (with respect to the measure μ from (1.8)), the functions $m_\pm^\omega$ are reflectionless on $\mathcal{Z}$. In other words, for almost every $\omega \in \Omega$ and Lebesgue almost every $E \in \mathcal{Z}$,*

$$(4.1) \quad m_\omega^-(E + i0) = -\overline{m_\omega^+(E + i0)},$$

where $m_\omega^\pm(E + i0) = \lim_{\epsilon \rightarrow 0^+} m_\omega^\pm(E + i\epsilon)$.

In other words, Kotani's theorem states that the set

$$(4.2) \quad \mathcal{D}(\mathcal{Z}) := \{\omega \in \Omega : m_\omega^\pm(z) \text{ are reflectionless on } \mathcal{Z}\},$$

is a set of full measure. We note that while Kotani's original result was established for ergodic potentials on $\mathbb{R}$, it can be directly extended to tiling graphs by following essentially the same steps, adapted to metric graphs (see [?, sec. 2]).

For the sake of the next definition we introduce the notations $\mathbb{N}^+ := \{0, 1, 2, \dots\}$ and $\mathbb{N}^- := \{-1, -2, -3, \dots\}$.

Definition 4.2. Denote by $\mathcal{R}_\pm$ the restriction maps of a sequence ω from $\mathbb{Z}$ to $\mathbb{N}^\pm$:

$$(4.3) \quad \mathcal{R}_\pm : \mathcal{A}^\mathbb{Z} \rightarrow \mathcal{A}^{\mathbb{N}^\pm},$$

$$(4.4) \quad \mathcal{R}_\pm((\omega(n))_{n \in \mathbb{Z}}) = (\omega(n))_{n \in \mathbb{N}^\pm}.$$

Theorem 4.3 (Oracle theorem). *Suppose that $\mathcal{Z}$ has positive Lebesgue measure. Then for the set $\mathcal{D}(\mathcal{Z})$ in (4.2), the mapping*

$$(4.5) \quad \mathcal{R} : \mathcal{R}_-(\mathcal{D}(\mathcal{Z})) \rightarrow \mathcal{R}_+(\mathcal{D}(\mathcal{Z})),$$

$$(4.6) \quad \mathcal{R}_-(\omega) \mapsto \mathcal{R}_+(\omega), \quad \omega \in \mathcal{D}(\mathcal{Z}),$$

is well-defined, bijective, and uniformly continuous. The same holds for $\mathcal{R}^{-1}$.

In a nutshell, the oracle theorem states that if $|\mathcal{Z}| > 0$, then the set $\mathcal{D}(\mathcal{Z})$ is in some sense “deterministic”, and one can predict one half of the sequence $\omega \in \mathcal{D}(\mathcal{Z})$ by knowing the other (hence the name ‘oracle’). This is analogous to Remling's Oracle Theorem for one-dimensional Schrödinger and Jacobi operators [?]. Our result is also closely related to [?, thm. 3.7].

Proof. By the Borg-Marchenko Theorem 2.1, we know that the restrictions $\mathcal{R}_\pm(\omega)$ determine $m_\omega^\pm$ uniquely, and this correspondence is continuous by Corollary 2.5. Moreover, since we assume that $m_\omega^-(E + i0) = -\overline{m_\omega^+(E + i0)}$ on the positive measure set $\mathcal{Z}$, and Herglotz functions are uniquely determined by their boundary values on a set of positive measure, we obtain a 1-1 correspondence between the functions $m_\omega^\pm(z)$ for $\omega \in \mathcal{D}(\mathcal{Z})$. We thus see that $\mathcal{R}$ is well-defined, bijective, and continuous by the chain of continuous bijections

$$(4.7) \quad \mathcal{R}_+(\omega) \iff m_\omega^+(z) \iff m_\omega^-(z) \iff \mathcal{R}_-(\omega).$$

Due to [?, lem. 5] and Corollary 2.5, we also know that $\mathcal{D}(\mathcal{Z})$ is closed, and thus compact. Since $\mathcal{R}_\pm$ are continuous, $\mathcal{R}_\pm(\mathcal{D}(\mathcal{Z}))$ are compact as well. Thus the continuous bijection $\mathcal{R} : \mathcal{R}_-(\mathcal{D}(\mathcal{Z})) \rightarrow \mathcal{R}_+(\mathcal{D}(\mathcal{Z}))$ is a homeomorphism. Consequently, both $\mathcal{R}$ and $\mathcal{R}^{-1}$ are uniformly continuous. $\square$

The following lemma will allow us to apply the oracle theorem simultaneously for entire orbits in Ω :

Lemma 4.4. *The set $\mathcal{D}(\mathcal{Z})$ is shift-invariant.*

Proof. Let $z \in \mathbb{C}_+$. For $\omega \in \Omega$ fixed, we denote the elements of the transfer matrix $\mathcal{M}(\omega, E)$ by $\mathcal{M}_{ij}$. Then, combining the definition (2.3) for $m_\omega^\pm(z)$ with definition (3.3) for the transfer matrix (and carefully keeping notice of the orientation of the derivative with respect to the origin), direct computation shows that,

$$\begin{aligned}
m_{S\omega}^-(z) &= \frac{\sum_{e \sim o(\Gamma_{S\omega}^-)} \frac{d}{dx} f_{S\omega}|_e(o(\Gamma_{S\omega}^-); z)}{f_{S\omega}(o(\Gamma_{S\omega}^-); z)} = - \frac{\sum_{e \sim o(\Gamma_{S\omega}^-)} \frac{d}{dx} f_{S\omega}|_e(o(\Gamma_{S\omega}^+); z)}{f_{S\omega}(o(\Gamma_{S\omega}^+); z)} \\
&= - \frac{\mathcal{M}_{21} f_\omega(o(\Gamma_{S\omega}^+); z) + \mathcal{M}_{22} \sum_{e \sim o(\Gamma_\omega^+)} \frac{d}{dx} f_\omega|_e(o(\Gamma_\omega^+); z)}{\mathcal{M}_{11} f_\omega(o(\Gamma_{S\omega}^+); z) + \mathcal{M}_{12} \sum_{e \sim o(\Gamma_\omega^+)} \frac{d}{dx} f_\omega|_e(o(\Gamma_\omega^+); z)} \\
&= - \frac{\mathcal{M}_{21} f_\omega(o(\Gamma_{S\omega}^-); z) - \mathcal{M}_{22} \sum_{e \sim o(\Gamma_\omega^-)} \frac{d}{dx} f_\omega|_e(o(\Gamma_\omega^-); z)}{\mathcal{M}_{11} f_\omega(o(\Gamma_{S\omega}^-); z) - \mathcal{M}_{12} \sum_{e \sim o(\Gamma_\omega^-)} \frac{d}{dx} f_\omega|_e(o(\Gamma_\omega^-); z)} \\
(4.8) \quad &= - \frac{\mathcal{M}_{21} - \mathcal{M}_{22} m_\omega^-(z)}{\mathcal{M}_{11} - \mathcal{M}_{12} m_\omega^-(z)},
\end{aligned}$$

where we note that $f_\omega|_e(o(\Gamma_\omega^-); z) \neq 0$, otherwise m_ω^- would have a pole at z (which cannot happen for $z \in \mathbb{C}_+$). Similarly (and noting that the value of $f_\omega(v; z)$ does not depend on the orientation with respect to v), one can show that

$$(4.9) \quad m_{S\omega}^+(z) = \frac{\mathcal{M}_{21} + \mathcal{M}_{22} m_\omega^+(z)}{\mathcal{M}_{11} + \mathcal{M}_{12} m_\omega^+(z)}.$$

Now, let $\omega \in \mathcal{D}(\mathcal{Z})$, i.e., there exists $\tilde{\mathcal{Z}} \subset \mathcal{Z}$ of full measure such that $m_\omega^-(E + i0) = -\overline{m_\omega^+(E + i0)}$ for all $E \in \tilde{\mathcal{Z}}$. Taking $z = E + i\epsilon$ with $\epsilon \rightarrow 0$ in (4.9) above, one obtains that for all $E \in \tilde{\mathcal{Z}}$,

$$\begin{aligned}
& -\overline{m_{S\omega}^+(E + i0)} \\
&= - \frac{\mathcal{M}_{21}(\omega, E + i0) + \mathcal{M}_{22}(\omega, E + i0) \overline{m_\omega^+(E + i0)}}{\mathcal{M}_{11}(\omega, E + i0) + \mathcal{M}_{12}(\omega, E + i0) \overline{m_\omega^+(E + i0)}} \\
&= - \frac{\mathcal{M}_{21}(\omega, E + i0) - \mathcal{M}_{22}(\omega, E + i0) m_\omega^-(E + i0)}{\mathcal{M}_{11}(\omega, E + i0) - \mathcal{M}_{12}(\omega, E + i0) m_\omega^-(E + i0)} \\
(4.10) \quad &= m_{S\omega}^-(E + i0),
\end{aligned}$$

where we have used the fact that $\mathcal{M}(\omega, E + i0)$ is a real-valued matrix, as the ODE (3.1) has real-valued solutions at the limit $\epsilon \rightarrow 0$. This completes the proof. $\square$

Theorem 4.5. $\sigma(H_\Omega)$ has Lebesgue measure zero. Moreover, all isolated points of $\sigma(H_\Omega)$ (if any exist) are eigenvalues from the discrete set $\mathcal{B} \cap \sigma(H_\Omega)$. In particular, up to a possible discrete set of eigenvalues, $\sigma(H_\Omega)$ is a generalized Cantor set.

Remark 4.6. This is exactly Theorem 1.6, where the set F is the set of isolated points in $\mathcal{B} \cap \sigma(H_\Omega)$.

Proof. We know by Theorem 3.3 that $\sigma(H_\omega) \setminus \mathcal{B} = \sigma(H_\Omega) \setminus \mathcal{B} = \mathcal{Z}$ for all $\omega \in \Omega$. Since $\mathcal{B}$ is discrete, we can simply show that $|\mathcal{Z}| = 0$ to deduce that $|\sigma(H_\Omega)| = 0$. Assume

by contradiction that this is not the case. Since $|\mathcal{Z}| > 0$, the oracle theorem gives that the mapping

$$(4.11) \quad \mathcal{R} : (\omega(n))_{n \in \mathbb{N}^-} \mapsto (\omega(n))_{n \in \mathbb{N}^+}, \quad \omega \in \mathcal{D}(\mathcal{Z}),$$

is well-defined and uniformly continuous. Moreover, Kotani's theorem 4.1 implies that the set $\mathcal{D}(\mathcal{Z}) \subset \Omega$ is of full measure.

By uniform continuity of $\mathcal{R}$ there exists $k = k(\mathcal{A}) \in \mathbb{N}$ (independent of ω) such that for all $\omega \in \mathcal{D}(\mathcal{Z})$, the collection $(\omega(-k), \dots, \omega(-1))$ uniquely determines $\omega(0)$. Using the shift invariance of $\mathcal{D}(\mathcal{Z})$ (Lemma 4.4), we repeat this argument for all positive shifts $\{S^n \omega\}_{n \in \mathbb{N}}$, deducing that the collection $(\omega(-k), \dots, \omega(-1))$ uniquely determines $(\omega(n))_{n \in \mathbb{N}^+}$. Similarly, using the uniform continuity of $\mathcal{R}^{-1}$ together with the shift invariance of $\mathcal{D}(\mathcal{Z})$ and applying negative shifts, we get that $(\omega(n))_{n \in \mathbb{N}^+}$ uniquely determines $(\omega(n))_{n \in \mathbb{N}^-}$. Summing up, we deduce that the collection $(\omega(-k), \dots, \omega(-1))$ determines the entire sequence ω . Thus,

$$(4.12) \quad \#\mathcal{D}(\mathcal{Z}) \leq |\mathcal{A}|^k < \infty.$$

We conclude that every sequence $\omega \in \mathcal{D}(\mathcal{Z})$ has only finitely many possible shifts, and in particular it is periodic – contradicting the aperiodicity of Ω . This gives that $|\mathcal{Z}| = |\sigma(H_\Omega) \setminus \mathcal{B}| = |\sigma(H_\Omega)| = 0$.

In particular, $\sigma(H_\Omega)$ contains no intervals. Since it is also closed, then it is nowhere dense in $\mathbb{R}$. Moreover, standard theory [?, thm. 4.2.4] implies the discrete spectrum $\sigma_{\text{disc}}(H_\Omega) \setminus \mathcal{B}$ is empty. Hence, $\sigma(H_\Omega)$ is closed, nowhere dense, and contains no isolated points, up to possible isolated points in $\mathcal{B} \cap \sigma(H_\Omega)$ (which are automatically eigenvalues as isolated spectral points), completing the proof. $\square$

Remark 4.7. We emphasize that energies in $\mathcal{B} \cap \sigma(H_\Omega)$ may correspond either to isolated eigenvalues or to embedded eigenvalues. In the latter case, these energies are accumulation points of the spectrum and therefore do not affect its Cantor nature. The question of whether attaching graph decorations or introducing internal vertex structures necessarily leads to the opening of spectral gaps (and hence to isolated eigenvalues) has been studied in various settings, see for instance [?, ?, ?]. These works identify conditions under which such local structures produce genuine spectral gaps (and not embedded eigenvalues).

4.2. Generic Cantor spectrum for decorated $\mathbb{Z}$ -graphs – proof of Theorem 1.9.

By Theorem 1.6, $\sigma(H_\Omega) \setminus \mathcal{B}$ is a generalized Cantor set. We have also seen that $\mathcal{B} \cap \sigma(H_\Omega)$ is discrete and now wish to examine the cases for which $\sigma(H_\Omega) \cap \mathcal{B} = \emptyset$, so that the actual spectrum $\sigma(H_\Omega)$ is a generalized Cantor set. In this subsection we prove that for the class of decorated $\mathbb{Z}$ -graphs (Definition 1.3), $\sigma(H_\Omega) \cap \mathcal{B} = \emptyset$ holds Baire-genericly for the edge length set, which forms part of Theorem 1.9. We conjecture that a similar genericity result should hold for a broader class within tiling graphs, leading to generic Cantor spectrum, but the proof below uses the decorated $\mathbb{Z}$ -graph geometry in an essential way.

For the remainder of this subsection, we fix a decorated $\mathbb{Z}$ -graph Γ_ω with no loops. As previously noted in Subsection 3.1, the discrete set $\mathcal{B} \cap \sigma(H_\omega)$ consists of points $E \in \sigma(H_\omega)$ admitting a nontrivial solution to (3.1) with initial condition $\Phi_\omega = \vec{0}$. Consider

the generalized solution to the eigenvalue equation $H_\omega f = Ef$, with $E \in \mathcal{B} \cap \sigma(H_\omega)$. Since for decorated $\mathbb{Z}$ -graphs the vector $\Phi_\omega = \vec{0}$ is evaluated at the center of an edge, the ODE uniqueness principle on intervals implies that f vanishes at the horizontal infinite line graph. This also means that f can be chosen to be a compactly supported eigenfunction.

Lemma 4.8. *Assume that (E, f) is an eigenpair of a decorated $\mathbb{Z}$ -graph Γ_ω with f compactly supported. Then there exists a pair of decorations $\mathcal{D}_1, \mathcal{D}_2$ (possibly $\mathcal{D}_1 = \mathcal{D}_2$) such that for $i \in \{1, 2\}$, $(E, f|_{\mathcal{D}_i})$ is an eigenpair of the Laplacian $H|_{\mathcal{D}_i}$, with a Dirichlet condition imposed at the base vertex of the decoration $\mathcal{D}_i$ and the Kirchhoff condition everywhere else. Furthermore, f vanishes on all edges of Γ_ω which are either to the left of $\mathcal{D}_1$ or to the right of $\mathcal{D}_2$.*

Proof. Let (E, f) be as above. Since the line graph itself does not admit compactly supported eigenfunctions, f must be nonzero at some decoration $\mathcal{D}$. Choose $\mathcal{D}_1$ to be the left-most decoration of the comb such that this happens. Since $f \equiv 0$ to the left of $\mathcal{D}_1$, continuity then implies that $f(v_a) = 0$ at the base vertex v_a of the decoration $\mathcal{D}_1$. Now, by construction, $f|_{\mathcal{D}_1}$ is a solution to the eigenvalue equation $-f'' = Ef$, satisfies the Dirichlet condition at the base of $\mathcal{D}_1$, and the Kirchhoff condition everywhere else, as required. Similarly we choose $\mathcal{D}_2$ to be the right-most decoration of the comb at which f is non-zero. We get that the assertion of the lemma is satisfied for $\mathcal{D}_1$ and $\mathcal{D}_2$. It might also be that $\mathcal{D}_1 = \mathcal{D}_2$. $\square$

Thanks to Theorem 3.3, there exists a closed subset $\sigma(H_\Omega) \subset \mathbb{R}$ such that $\sigma(H_\omega) = \sigma(H_\Omega)$ for all $\omega \in \Omega$. We now prove

Theorem 4.9. *There exists a Baire-generic set A of edge lengths for which $\mathcal{B} \cap \sigma(H_\Omega) = \emptyset$ for all $\vec{\ell} \in A$.*

Remark 4.10. In light of the discussion above, Theorem 4.9 states that Baire-genericly (over the edge length set) $\sigma(H_\Omega)$ admits no compactly supported eigenfunctions. We note that a generic exclusion of compactly supported eigenfunctions was proven for periodic discrete graphs in [?, ?] (where it is also referred to as flat bands or eigenvalues of infinite multiplicity).

Proof. By the discussion preceding Lemma 4.8, it suffices to construct a Baire-generic set of edge lengths such that the corresponding decorated $\mathbb{Z}$ -graphs do not admit compactly supported eigenfunctions. For a compactly supported eigenfunction f we may apply Lemma 4.8 and get the existence of decorations $\mathcal{D}_1, \mathcal{D}_2$ as in the lemma. Denote by $[\mathcal{D}_1, \mathcal{D}_2]$ the subgraph of Γ_ω which lies between $\mathcal{D}_1, \mathcal{D}_2$, including these two decorations. Denote by $H_{[\mathcal{D}_1, \mathcal{D}_2]}$ the Laplacian on $[\mathcal{D}_1, \mathcal{D}_2]$ with Kirchhoff conditions at all vertices. We get that f is an eigenfunction of $H_{[\mathcal{D}_1, \mathcal{D}_2]}$ and in addition vanishes at the base vertices of $\mathcal{D}_1$ and $\mathcal{D}_2$. Since the decorations are assumed to contain no loops, applying [?, thm. 3.6] (see also [?, thm. 3.2]) yields that there exists a Baire-generic set of edge lengths such that no eigenfunction of $H_{[\mathcal{D}_1, \mathcal{D}_2]}$ vanishes at vertices. Hence, such a choice of edge lengths excludes the possibility of an eigenfunction f as above which is supported on $[\mathcal{D}_1, \mathcal{D}_2]$. Now, we apply this argument for all possible (countably many)

subgraphs of Γ_ω of the type $[\mathcal{D}_3, \mathcal{D}_4]$, i.e., subgraphs which are bounded between two arbitrary decorations. Take the intersection of all Baire-generic sets of edge lengths which by [?, thm. 3.6] ensure that no eigenfunction of the Kirchhoff Laplacian on any of these subgraphs vanishes at a vertex. This (countable) intersection is a Baire-generic set of edge lengths which (by Lemma 4.8) permits no compactly supported eigenfunction. $\square$

The proof of Theorem 1.9 now follows immediately:

Proof of Theorem 1.9. Theorem 4.5 for decorated $\mathbb{Z}$ -graphs can be proven by repeating verbatim the proof of Theorem 1.6, using the adapted Borg-Marchenko result from Proposition 2.7, under the generic condition on decoration edge lengths assumed in the theorem (see also Corollary 2.8). In this case, we immediately get that $\sigma(H_\Omega)$ is of zero Lebesgue measure. Furthermore, by Theorem 4.9, we have that $\mathcal{B} \cap \sigma(H_\Omega) = \emptyset$ under a Baire-generic choice of the edge lengths (and since the intersection of generic sets is generic, we can also ensure that the conditions of Proposition 2.7 still hold). For such generic choices of the edge lengths, Theorem 4.5 then also implies that $\sigma(H_\Omega)$ is a Cantor set. $\square$

5. DETAILED EXAMPLE – STURMIAN COMBS

To illustrate the results developed in this paper, we now turn to the concrete example of Sturmian comb graphs. These are the graphs presented in Example 1.4 (cf. Figure 1.2), which consist of two possible tiles – a trivial tile (a horizontal line) and a nontrivial tile (a star graph with three edges). Following the notation of the Sturmian sequences, (1.9), we denote the corresponding operators by $(H_{\alpha,\theta})_{\alpha \in [0,1)}^{\theta \in [0,1)}$. By Theorem 1.6, $\sigma(H_{\alpha,\theta})$ is independent of θ , and we thus focus on $H_{\alpha,0} =: H_\alpha$. Concretely, let Γ_α denote the chain graph on $\mathbb{Z}$ with an attached dangling edge of length $\ell > 0$ for each $n \in \mathbb{Z}$ such that $\omega_\alpha(n) = 1$, where

$$(5.1) \quad \omega_\alpha(n) = \chi_{(1-\alpha,1]}(n\alpha \bmod 1).$$

Given $\alpha \notin \mathbb{Q}$, we can define a family of periodic approximations for H_α using the (infinite) continued fraction expansion for α :

$$(5.2) \quad \alpha = \frac{1}{a_1 + \frac{1}{a_2 + \dots}}.$$

By truncating the continued fraction expansion at the n -th step, we obtain a rational number $\alpha_n := \frac{p_n}{q_n}$, which through (5.1) corresponds to a periodic quantum graph operator H_{α_n} (see Figure 5.1).

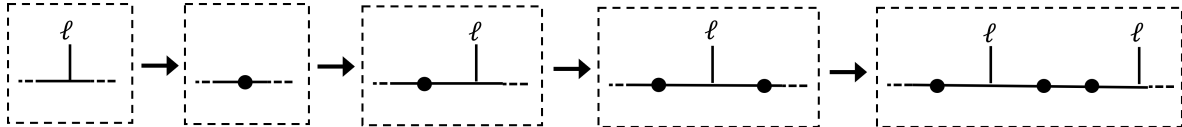


FIGURE 5.1. Fundamental domains for the sequence of periodic approximations for the Fibonacci comb with $\alpha = \frac{\sqrt{5}-1}{2}$. The n th fundamental domain contains F_n tiles, where F_n is the n th Fibonacci number.

To analyze the spectra, we compute the one-step transfer matrix of the possible tiles. They are:

$$(5.3) \quad \mathcal{M}_a(E = k^2) = \begin{pmatrix} \cos(kL) & \frac{1}{k} \sin(kL) \\ -\frac{1}{k} \sin(kL) & \cos(kL) \end{pmatrix},$$

for the trivial tile, and

$$(5.4) \quad \mathcal{M}_b(E = k^2) = \begin{pmatrix} \cos(2kL) - \frac{1}{2} \sin(2kL) \tan(k\ell) & -k \sin(2kL) - k \cos(kL)^2 \tan(k\ell) \\ \frac{1}{k} \sin(2kL) - \frac{1}{k} \sin^2(kL) \tan(k\ell) & \cos(2kL) - \frac{1}{2} \sin(2kL) \tan(k\ell) \end{pmatrix},$$

for the nontrivial tile. Note that the cocycle $\mathcal{M}(k^2)$ is well-defined for all k , except for a discrete subset of energies, as in Definition 3.2:

$$(5.5) \quad \mathcal{B}_\ell := \left\{ \left(\frac{\pi/2 + \pi m}{\ell} \right)^2 : m \in \mathbb{Z} \right\}.$$

To make our analysis more convenient, we assume from now on that these “forbidden” values of k^2 are excluded from the spectrum, similar to the Baire-generic assumption in Theorem 4.9:

Assumption 5.1. $\sigma(H_\alpha) \cap \mathcal{B}_\ell = \emptyset$.

Under this generic assumption, Theorem 1.6 ensures that $\sigma(H_\alpha)$ is of zero Lebesgue measure. In addition, the standard Gordon lemma (see, e.g., [?, prop. 2]) also guarantees that $\sigma(H_\alpha)$ contains no eigenvalues, and so the spectral measures are purely singular continuous and the spectrum is a generalized Cantor set.

On the other hand, if Assumption 5.1 is not satisfied, then one can show that $\sigma(H_\alpha)$ will in fact contain isolated eigenvalues of infinite multiplicity at the spectral points where $\mathcal{M}(\omega, E)$ does not exist (and in particular, will not be a Cantor set). In this case, the integrated density of states is discontinuous at these spectral points, similarly to the case of the Penrose and Ammann-Beenker tilings studied in [?]. In our parallel work, [?] (see also [?]), about the IDS gap labels we also examine the Sturmian comb example. We completely characterize there for which energies and tiles the infinite multiplicity eigenvalues appear, and compute the associated jump discontinuities of the IDS explicitly.

We return to the periodic approximants which are defined via the rational approximants $\alpha_n = \frac{p_n}{q_n}$, with the corresponding graphs Γ_{α_n} and operators H_{α_n} (as in Figure 5.1). With some careful analysis of the corresponding transfer matrices and applying the methods presented in [?, ?] and the result in [?, thm. 2.5.1], we get the following:

Theorem 5.2. *Denoting $\Sigma_{\alpha_n} := \sigma(H_{\alpha_n}) \cup \sigma(H_{\alpha_{n+1}})$, we have*

$$(5.6) \quad \sigma(H_\alpha) = \bigcap_{n \in \mathbb{N}} \Sigma_{\alpha_n}$$

and

$$(5.7) \quad \Sigma_{\alpha_n} \rightarrow \sigma(H_\alpha)$$

in the Hausdorff topology.

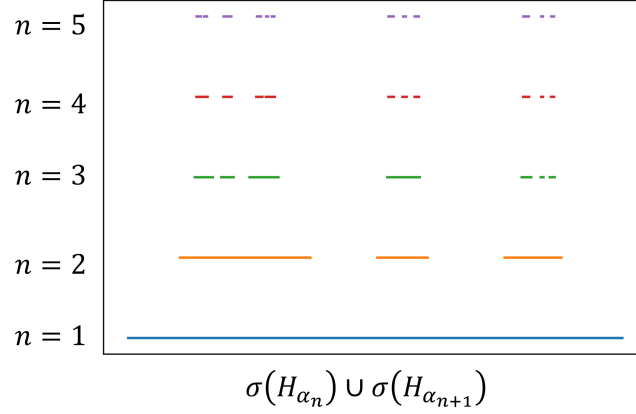


FIGURE 5.2. Parts of the sequence of periodic approximations $\sigma(H_{\alpha_n}) \cup \sigma(H_{\alpha_{n+1}})$ for the spectrum $\sigma(H_\alpha)$ of the Fibonacci comb.

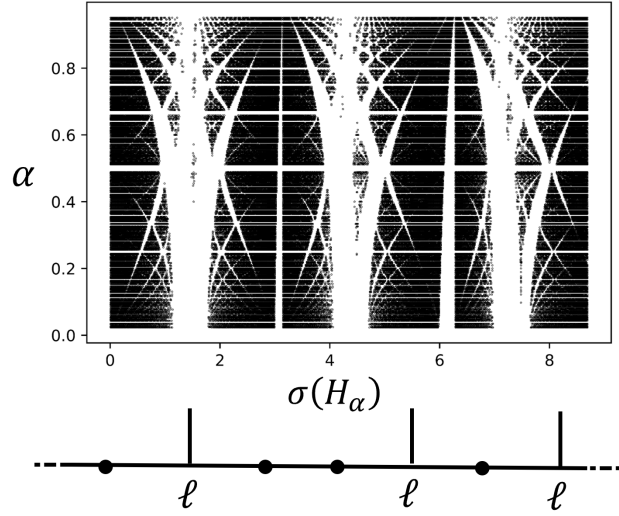


FIGURE 5.3. The “Kohmoto butterfly” for Sturmian combs, showing the fractal structure of $\sigma(H_\alpha)$ changing gradually as the energy increases.

The result is interesting in particular as the sets Σ_{α_n} are not compact (though closed). A detailed analysis and proof appear in the PhD thesis [?, theorems 7.3, 7.4]. Figure 5.2 demonstrates how the sequence Σ_{α_n} converges to the limiting set $\sigma(H_\alpha)$. It also sheds light on the Cantor set nature of the spectrum, and to it being of zero measure. Figures 1.1 and 5.3 display the fractal nature of the spectrum $\sigma(H_\alpha)$ as a function of α ; this can be thought of as a version of the Kohmoto butterfly (cf. [?, ?, ?]) for quantum graphs. We also refer to [?], where a plot similar to 5.3 was produced for a Harper-like quantum graph model, showing resemblance to the Hofstadter butterfly [?].

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