COMPLETE HIERARCHICAL STRUCTURE OF THE SPECTRAL BANDS IN THE KOHMOTO MODEL

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Abstract. We study the Kohmoto model, a family of discrete Schrödinger operators with Sturmian potentials depending on a frequency and a coupling constant. We prove that, for all non-vanishing coupling constants, all spectral bands admit a hierarchical structure. This structure offers a variety of applications, including a detailed description of the Kohmoto butterfly and a central step towards the resolution of the dry ten Martini problem for Sturmian Hamiltonians, which we carry out in a subsequent work.

Contents

1. Introduction and main results

For $\alpha \in [0,1]$ and $V \in \mathbb{R}$, consider the self-adjoint operator $H_{\alpha,V} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ defined by

$$(H_{\alpha,V}\psi)(n) := \psi(n+1) + \psi(n-1) + V\chi_{[1-\alpha,1)}(n\alpha \mod 1)\,\psi(n), \tag{1.1}$$

where $\chi_{[1-\alpha,1)}$ is the characteristic function of the interval $[1-\alpha,1)$, $V \in \mathbb{R}$ is the strength of the potential, called the *coupling constant*, and α is called the *frequency*. Here, $n\alpha \mod 1$ is the fractional part of $n\alpha$ in $[0,1)$.

When $\alpha \notin \mathbb{Q}$, this operator $H_{\alpha,V}$ is called a *Sturmian Hamiltonian*, since the sequence

$$\omega_\alpha \in \{0,1\}^{\mathbb{Z}}, \qquad \omega_\alpha(n) := \chi_{[1-\alpha,1)}(n\alpha \mod 1),\ n \in \mathbb{Z},$$

is called a *Sturmian sequence*. When $\alpha = \frac{p}{q} \in \mathbb{Q}$, the sequence $\omega_\alpha \in \{0,1\}^{\mathbb{Z}}$ is q-periodic, that is, $\omega_\alpha(n) = \omega_\alpha(n+q)$ for all $n \in \mathbb{Z}$. The associated operator $H_{\frac{p}{q},V}$ is periodic and assuming that $V \neq 0$ and p and q are coprime, its spectrum, $\sigma(H_{\frac{p}{q},V})$, consists of exactly q closed intervals, which are called *spectral bands*; see, e.g., [Tes00, Ray95, DF25, BBB$^+$26]. These periodic operators may also serve as approximations of Sturmian Hamiltonians with irrational frequencies $\alpha \in [0,1]\backslash\mathbb{Q}$ [Ray95, BIST89, DF25, BBL24]. The family of all operators $(H_{\alpha,V})_{\alpha\in[0,1]}$ as α ranges over $[0,1]$ is called the *Kohmoto model* [KKT83, OK85]. Plotting the spectra $\sigma(H_{\alpha,V})$ for various values of $\alpha \in \mathbb{Q}$ gives rise to the Kohmoto butterfly; see Figure 2.1. This structure exhibits a striking self-similar and fractal nature, attracting interest from both mathematicians and physicists.

We start by presenting some notations which are needed to state our main theorem. Finite continued fraction expansions are commonly used to represent rational numbers via

$$c_0 + \cfrac{1}{c_1 + \cfrac{1}{\ddots + \frac{1}{c_k}}} \in \mathbb{Q}.$$

We extend the standard definitions and introduce the space of *augmented finite continued fraction expansions,*

$$\mathcal{C} := \{[0],\ [0,0]\} \cup \bigcup_{k\in\mathbb{N}} \{[0,0,c_1,\dots,c_k] \,:\, c_1,\dots,c_{k-1}\in\mathbb{N},\ c_k\in\mathbb{N}_{-1}\},$$

where $\mathbb{N}_{-1} := \mathbb{N}\cup\{-1,0\}$. This notation uses the convention that the two first entries of all $\mathbf{c}\in\mathcal{C}$, satisfy $c_{-1}=c_0=0$. The connection between the finite continued fraction expansions and the rational numbers is as follows.

The *evaluation map* $\varphi : \mathcal{C} \to \mathbb{R}\cup\{\infty\}$ is defined for all $\mathbf{c} = [0,c_0,c_1,\dots,c_k] \in \mathcal{C}\backslash\{[0]\}$ by

$$\varphi([0,c_0,c_1,\dots,c_k]) := \begin{cases} \varphi([0,c_0,c_1,\dots,c_{k-2}]), & k\in\mathbb{N} \text{ and } c_k = 0,\\ c_0 + \cfrac{1}{c_1+\cfrac{1}{\ddots+\frac{1}{c_k}}}, & \text{otherwise,} \end{cases} \tag{1.2}$$

and $\varphi([0]) := \infty$.

We use the $\mathcal{C}$-space to show that there is a hierarchical structure which involves both nesting and interlacing of spectral bands. In particular, we define two types (A and B) of spectral bands (Definition 2.10). In order not to delay the presentation of the main theorem, we defer the introduction of spectral bands types and the related hierarchical structure to Section 2. We only mention here that the definition of these types depends on $\mathbf{c}\in\mathcal{C}$, rather than on the rational value $\varphi(\mathbf{c})$; noting that there exist $\mathbf{c},\tilde{\mathbf{c}}\in\mathcal{C}$ with $\varphi(\mathbf{c})=\varphi(\tilde{\mathbf{c}})$, see a discussion in Section 2. Our main theorem states the following.

Theorem 1.1. *For all $V\neq 0$ and $\boldsymbol{c}\in\mathcal{C}$ with $\varphi(\boldsymbol{c})\in[0,1]\cap\mathbb{Q}$, every spectral band in $\sigma\left(H_{\varphi(\boldsymbol{c}),V}\right)$ is either of type A or B and its type is independent of the value of $V>0$ respectively $V<0$.*

This theorem forms a central step towards our resolution of the dry ten Martini problem [BBL24]. In the large coupling regime $V>4$, an analogue of Theorem 1.1 was proven by Raymond [Ray95] in a slightly different terminology; see the review [BBB+26], which is adapted to the present framework. The hierarchical structure which that result provided was a powerful tool for estimating the fractal dimensions in the large coupling regime; see [KKL03, DEGT08, LQW14, DG15, CQ25]. Hence, the result above for all $V\neq 0$ might enhance the analysis of Kohmoto model's fractal dimensions. Moreover, we expect that this structure provides new insights into the self-similarity of the Kohmoto butterfly in terms of number-theoretic properties of α, such as its continued fraction expansion; see [BBT25] for progress in this direction.

It is not trivial to extend the original result of Raymond [Ray95] from the large coupling regime to $0<V\leq 4$ is more delicate. The difficulty arises since spectral bands start to overlap and classical approaches based on trace maps are no longer sufficient to control the spectral band relative structure. We overcome this challenge by

- studying the whole $\mathcal{C}$-space of finite continued fractions expansions simultaneously, see Section 3.
- establishing an interlacing theorem for the Floquet–Bloch matrices, see Section 4.1. For this sake, a new concept of eigenvalue admissibility plays a central role.
- using a uniform Lipschitz bound in V of the spectral bands, see Section 2.

The paper is organized as follows. Section 2 introduces the formal definitions of spectral bands of type A and B. Section 3 contains the proof of the main theorem via induction over the $\mathcal{C}$-space, subject to two key ingredients: that the backward type implies the forward type, and the induction base. The necessary spectral descriptions via Floquet-Bloch matrices and transfer matrix traces are developed in Section 4, where we also show a symmetry of the spectrum – reducing the study to $V > 0$ – and state an interlacing theorem, whose proof is deferred to the Appendix III. The assumptions of this theorem are then verified in Section 5, where we also develop the necessary tools and the partial results which support the proof of the main theorem. Sections 6 and 7 are devoted to proving these two key ingredients: Section 6 shows that the backward type implies the forward type, while Section 7 establishes the induction base. Basic combinatorial properties of Sturmian sequences are collected in Appendix I, while standard techniques for trace identities are deferred to Appendix II.

2. The A/B type of spectral bands

We start by returning to the definition of the $\mathcal{C}$-space and further exploring it.

From (1.2) follows that for $k \geq 1$,

$$\varphi([0, c_0, c_1, \ldots, c_k, -1]) = \varphi([0, c_0, c_1, \ldots, c_k - 1]) \quad \text{and}$$
$$\varphi([0, c_0, \ldots, c_{k-1}, 0]) = \varphi([0, c_0, \ldots, c_{k-2}]).$$

The second identity may be intuitively understood if one allows c_k to take real values and then consider the limit $c_k \to 0$. Such non-standard continued fraction expansions, ending with 0 or -1 play a special role in the theory presented here, see e.g. Definition 2.6.

Remark. Observe that $\text{Image}(\varphi) \subseteq (\mathbb{Q} \cap [0, 1]) \cup \{-1, \infty\}$. The values -1 and ∞ deserve a special treatment and this is done in the forthcoming statements, see e.g. Definition 2.5 (see also [BBB+26, Sec. 2.1]). Currently, just note that $\varphi(\mathbf{c}) = -1 \Leftrightarrow \mathbf{c} = [0, 0, -1]$, and

$$\varphi(\mathbf{c}) = \infty \quad \Leftrightarrow \quad \mathbf{c} \in \{[0],\ [0, 0, 0],\ [0, 0, 1, -1]\}.$$

It should be noted that the evaluation map φ is surjective but not injective. In fact for each rational $\frac{p}{q} \in (0, 1)$, there exist exactly two $\mathbf{c}, \tilde{\mathbf{c}} \in \mathcal{C}$ whose last digit is not 0 nor -1 and $\varphi(\mathbf{c}) = \frac{p}{q} = \varphi(\tilde{\mathbf{c}})$, see [Khi64, Ch. I.4]. This is used in Proposition 2.7 below.

The following are well-known properties of the periodic Schrödinger operators $H_{\frac{p}{q}, V}$ with p, q coprime, see e.g. [Tes00, Ray95, DF25, BBB+26].

Proposition 2.1. *Let $V \in \mathbb{R} \setminus \{0\}$ and $\frac{p}{q} \in [0, 1]$ such that p and q are coprime. Then $H_{\frac{p}{q}, V}$ has absolutely continuous spectrum and the spectrum $\sigma\left(H_{\frac{p}{q}, V}\right)$ consists of exactly q connected components, each being a closed interval I.*

Let $I \subseteq \mathbb{R}$ be a closed interval. We define its left and right endpoints by

$$L(I) := \inf_{x \in I} x \quad \text{respectively} \quad R(I) := \sup_{x \in I} x.$$

An antisymmetry of the Floquet–Bloch matrices yields $\sigma\left(H_{\frac{p}{q}, V}\right) = -\sigma\left(H_{\frac{p}{q}, -V}\right)$, as shown in Lemma 4.1 below, reducing the study to $V > 0$.

Definition 2.2. For p, q coprime, a map $I : V \mapsto I(V) = [L(I(V)), R(I(V))]$, $V > 0$, is called a *spectral band* in $\sigma\left(H_{\frac{p}{q},V}\right)$ if there is a $0 \le j < q$, such that for all $V > 0$, $I(V)$ is the j-th connected component (counted from the left) of $\sigma\left(H_{\frac{p}{q},V}\right)$.

Remark. In the following, we will abuse terminology and also refer to the evaluation of that map, i.e., $I(V)$, as a spectral band. This is a common terminology in the literature. Whether a spectral band means the map itself or its evaluation will be either understood from the context or explicitly mentioned.

The spectral bands vary continuously with respect to the Hausdorff metric d_H on the compact subsets of $\mathbb{R}$ induced by the Euclidean distance,

$$d_H(X, Y) := \max\left\{ \sup_{x\in X} d(x, Y),\ \sup_{y\in Y} d(y, X) \right\}.$$

Proposition 2.3. *Let $\alpha \in [0, 1]$ and $V, V' \in \mathbb{R}$. Then*

$$d_H\big(\sigma(H_{\alpha,V}), \sigma(H_{\alpha,V'})\big) \le |V - V'|.$$

In particular, if $I : V \mapsto I(V)$ is a spectral band of $\sigma\left(H_{\frac{p}{q},V}\right)$), then for all $V, V' > 0$,

$$\max\big\{|L(I(V)) - L(I(V'))|,\ |R(I(V)) - R(I(V'))|\big\} \le |V - V'|.$$

Proof. The first statement follows from the operator norm estimate $\|H_{\alpha,V} - H_{\alpha,V'}\| \le |V - V'|$. The second follows directly from the definition of the Hausdorff metric. □

We continue introducing basic relations on spectral bands.

Definition 2.4. For two closed intervals I and J define the following order relations.

(a) The interval I *is* contained *in* J:
$$I \subseteq J \quad \Leftrightarrow \quad L(J) \le L(I) < \mathrm{R(I)} \le \mathrm{R(J)}.$$
(b) The interval I *is* strictly contained *in* J:
$$I \subseteq_{\mathrm{str}} J \quad \Leftrightarrow \quad L(J) < L(I) < \mathrm{R(I)} < \mathrm{R(J)}.$$
(c) The interval I *is* to the left of J *(respectively* J *is* to the right of I*):*
$$I \prec J \quad \Leftrightarrow \quad L(I) < L(J) \text{ and } \mathrm{R(I)} < \mathrm{R(J)}.$$
(d) The interval I *is* strictly to the left of J *(respectively* J *is* strictly to the right of I*):*
$$I \prec_{\mathrm{str}} J \quad \Leftrightarrow \quad R(I) < L(J).$$

For two spectral bands $I : V \mapsto [L(I(V)), R(I(V))]$ and $J : V \mapsto [L(J(V)), R(J(V))]$, we extend these strict (i.e., irreflexive) order relations if they hold for all $V > 0$.

(a) The spectral band I *is* strictly contained *in* J:
$$I \subseteq_{\mathrm{str}} J \quad \Leftrightarrow \quad \forall V > 0 : \quad I(V) \subseteq_{\mathrm{str}} J(V)$$
(b) The spectral band I *is* to the left of J *(respectively* J *is* to the right of I*):*
$$I \prec J \quad \Leftrightarrow \quad \forall V > 0 : \quad I(V) \prec J(V).$$

Note that it is possible that I is to the left of J even if $I(V) \cap J(V) \ne \emptyset$ for some value of V.

We now have all tools to define the types of spectral bands. As discussed in Section 1 the types depend on the finite continued fraction expansion and so the following terminology will be used.

Definition 2.5. For all $V \in \mathbb{R}$, and $\mathbf{c} \in \mathcal{C}$ define

$$\sigma_{\mathbf{c}}(V) := \begin{cases} \sigma\left(H_{\varphi(\mathbf{c}),V}\right), & \varphi(\mathbf{c}) \in [0,1], \\ \sigma\left(H_{1,-V}\right) = [-2-V, 2-V], & \varphi(\mathbf{c}) = -1, \\ \mathbb{R} & \varphi(\mathbf{c}) = \infty. \end{cases}$$

We will see in Section 4.2 that these sets are naturally defined by traces associated with $\mathbf{c} \in \mathcal{C}$. Towards the next definition we introduce the following notation:

$$[\mathbf{c}, m] := [0, 0, c_1, \dots, c_k, m], \qquad m \in \mathbb{N}_{-1},$$

which is employed only when $[\mathbf{c}, m] \in \mathcal{C}$. This notation will be used in several statements and proofs below.

Definition 2.6. Let $V \in \mathbb{R}$ and $\mathbf{c} \in \mathcal{C}$ be such that $\varphi(\mathbf{c}) \notin \{-1, \infty\}$ and $[\mathbf{c}, 0], [\mathbf{c}, -1] \in \mathcal{C}$. A spectral band $I(V)$ of $\sigma_{\mathbf{c}}(V)$ is called

- *backward type A*
  if there exists a spectral band $J(V)$ in $\sigma_{[\mathbf{c},0]}(V)$ such that $I(V) \subseteq_{\text{str}} J(V)$.
- *weak backward type A*
  if there exists a spectral band $J(V)$ in $\sigma_{[\mathbf{c},0]}(V)$ such that $I(V) \subseteq J(V)$.
- *backward type B*
  if there exists a spectral band $J(V)$ in $\sigma_{[\mathbf{c},-1]}(V)$ such that $I(V) \subseteq_{\text{str}} J(V)$.
- *weak backward type B*
  if there exists a spectral band $J(V)$ in $\sigma_{[\mathbf{c},-1]}(V)$ such that $I(V) \subseteq J(V)$.

We note that, by definition, whether a spectral band is of (weak) backward type A or B depends on V, since bands are treated as intervals here. In Theorem 1.1, we state that each spectral band in $\sigma_{\mathbf{c}}(V)$ has a unique type (either A or B), independent of $V > 0$ (and likewise for $V < 0$). For $V = 0$, all spectra coincide with $[-2, 2]$, so this case is excluded. The notation A and B is adopted from [KKL03], where it appeared for the specific case $\alpha = \frac{\sqrt{5}-1}{2}$. For visual reasons we do not use the II, III notation as in [Ray95], see also a discussion in [BBB$^+$26, Sec. 4.2].

By definition $\sigma_{\widetilde{\mathbf{c}}}(V) = \sigma_{\mathbf{c}}(V)$ for $\mathbf{c}, \widetilde{\mathbf{c}} \in \mathcal{C}$ with $\varphi(\mathbf{c}) = \varphi(\widetilde{\mathbf{c}})$. Nevertheless, we emphasize that the backward type of a spectral band $I(V)$ of $\sigma_{\mathbf{c}}(V)$ depends on $\mathbf{c} \in \mathcal{C}$ and not on its evaluation $\varphi(\mathbf{c}) \in [0,1]$. This is demonstrated in the next proposition, which shows why it is advantageous to use the space $\mathcal{C}$ and not just rational numbers.

Proposition 2.7. *Let $V \in \mathbb{R}$ and $\mathbf{c} \in \mathcal{C}$ with $\varphi(\mathbf{c}) \notin \{-1, \infty\}$ and $[\mathbf{c}, m] \in \mathcal{C}$ for all $m \in \mathbb{N}_{-1}$. For $m \geq 2$, we have $\sigma_{[\mathbf{c},m]}(V) = \sigma_{[\mathbf{c},m-1,1]}(V)$. Moreover, if $I(V)$ is a spectral band in $\sigma_{[\mathbf{c},m]}(V) = \sigma_{[\mathbf{c},m-1,1]}(V)$, then both of the following hold*

- *$I(V)$ is of (weak) backward type A in $\sigma_{[\mathbf{c},m]}(V)$ if and only if $I(V)$ is of (weak) backward type B in $\sigma_{[\mathbf{c},m-1,1]}(V)$.*
- *$I(V)$ is of (weak) backward type B in $\sigma_{[\mathbf{c},m]}(V)$ if and only if $I(V)$ is of (weak) backward type A in $\sigma_{[\mathbf{c},m-1,1]}(V)$.*

Proof. If $m \geq 2$, then $\varphi([\mathbf{c}, m]) = \varphi([\mathbf{c}, m-1, 1])$ follows by the definition of φ (this is actually a well-known duality for finite continued fraction expansions [Khi64, Ch. I.4]). Now, $\sigma_{[\mathbf{c},m]}(V) = \sigma_{[\mathbf{c},m-1,1]}(V)$ follows. We suppress the V dependence in the following.

Let I be a spectral band in $\sigma_{[\mathbf{c},m]} = \sigma_{[\mathbf{c},m-1,1]}$. By definition, I is of backward type A in $\sigma_{[\mathbf{c},m]}$ if and only if it is strictly contained in a spectral band of $\sigma_{[\mathbf{c},m,0]} = \sigma_{\mathbf{c}}$ (where we used $\varphi([\mathbf{c}, m, 0]) = \varphi(\mathbf{c})$). On the other hand, I is of backward type B in $\sigma_{[\mathbf{c},m-1,1]}$ if and only if it is strictly contained in a spectral band of $\sigma_{[\mathbf{c},m-1,1,-1]} = \sigma_{\mathbf{c}}$ (where we used

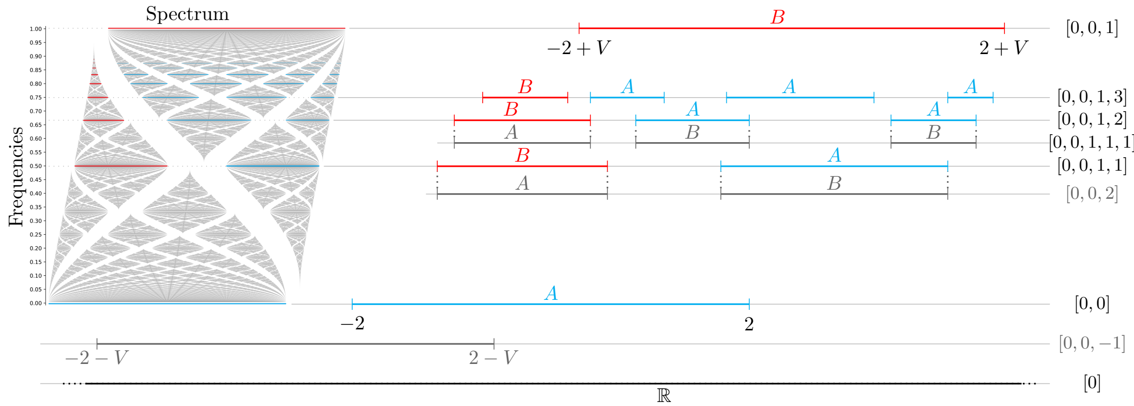

FIGURE 2.1. A plot of various spectra $\sigma_{\mathbf{c}}(V)$ for $\mathbf{c} \in \mathcal{C}$. The spectral bands are colored according to their backward types (A in blue and B in red). The embedding of these spectral bands within the Kohmoto butterfly is highlighted.

$\varphi([\mathbf{c}, m-1, 1, -1]) = \varphi([\mathbf{c}, m-1, 0]) = \varphi(\mathbf{c})$). This proves the first equivalence and the second one follows similarly. □

Example 2.8. A short computation shows

$$\sigma_{[0,0]}(V) = [-2, 2] =: I_{[0,0]}(V) \quad \text{and} \quad \sigma_{[0,0,1]}(V) = [-2+V, 2+V] =: I_{[0,0,1]}(V)$$

Moreover, $I_{[0,0]}(V)$ is of backward type A but not of weak backward type B, while $I_{[0,0,1]}(V)$ is of backward type B but not of weak backward type A, for all $V > 0$.

Since $0 = \varphi([0,0])$ and $1 = \varphi([0,0,1])$ have unique finite continued fraction expansions, their spectral bands have a unique type. In contrast, each rational $\frac{p}{q} \in (0,1)$ admits exactly two representations, so Proposition 2.7 applies. For example, $\frac{1}{2} = \varphi([0,0,2]) = \varphi([0,0,1,1])$ and $\frac{2}{3} = \varphi([0,0,1,2]) = \varphi([0,0,1,1,1])$, see Figure 2.1.

In addition, we observe in Figure 2.1 that $I_{[0,0]}$ contains exactly one band from each of $\sigma_{[0,0,1,1]}$, $\sigma_{[0,0,1,2]}$ and $\sigma_{[0,0,1,3]}$. These bands form a nested sequence in $\sigma_{[0,0,1,n]}$ for $n \in \mathbb{N}$ and are therefore of backward type B. This spectral band nesting is part of a hierarchical structure of the spectral bands which is proved in the current paper. We formalize this next (the spectral band nesting mentioned above appears as Property (B1)).

Definition 2.9. Let $V \in \mathbb{R}$. Let $\mathbf{c} \in \mathcal{C}$ and $m \in \mathbb{N}$ be such that $\varphi(\mathbf{c}) \notin \{-1, \infty\}$ and $[\mathbf{c}, m] \in \mathcal{C}$. A spectral band $I_{\mathbf{c}}(V)$ of $\sigma_{\mathbf{c}}(V)$ is called of *m-forward type* A with $M = m-1$ (respectively *m-forward type* B with $M = m$) if the following holds.

(A) There exist M spectral bands in $\sigma_{[\mathbf{c},m]}(V)$ (denoted $I^1_{[\mathbf{c},m]}(V), \dots, I^M_{[\mathbf{c},m]}(V)$) such that

  (A1) $I^i_{[\mathbf{c},m]}(V) \subseteq_{\mathrm{str}} I_{\mathbf{c}}(V)$ for all $1 \le i \le M$.
  In particular, these bands are of backward type A.

  (A2) $I^i_{[\mathbf{c},m]}(V)$ is not of weak backward type B for all $1 \le i \le M$.

(B) For each $n \in \mathbb{N}$, there exist $M+1$ spectral bands in $\sigma_{[\mathbf{c},m,n]}(V)$ (denoted $I^1_{[\mathbf{c},m,n]}(V), \dots, I^{M+1}_{[\mathbf{c},m,n]}(V)$) such that

  (B1) $I^j_{[\mathbf{c},m,n]}(V) \subseteq_{\mathrm{str}} I^j_{[\mathbf{c},m,n-1]}(V)$ for all $1 \le j \le M+1$, where $I^j_{[\mathbf{c},m,0]}(V) := I_{\mathbf{c}}(V)$.
  In particular, these bands are of backward type B.

(B2) $I^j_{[\mathbf{c},m,n]}$ is not of weak backward type A for all $1 \le j \le M+1$.

(I) For each $n \in \mathbb{N}$, we have
$$I^1_{[\mathbf{c},m,n]} \prec I^1_{[\mathbf{c},m]} \prec I^2_{[\mathbf{c},m,n]} \prec I^2_{[\mathbf{c},m]} \dots \prec I^M_{[\mathbf{c},m]} \prec I^{M+1}_{[\mathbf{c},m,n]}.$$

Remark. The nested spectral band structure given by Property (B1) will be called the *tower property.*

Finally, the notions above are combined to define the type A and B spectral bands.

Definition 2.10. Let $V \in \mathbb{R}$. Let $\mathbf{c} \in \mathcal{C}$ and $m \in \mathbb{N}$ such that $\varphi(\mathbf{c}) \notin \{-1, \infty\}$ and $[\mathbf{c}, m] \in \mathcal{C}$. A spectral band $I_{\mathbf{c}}(V)$ of $\sigma_{\mathbf{c}}(V)$ is called

- *m-type A* if $I_{\mathbf{c}}(V)$ is of backward type A and of m-forward type A.
- *m-type B* if $I_{\mathbf{c}}(V)$ is of backward type B and of m-forward type B.
- *type A* if $I_{\mathbf{c}}(V)$ is of m-type A for all $m \in \mathbb{N}$.
- *type B* if $I_{\mathbf{c}}(V)$ is of m-type B for all $m \in \mathbb{N}$.

The starting point of this work is the following substantial result of Raymond which appeared more than three decades ago.

Theorem 2.11. [Ray95] *For all $V > 4$ and $\mathbf{c} \in \mathcal{C}$ with $\varphi(\mathbf{c}) \notin \{-1, \infty\}$, every spectral band in $\sigma_{\mathbf{c}}(V)$ is either of type A or B and its type is independent of the value of $V > 4$.*

Moreover, for a spectral band $I_{\mathbf{c}}(V)$ and $m, n \in \mathbb{N}$, the spectral bands $I^i_{[\mathbf{c},m]}(V)$ and $I^j_{[\mathbf{c},m,n]}(V)$ introduced in the forward property (A) and (B) are unique for $V > 4$, i.e. $I_{\mathbf{c}}(V)$ does not contain any other spectral band of $\sigma_{[\mathbf{c},m]}(V)$ respectively $\sigma_{[\mathbf{c},m,n]}(V)$.

We took liberty with phrasing Theorem 2.11 differently than it originally appeared in [Ray95] (it actually did not appear there as a single theorem). In particular, the notation used in [Ray95] is different than ours; we had to adapt the notation for the sake of our proofs. We have done such an adaptation already in the review [BBB+26, Thm. 4.22], as a preliminary step towards the current paper.

Definition 2.12. Let $m, n \in \mathbb{N}$, $\mathbf{c} \in \mathcal{C}$ be such that $\varphi(\mathbf{c}) \notin \{-1, \infty\}$ and $[\mathbf{c}, m] \in \mathcal{C}$. For a spectral band $I_{\mathbf{c}}$ in $\sigma_{\mathbf{c}}$ define the associated unique value
$$M := \begin{cases} m-1, & I_{\mathbf{c}}(V) \text{ is of backward type } A \text{ for all } V > 4, \\ m, & I_{\mathbf{c}}(V) \text{ is of backward type } B \text{ for all } V > 4, \end{cases}$$
and the unique spectral bands $\left\{ I^i_{[\mathbf{c},m]} \right\}_{i=1}^{M}$ of $\sigma_{[\mathbf{c},m]}$ and the unique spectral bands $\left\{ I^j_{[\mathbf{c},m,n]} \right\}_{j=1}^{M+1}$ of $\sigma_{[\mathbf{c},m,n]}$ satisfy (A), (B) and (I) for all $V > 4$.

Note that M actually depends both on the backward type of $I_{\mathbf{c}}$ and on m, but we omit this dependence from the notation.

The existence and uniqueness of the spectral bands $\left\{ I^i_{[\mathbf{c},m]} \right\}_{i=1}^{M}$ and $\left\{ I^j_{[\mathbf{c},m,n]} \right\}_{j=1}^{M+1}$ are justified by Theorem 2.11. Due to Corollary 2.3, we may consider the continuous maps $V \mapsto I^i_{[\mathbf{c},m]}(V)$ and $V \mapsto I^j_{[\mathbf{c},m,n]}(V)$ on $V \in (0, \infty)$.

A word of caution is needed regarding the notation in Definition 2.12. In order to know to which spectral band the notation $I^i_{[\mathbf{c},m]}$ refers to within $\sigma_{[\mathbf{c},m]}$, one needs to know which spectral band $I_{\mathbf{c}}$ was designated. For different choices of spectral bands $I_{\mathbf{c}}$ within $\sigma_{\mathbf{c}}$, the spectral bands $I^i_{[\mathbf{c},m]}$ and $I^j_{[\mathbf{c},m,n]}$ will also differ. This should not lead to confusion, since in the beginning of each proof or discussion, the spectral band $I_{\mathbf{c}}$ will be explicitly indicated.

Definition 2.13. Let $m \in \mathbb{N}$, $\mathbf{c} \in \mathcal{C}$ be such that $\varphi(\mathbf{c}) \notin \{-1, \infty\}$ and $[\mathbf{c}, m] \in \mathcal{C}$. For $V > 0$, we say that a spectral band $I_{\mathbf{c}}$ in $\sigma_{\mathbf{c}}$ satisfies $P(m, V)$ (the (m, V)-*property*) if both of the following hold:

(a) either
- for all $V' \geq V$, $I_{\mathbf{c}}(V')$ is of backward type A with $M = m - 1$,

or
- for all $V' \geq V$, $I_{\mathbf{c}}(V')$ is of backward type B with $M = m$,

(b) for all $V' \geq V$ and for all $n \in \mathbb{N}$, the unique spectral bands $\left\{ I^i_{[\mathbf{c},m]}(V') \right\}_{i=1}^{M}$ of $\sigma_{[\mathbf{c},m]}(V')$ and the unique spectral bands $\left\{ I^j_{[\mathbf{c},m,n]}(V') \right\}_{j=1}^{M+1}$ of $\sigma_{[\mathbf{c},m,n]}(V')$, as in Definition 2.12, satisfy (A), (B) and (I).

3. Proof of the main theorem and further consequences

In this section we prove Theorem 1.1 by induction over the space $\mathcal{C}$ of finite continued fractions. For all $\mathbf{c} \in \mathcal{C}$ and $m \in \mathbb{N}$ such that $\varphi(\mathbf{c}) \notin \{-1, \infty\}$ and $[\mathbf{c}, m] \in \mathcal{C}$, define

$$V_{\mathrm{crit}}([\mathbf{c}, m]) := \inf \left\{ V \geq 0 : \text{ each spectral band } I_{\mathbf{c}} \text{ in } \sigma_{\mathbf{c}} \text{ satisfies } P(m, V) \right\}.$$

Hence, to prove Theorem 1.1 we should then show that $V_{\mathrm{crit}}([\mathbf{c}, m]) = 0$ for all relevant $\mathbf{c} \in \mathcal{C}$ and $m \in \mathbb{N}$. Theorem 1.1 is stated for both $V > 0$ and $V < 0$, but it is possible to restrict to the case $V > 0$ thanks to an antisymmetry of the Floquet-Bloch matrices, stated in Lemma 4.1. The proof of Theorem 1.1 is carried by induction over the space $\mathcal{C}$, and it relies on two key ingredients: Proposition 3.1 and the induction base. Both ingredients require some detailed analysis and their proofs are postponed to Sections 6 and 7, respectively. In the current section we prove Theorem 1.1 assuming Proposition 3.1 and the induction base.

Proposition 3.1 (Backward implies forward type)**.** *Let $\boldsymbol{c} \in \mathcal{C}$ and $\varphi(\boldsymbol{c}) \in (0, 1)$. If each spectral band $I_{\boldsymbol{c}}(V)$ in $\sigma_{\boldsymbol{c}}(V)$ is either of backward type A for all $V > 0$ or of backward type B for all $V > 0$, then $V_{\mathrm{crit}}([\boldsymbol{c}, m]) = 0$ for all $m \in \mathbb{N}$.*

The induction step in the proof of of Theorem 1.1 is divided into two: increasing the number of digits of $\mathbf{c} \in \mathcal{C}$ for which the statement is valid (call it a horizontal step), and showing the validity for all values of the last digit of $\mathbf{c}$ (vertical step). These steps are stated and proven in the next two lemmata with the aid of Proposition 3.1 and afterwards we are able to prove Theorem 1.1. Throughout this section we use the notational conventions of Definition 2.12.

Lemma 3.2 (horizontal induction step)**.** *Let $m \in \mathbb{N}$ and $\boldsymbol{c} \in \mathcal{C}$ be such that $\varphi(\boldsymbol{c}) \notin \{-1, \infty\}$ and $[\boldsymbol{c}, m] \in \mathcal{C}$. If $V_{\mathrm{crit}}([\boldsymbol{c}, m]) = 0$ and $V_{\mathrm{crit}}([\boldsymbol{c}, m, 1]) = 0$, then $V_{\mathrm{crit}}([\boldsymbol{c}, m, 1, n]) = 0$ for all $n \in \mathbb{N}$.*

Proof. Let $\mathbf{c}' := [\mathbf{c}, m, 1]$. We have to show that $V_{\mathrm{crit}}([\mathbf{c}', n]) = 0$ for all $n \in \mathbb{N}$. Since $m \in \mathbb{N}$, we conclude $\varphi(\mathbf{c}') \in (0, 1)$. Thus, Proposition 3.1 implies that it suffices to prove that each spectral band in $\sigma_{\mathbf{c}'}(V)$ is either of backward type A for all $V > 0$ or of backward type B for all $V > 0$. Let $I_{\mathbf{c}'}$ be a spectral band in $\sigma_{\mathbf{c}'}$. By Theorem 2.11, we already have that $I_{\mathbf{c}'}(V)$ is either of backward type A for all $V > 4$ or of backward type B for all $V > 4$. We treat each of these two cases separately.

<u>Case 1:</u> (For all $V > 4$, $I_{\mathbf{c}'}(V)$ is of backward type A). In this case, using $\sigma_{[\mathbf{c}',0]}(V) = \sigma_{[\mathbf{c},m]}(V)$ (as $c' := [\mathbf{c}, m, 1]$ and so $\varphi([\mathbf{c}', 0]) = \varphi([\mathbf{c}, m])$) we conclude that $I_{\mathbf{c}'}(V)$ is strictly included in a spectral band of $\sigma_{[\mathbf{c},m]}(V)$ for all $V > 4$. By Theorem 2.11, there is a unique spectral band $I_{[\mathbf{c},m]}(V)$ such that $I_{\mathbf{c}'}(V) = I^i_{[\mathbf{c},m,1]}(V) \subseteq_{\mathrm{str}} I_{[\mathbf{c},m]}(V)$, for all $V > 4$. Since $V_{\mathrm{crit}}([\mathbf{c}, m, 1]) = 0$, we conclude $I_{\mathbf{c}'}(V) = I^i_{[\mathbf{c},m,1]}(V) \subseteq_{\mathrm{str}} I_{[\mathbf{c},m]}(V)$ for all $V > 0$ implying that $I_{\mathbf{c}'}(V)$ is of backward type A for all $V > 0$.

Case 2: (For all $V > 4$, $I_{\mathbf{c}'}(V)$ is of backward type B). In this case, using $\sigma_{[\mathbf{c}',-1]}(V) = \sigma_{[\mathbf{c},m,0]}(V) = \sigma_{\mathbf{c}}(V)$ (as $c' := [\mathbf{c}, m, 1]$ and so $\varphi([\mathbf{c}', -1]) = \varphi([\mathbf{c}, m, 0])$) we conclude that $I_{\mathbf{c}'}(V)$ is strictly included in a spectral band of $\sigma_{\mathbf{c}}(V)$ for all $V > 4$. By Theorem 2.11, there is a unique spectral band $I_{\mathbf{c}}(V)$ such that $I_{\mathbf{c}'}(V) = I^j_{[\mathbf{c},m,1]}(V) \subseteq_{\text{str}} I_{\mathbf{c}}(V)$, for all $V > 4$. Since $V_{\text{crit}}([\mathbf{c}, m]) = 0$, we conclude $I_{\mathbf{c}'}(V) = I^j_{[\mathbf{c},m,1]}(V) \subseteq_{\text{str}} I_{\mathbf{c}}(V)$ for all $V > 0$ implying that $I_{\mathbf{c}'}(V)$ is of backward type B for all $V > 0$. □

Lemma 3.3 (vertical induction step)**.** *Let $\mathbf{c} \in \mathcal{C}$ be such that $\varphi(\mathbf{c}) \notin \{-1, \infty\}$ and $[\mathbf{c}, m] \in \mathcal{C}$ for all $m \in \mathbb{N}$. If $V_{\text{crit}}([\mathbf{c}, m]) = 0$ for all $m \in \mathbb{N}$ and $V_{\text{crit}}([\mathbf{c}, 1, n]) = 0$ for all $n \in \mathbb{N}$, then $V_{\text{crit}}([\mathbf{c}, m, n]) = 0$ for all $m, n \in \mathbb{N}$.*

Proof. We denote by $T(m)$ the statement that $V_{\text{crit}}([\mathbf{c}, m, n]) = 0$ for all $n \in \mathbb{N}$. We use induction over $m \in \mathbb{N}$ to prove that $T(m)$ holds for all $m \in \mathbb{N}$ and the lemma follows. The induction base $T(1)$ is true by the assumption in the lemma.

Suppose $T(m)$ holds. Denote $\mathbf{c}' := [\mathbf{c}, m + 1]$. We need to show $V_{\text{crit}}([\mathbf{c}, m + 1, n]) = V_{\text{crit}}([\mathbf{c}', n]) = 0$ for all $n \in \mathbb{N}$. Since $m \in \mathbb{N}$, we have $m + 1 \geq 2$ and so $\varphi(\mathbf{c}') \in (0, 1)$. Thus, Proposition 3.1 implies that it suffices to prove that each spectral band in $\sigma_{\mathbf{c}'}(V)$ is either of backward type A for all $V > 0$ or of backward type B for all $V > 0$. Let $I_{\mathbf{c}'}$ be a spectral band in $\sigma_{\mathbf{c}'}$. By Theorem 2.11, $I_{\mathbf{c}'}(V)$ is either of backward type A for all $V > 4$ or of backward type B for all $V > 4$. We treat each of these two cases separately.

Case 1: (For all $V > 4$, $I_{\mathbf{c}'}(V)$ is of backward type A). In this case, using $\sigma_{[\mathbf{c}',0]}(V) = \sigma_{\mathbf{c}}(V)$ (as $c' := [\mathbf{c}, m + 1]$ and so $\varphi([\mathbf{c}', 0]) = \varphi([\mathbf{c}])$) we conclude that $I_{\mathbf{c}'}(V)$ is strictly included in a spectral band of $\sigma_{\mathbf{c}}(V)$ for all $V > 4$. By Theorem 2.11 there is a unique spectral band $I_{\mathbf{c}}(V)$ such that $I_{\mathbf{c}'}(V) = I^i_{[\mathbf{c},m+1]}(V) \subseteq_{\text{str}} I_{\mathbf{c}}(V)$, for all $V > 4$. Since $V_{\text{crit}}([\mathbf{c}, m + 1]) = 0$, we conclude $I_{\mathbf{c}'}(V) = I^i_{[\mathbf{c},m+1]}(V) \subseteq_{\text{str}} I_{\mathbf{c}}(V)$ for all $V > 0$ implying that $I_{\mathbf{c}'}(V)$ is of backward type A for all $V > 0$.

Case 2: (For all $V > 4$, $I_{\mathbf{c}'}(V)$ is of backward type B). In this case, using $\sigma_{[\mathbf{c}',-1]}(V) = \sigma_{[\mathbf{c},m]}(V)$ (as $c' := [\mathbf{c}, m+1]$ and so $\varphi([\mathbf{c}', -1]) = \varphi([\mathbf{c}, m])$) we conclude that $I_{\mathbf{c}'}(V)$ is strictly included in a spectral band of $\sigma_{[\mathbf{c},m]}(V)$ for all $V > 4$. Recalling that $\sigma_{\mathbf{c}'}(V) = \sigma_{[\mathbf{c},m,1]}(V)$ (again by $\varphi(\mathbf{c}') = \varphi([\mathbf{c}, m, 1])$) and applying Theorem 2.11, we conclude that there is a unique spectral band $I_{[\mathbf{c},m]}(V)$ such that $I_{\mathbf{c}'}(V) = I^i_{[\mathbf{c},m,1]}(V) \subseteq_{\text{str}} I_{[\mathbf{c},m]}(V)$, for all $V > 4$. Since by the induction hypothesis $V_{\text{crit}}([\mathbf{c}, m, 1]) = 0$, we conclude $I_{\mathbf{c}'}(V) = I^i_{[\mathbf{c},m,1]}(V) \subseteq_{\text{str}} I_{[\mathbf{c},m]}(V)$ for all $V > 0$ implying that $I_{\mathbf{c}'}(V)$ is of backward type B for all $V > 0$. □

We are ready to prove Theorem 1.1, using Lemmata 3.2 and 3.3 for the induction step and where the ingredients needed for the proof of the induction base are postponed to Section 7.

Proof of Theorem 1.1. Thanks to the anti-symmetric property, $\sigma_{\mathbf{c}}(V) = -\sigma_{\mathbf{c}}(-V)$, proven in Lemma 4.1, it suffices to consider the case $V > 0$. We should therefore prove that $V_{\text{crit}}([\mathbf{c}, m]) = 0$ for all $\mathbf{c} \in \mathcal{C}$ with $\varphi(\mathbf{c}) \in [0, 1]$ and $[\mathbf{c}, m] \in \mathcal{C}$, where $m \in \mathbb{N}$.

For $l \in \mathbb{N}$, we denote by $T(l)$ the statement that

$$V_{\text{crit}}([0, 0, c_1, \dots, c_l]) = 0 \qquad \text{and} \qquad V_{\text{crit}}([0, 0, c_1, \dots, c_{l+1}]) = 0, \tag{3.1}$$

for all $[0, 0, c_1, \dots, c_l, c_{l+1}] \in \mathcal{C}$ with $c_{l+1} \in \mathbb{N}$.

We start from the induction base, $T(1)$. By Lemma 7.4 and Lemma 7.5,

$$V_{\text{crit}}([0, 0, c_1]) = 0 \qquad \text{and} \qquad V_{\text{crit}}([0, 0, 1, c_2]) = 0,$$

hold for all $c_1, c_2 \in \mathbb{N}$. Then Lemma 3.3 (with $\mathbf{c} = [0, 0]$, $m = c_1$, $n = c_2$) gives that $V_{\text{crit}}([0, 0, c_1, c_2]) = 0$ for all $c_1, c_2 \in \mathbb{N}$, proving the induction base.

Now, suppose $T(l)$ holds for $l \in \mathbb{N}$. In order to prove $T(l+1)$, it suffices to show that $V_{\text{crit}}([0,0,c_1,\ldots,c_{l+1},c_{l+2}]) = 0$ for all $c_{l+2} \in \mathbb{N}$. Apply Lemma 3.2 (for $\mathbf{c} = [0,0,c_1,\ldots,c_{l-1}]$ and $m = c_l$) to the induction hypothesis (3.1) with $c_{l+1} = 1$ to conclude

$$V_{\text{crit}}([0,0,c_1,\ldots,c_l,1,c_{l+2}]) = 0,$$

for all $c_{l+2} \in \mathbb{N}$. Using this and the induction hypothesis (3.1), we apply Lemma 3.3 (for $\mathbf{c} = [0,0,c_1,\ldots,c_l]$, $m = c_{l+1}$, $n = c_{l+2}$) and get that

$$V_{\text{crit}}([0,0,c_1,\ldots,c_l,c_{l+1},c_{l+2}]) = 0,$$

for all $c_{l+1}, c_{l+2} \in \mathbb{N}$. Hence, $T(l+1)$ holds. □

Having proven Theorem 1.1 we devote the rest of this section for drawing some interesting conclusions from this classification of all spectral bands to types A and B. The next proposition presents an equivalent formulation of this dichotomy.

Proposition 3.4. *Let $\mathbf{c} = [0,c_0,c_1,c_2,\ldots,c_k] \in \mathcal{C}$ with $\varphi(\mathbf{c}) \notin \{-1,\infty\}$ and $V \neq 0$. For a spectral band $I_{\mathbf{c}}$ in $\sigma_{\mathbf{c}}$, we have the following equivalences*

$$I_{\mathbf{c}}(V) \text{ is of type } A \quad \Leftrightarrow \quad \begin{array}{c} I_{\mathbf{c}}(V) \text{ is of} \\ \text{backward type } A \end{array} \quad \Leftrightarrow \quad I_{\mathbf{c}}(V) \subseteq_{\text{str}} \sigma_{[0,c_0,c_1,\ldots,c_{k-1}]}(V)$$

and

$$I_{\mathbf{c}}(V) \text{ is of type } B \quad \Leftrightarrow \quad \begin{array}{c} I_{\mathbf{c}}(V) \text{ is of} \\ \text{backward type } B \end{array} \quad \Leftrightarrow \quad \begin{array}{c} I_{\mathbf{c}}(V) \nsubseteq \sigma_{[0,c_0,c_1,\ldots,c_{k-1}]}(V) \text{ and} \\ I_{\mathbf{c}}(V) \subseteq_{\text{str}} \sigma_{[0,c_0,c_1,\ldots,c_{k-2}]}(V). \end{array}$$

Proof. By Lemma 4.1, it suffices to consider $V > 0$. The cases $\varphi(\mathbf{c}) \in \{0,1\}$ follow from Lemma 7.4 and Lemma 7.5.

Let $\varphi(\mathbf{c}) \in (0,1)$. By Theorem 1.1, each band is of type A or B. The left-most equivalences in the statement, namely the equivalence between type $A \backslash B$ and backward type $A \backslash B$ follows from Proposition 3.1.

By Definition 2.6, $I_{\mathbf{c}}(V)$ is of backward type A if and only if $I_{\mathbf{c}}(V) \subseteq_{\text{str}} \sigma_{[0,c_0,\ldots,c_{k-1}]}(V)$, proving the right-most equivalence in the first line.

For type B, if $I_{\mathbf{c}}(V) \nsubseteq \sigma_{[0,c_0,\ldots,c_{k-1}]}(V)$, then $I_{\mathbf{c}}(V)$ is not of backward type A and hence of type B by Theorem 1.1. Conversely, type B implies this non-inclusion. It remains to show $I_{\mathbf{c}}(V) \subseteq_{\text{str}} \sigma_{[0,c_0,\ldots,c_{k-2}]}(V)$ if $I_{\mathbf{c}}(V)$ is of type B.

By definition, $I_{\mathbf{c}}(V) \subseteq_{\text{str}} J_1(V)$ for some band in $\sigma_{[0,c_0,\ldots,c_k-1]}(V)$. If $c_k = 1$, we are finished since then $\sigma_{[0,0,c_1,\ldots,c_k-1]}(V) = \sigma_{[0,0,c_1,\ldots,c_{k-2}]}(V)$. Otherwise, $J_1(V)$ cannot be of backward type A. Indeed, if it were, then $J_1(V) \subseteq_{\text{str}} \sigma_{[0,c_0,\ldots,c_k-1,0]}(V) = \sigma_{[0,c_0,\ldots,c_{k-1}]}(V)$, which contradicts $I_{\mathbf{c}}(V) \nsubseteq \sigma_{[0,c_0,\ldots,c_{k-1}]}(V)$.

Iterating this argument yields a chain of spectral bands $J_n(V)$ in $\sigma_{[0,0,c_1,\ldots,c_k-n]}(V)$ for $n \in \{1,\ldots,c_k-1\}$ of backward type B such that

$$I_{\mathbf{c}}(V) \subseteq_{\text{str}} J_1(V) \subseteq_{\text{str}} J_2(V) \subseteq_{\text{str}} \ldots \subseteq_{\text{str}} J_{c_k-1}(V).$$

In particular, $I_{\mathbf{c}}(V) \subseteq_{\text{str}} J_{c_k-1}(V) \subseteq_{\text{str}} \sigma_{[0,c_0,c_1,\ldots,c_{k-1},1,-1]}(V) = \sigma_{[0,c_0,\ldots,c_{k-2}]}(V)$. □

Remark. A spectral band $I_{\mathbf{c}}(V)$ of type A may also satisfy $I_{\mathbf{c}}(V) \subseteq_{\text{str}} \sigma_{[0,c_0,c_1,\ldots,c_{k-2}]}(V)$ for some values of V. Such examples of spectral bands of type A can be found in the spectrum $\sigma_{[0,0,1,2]}$. This explains why the classification of type B includes also the condition $I_{\mathbf{c}}(V) \nsubseteq \sigma_{[0,c_0,c_1,\ldots,c_{k-1}]}(V)$.

The rightmost characterization of spectral bands in Proposition 3.4 is of particular interest when considering a sequence of rational approximations of an irrational frequency $\alpha \in [0,1]\backslash\mathbb{Q}$. This is used in [BBL24] for the solution of the dry ten Martini problem for Sturmian Hamiltonians (see also [BBL23] for a brief description). We bring next two additional corollaries which provide interesting information on the spectral band types from the perspective of rational approximations.

Corollary 3.5. *Let $V \neq 0$ and $\mathbf{c} := [0, c_0, c_1, \dots, c_k] \in \mathcal{C}$ for $k \in \mathbb{N}_0$ be such that $\varphi(\mathbf{c}) \in [0,1]$ and $c_k \in \mathbb{N}$ if $k \geq 1$. Define the rational numbers $\alpha_j := \varphi([0, c_0, c_1, \dots, c_j]) = \frac{p_j}{q_j} \in [0,1] \cap \mathbb{Q}$ for $0 \leq j \leq k$ with coprime p_j, q_j and $q_{-1} = 0$, $q_0 = 1$. Then the spectrum σ_k contains $q_k - q_{k-1}$ spectral bands of type A, and q_{k-1} spectral bands of type B.*

Proof. Due to $\sigma_{\mathbf{c}}(V) = -\sigma_{\mathbf{c}}(-V)$, proven in Lemma 4.1, it suffices to consider the case $V > 0$. For $k \in \mathbb{N}_0$ and $\mathbf{c} := [0, c_0, c_1, \dots, c_k] \in \mathcal{C}$, define

$$\mathcal{N}_k^{(A)} := \text{number of spectral bands of type } A \text{ in } \sigma_{\mathbf{c}},$$
$$\mathcal{N}_k^{(B)} := \text{number of spectral bands of type } B \text{ in } \sigma_{\mathbf{c}}.$$

By standard properties of continued fractions [Khi64, Thm. 1], we have

$$q_{-1} = 0,\ q_0 = 1,\ q_j = c_j q_{j-1} + q_{j-2}, \qquad 1 \leq j \leq k. \tag{3.2}$$

We will inductively over $k \in \mathbb{N}_0$ prove

$$\mathcal{N}_k^{(A)} \geq q_k - q_{k-1} \quad \text{and} \quad \mathcal{N}_k^{(B)} \geq q_{k-1}. \tag{3.3}$$

For the induction base, we show the estimate for $k \in \{0, 1\}$: First let $\mathbf{c} = [0, 0]$ with $k = 0$. Then $\sigma_{\mathbf{c}}$ consist of exactly $q_0 - q_{-1} = 1$ spectral band $I = [-2, 2]$ of type A and $q_{-1} = 0$ spectral bands of type B, see Example 2.8. Now let $k = 1$ and consider $\mathbf{c} = [0, 0, m]$ for some $m \in \mathbb{N}$. We have $q_1 = m$ and $q_0 = 1$. Then Lemma 7.4 (c) asserts that $\sigma_{[0,0,m]}$ consists of exactly $m - 1 = q_1 - q_0$ spectral bands of backward type A and $q_0 = 1$ spectral bands of backward type B. Thus, these bands are of type A respectively B by Proposition 3.4. This finishes the induction base. Note that we actually proved equality.

For the induction step suppose (3.3) holds for k and $k - 1$. Let $\mathbf{c} = [0, c_0, c_1, \dots, c_{k+1}] \in \mathcal{C}$ with $c_{k+1} \in \mathbb{N}$. By forward type properties of a spectral band of type A and B in $\sigma_{[0,c_0,c_1,\dots,c_k]}$ and $\sigma_{[0,c_0,c_1,\dots,c_{k-1}]}$, we conclude with the induction hypothesis and (3.2) that

$$\mathcal{N}_{k+1}^{(A)} \geq (c_{k+1} - 1) \cdot \mathcal{N}_k^{(A)} + c_{k+1} \cdot \mathcal{N}_k^{(B)} \geq c_{k+1} q_k + q_{k-1} - q_k = q_{k+1} - q_k,$$
$$\mathcal{N}_{k+1}^{(B)} \geq c_k \cdot \mathcal{N}_{k-1}^{(A)} + (c_k + 1) \cdot \mathcal{N}_{k-1}^{(B)} \geq c_k q_{k-1} + q_{k-2} = q_k.$$

This proves (3.2) for all $\mathbf{c} := [0, c_0, c_1, \dots, c_k] \in \mathcal{C}$ and $k \in \mathbb{N}_0$. These inequalities are actually equalities since $\mathcal{N}_k^{(A)} + \mathcal{N}_k^{(B)}$ is bounded from above by the total number of spectral bands in $\sigma_{\mathbf{c}}$, which equals to q_k by Proposition 2.1. ☐

Corollary 3.6. *Let $V \neq 0$ and $\mathbf{c} := [0, c_0, c_1, \dots, c_k] \in \mathcal{C}$ for $k \in \mathbb{N}_0$ be such that $\varphi(\mathbf{c}) \in [0,1]$ and $c_k \in \mathbb{N}$ if $k \geq 1$. For a spectral band $I_{\mathbf{c}}(V)$ and $m, n \in \mathbb{N}$, the spectral bands $I^i_{[\mathbf{c},m]}(V)$ and $I^j_{[\mathbf{c},m,n]}(V)$ introduced in the forward property (A) and (B) are unique for $V \neq 0$, i.e. $I_{\mathbf{c}}(V)$ does not contain any other spectral band of $\sigma_{[\mathbf{c},m]}(V)$ respectively $\sigma_{[\mathbf{c},m,n]}(V)$.*

Proof. The uniqueness of the spectral bands follows immediately from our counting argument in Corollary 3.5 and the forward property of each spectral band. ☐

4. Two perspectives describing the spectra

We discuss two descriptions of the spectra $\sigma_{\mathbf{c}}(V)$: one via Floquet–Bloch matrices and one using transfer matrices. Floquet–Bloch theory reduces the spectral analysis to families of Hermitian matrices, providing a framework for which we develop a suitable interlacing theorem. We combine both descriptions to control spectral band edges.

4.1. **Floquet-Bloch matrices and an interlacing theorem.** Given an $n \times n$ matrix H and $\theta \in [0, 2\pi]$, define the $n \times n$ matrix

$$H(\theta) := H + e^{-i\theta}\mathbb{I}_{1,n} + e^{i\theta}\mathbb{I}_{n,1},$$

where $\mathbb{I}_{i,j}$ denotes the $n \times n$ matrix that has only zeros except at the (i, j)-th entry where it is equal to one.

For $\alpha \in [0, 1]$, we use in the following the notation $\omega_\alpha(n) := \chi_{[1-\alpha,1)}(n\alpha \mod 1)$ for the potential. Let $V \in \mathbb{R}$, $\mathbf{c} \in \mathcal{C}$ be such that $\{-1, \infty\} \not\ni \varphi(\mathbf{c}) = \frac{p}{q}$ with p, q coprime. Recall the self-adjoint operator $H_{\varphi(\mathbf{c}),V} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ introduced in Equation (1.1). The spectral analysis of $H_{\varphi(\mathbf{c}),V}$ is done via the following hermitian $q \times q$ matrix

$$H_{\mathbf{c},V} := H_{\varphi(\mathbf{c}),V}|_{[0,q-1]} = \begin{pmatrix} V\omega_{\varphi(\mathbf{c})}(0) & 1 & 0 & \dots & & 0 \\ 1 & V\omega_{\varphi(\mathbf{c})}(1) & 1 & \dots & & \\ 0 & 1 & \ddots & & & \\ \vdots & \ddots & & \ddots & & 0 \\ 0 & & & & & 1 \\ 0 & 0 & \dots & 0 & 1 & V\omega_{\varphi(\mathbf{c})}(q-1) \end{pmatrix}.$$

Note the ambiguity in the notation between the operator $H_{\varphi(\mathbf{c}),V}$ on $\ell^2(\mathbb{Z})$ and the $q \times q$ matrix $H_{\mathbf{c},V}$.

The Floquet-Bloch matrices $H_{\mathbf{c},V}(\theta)$ determine the spectrum of $H_{\varphi(\mathbf{c}),V}$ (see e.g. [Hoc75] and [DF25, Thm. 2.7]) by

$$\sigma_{\mathbf{c}}(V) = \sigma\left(H_{\varphi(\mathbf{c}),V}\right) = \bigcup_{\theta\in[0,\pi]} \sigma\left(H_{\mathbf{c},V}(\theta)\right). \tag{4.1}$$

Lemma 4.1. *For all $V \in \mathbb{R}$ and $\mathbf{c} \in \mathcal{C}$ with $\varphi(\mathbf{c}) \in [0, 1]$, we have $\sigma_{\mathbf{c}}(V) = -\sigma_{\mathbf{c}}(-V)$.*

Proof. Using the unitary $q \times q$ diagonal matrix, $D := \operatorname{diag}\{1, -1, 1, -1, \dots\}$, we obtain

- $D^{-1}H_{\mathbf{c},-V}(\theta)\, D = -H_{\mathbf{c},V}(\theta)$ if q is even,
- $D^{-1}H_{\mathbf{c},-V}(\theta)\, D = -H_{\mathbf{c},V}(\theta + \pi)$ if q is odd.

By this unitary equivalence and (4.1), this yields $\sigma_{\mathbf{c}}(V) = -\sigma_{\mathbf{c}}(-V)$. □

We have already mentioned (Proposition 2.1) that $\sigma_{\mathbf{c}}(V)$ consists of exactly q intervals (spectral bands). By standard arguments, the endpoints of these intervals are given by the eigenvalues of $H_{\mathbf{c},V}(0)$ and $H_{\mathbf{c},V}(\pi)$. Hence, the values $\theta \in \{0, \pi\}$ play a significance role in (4.1).

The spectral decomposition (4.1) may be also written in terms of the following $nq \times nq$-matrix

$$H^{\times n}_{\mathbf{c},V} := H_{\varphi(\mathbf{c}),V}|_{[0,nq-1]} = \begin{pmatrix} H_{\mathbf{c},V} & \mathbb{I}_{q,1} & 0 & \dots & & 0 \\ \mathbb{I}_{1,q} & H_{\mathbf{c},V} & \mathbb{I}_{q,1} & \dots & & \\ 0 & \mathbb{I}_{1,q} & \ddots & & & \\ \vdots & \ddots & & \ddots & & 0 \\ 0 & & & & & \mathbb{I}_{q,1} \\ 0 & 0 & \dots & 0 & \mathbb{I}_{1,q} & H_{\mathbf{c},V} \end{pmatrix}.$$

The diagonal of $H^{\times n}_{\mathbf{c},V}$ consists of n repetitions of the diagonal of $H_{\mathbf{c},V}$, corresponding to the minimal period of the potential sequence. Hence,

$$\sigma_{\mathbf{c}}(V) = \sigma\left(H_{\varphi(\mathbf{c}),V}\right) = \bigcup_{\theta\in[0,\pi]} \sigma\left(H^{\times n}_{\mathbf{c},V}(\theta)\right). \tag{4.2}$$

The eigenvalues of $H^{\times n}_{\mathbf{c},V}(0)$ and $H^{\times n}_{\mathbf{c},V}(\pi)$ determine the band edges, but also occur in the interior within these intervals (a detailed description appears in the proof of Lemma 5.5).

Although $H^{\times n}_{\mathbf{c},V}$ may seem redundant purely for determining the spectrum $\sigma_{\mathbf{c}}(V)$, it becomes essential in the sequel. The matrices $H_{\mathbf{c},V}, H^{\times n}_{[\mathbf{c},m],V}$, and $H_{[\mathbf{c},m,n],V}$ describe the spectra $\sigma_{\mathbf{c}}$, $\sigma_{[\mathbf{c},m]}$ and $\sigma_{[\mathbf{c},m,n]}$, respectively. By Lemma I.1, the diagonal of $H_{[\mathbf{c},m,n],V}$ is a concatenation of those of $H_{\mathbf{c},V}$ and $H^{\times n}_{[\mathbf{c},m]}$, so that

$$H_{[\mathbf{c},m,n],V} = H^{\times n}_{[\mathbf{c},m],V} \oplus H_{\mathbf{c},V} \quad \text{or} \quad H_{[\mathbf{c},m,n],V} = H_{\mathbf{c},V} \oplus H^{\times n}_{[\mathbf{c},m],V},$$

depending on the parity of the length of $\mathbf{c}$. This structure yields an eigenvalue interlacing theorem. Writing the eigenvalues of a Hermitian $q \times q$ matrix H as

$$\lambda_0(H) \le \lambda_1(H) \le \ldots \le \lambda_{q-1}(H), \tag{4.3}$$

we obtain the following.

Theorem 4.2 (Interlacing theorem)**.** *Let $V > 0$. Let $m, n \in \mathbb{N}$ and $\boldsymbol{c} \in \mathcal{C}$ be such that $\varphi(\boldsymbol{c}) \notin \{-1, \infty\}$ and $[\boldsymbol{c}, m] \in \mathcal{C}$. Let $\theta_{\boldsymbol{c}}, \theta_{[\boldsymbol{c},m]}, \theta_{[\boldsymbol{c},m,n]} \in \{0, \pi\}$ and denote*

$$Y = H_{[\boldsymbol{c},m,n],V}\left(\theta_{[\boldsymbol{c},m,n]}\right) \quad \text{and} \quad X = H^{\times n}_{[\boldsymbol{c},m],V}\left(\theta_{[\boldsymbol{c},m]}\right) \oplus H_{\boldsymbol{c},V}(\theta_{\boldsymbol{c}}).$$

If $\theta_{\boldsymbol{c}} + \theta_{[\boldsymbol{c},m]} + \theta_{[\boldsymbol{c},m,n]} \in \{0, 2\pi\}$, then

$$\lambda_{j-1}(Y) \le \lambda_j(X) \le \lambda_{j+1}(Y).$$

Furthermore, if $\lambda_j(X)$ is a simple eigenvalue of X, then both inequalities are strict.

Theorem 4.2 is proven in the Appendix III. Note that even though the eigenvalues depend on the parameter $V > 0$, the inequalities of the eigenvalues hold independently of the value $V > 0$ attains[1]. The condition in the previous theorem naturally leads to the following useful definition of admissibility.

Definition 4.3 (Admissibility)**.** Let $m, n \in \mathbb{N}$ and $\mathbf{c} \in \mathcal{C}$ be such that $\varphi(\mathbf{c}) \notin \{-1, \infty\}$ and $[\mathbf{c}, m] \in \mathcal{C}$.

(a) The values $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]} \in \{0, \pi\}$ are called *admissible* if $\theta_{\mathbf{c}} + \theta_{[\mathbf{c},m]} + \theta_{[\mathbf{c},m,n]} \in \{0, 2\pi\}$.

(b) For each $\tilde{\mathbf{c}} \in \{\mathbf{c}, [\mathbf{c}, m], [\mathbf{c}, m, n]\}$, let $I_{\tilde{\mathbf{c}}} : V \mapsto I_{\tilde{\mathbf{c}}}(V), V > 0$, be a spectral band in $\sigma_{\tilde{\mathbf{c}}}$, and let $\lambda_{\tilde{\mathbf{c}}} : V \mapsto \lambda_{\tilde{\mathbf{c}}}(V), V > 0$, satisfy either $\lambda_{\tilde{\mathbf{c}}}(V) = L\left(I_{\tilde{\mathbf{c}}}(V)\right)$ for all $V > 0$ or $\lambda_{\tilde{\mathbf{c}}}(V) = R\left(I_{\tilde{\mathbf{c}}}(V)\right)$ for all $V > 0$. Then we call $\lambda_{\mathbf{c}}, \lambda_{[\mathbf{c},m]}, \lambda_{[\mathbf{c},m,n]}$ admissible, if there exist admissible $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]} \in \{0, \pi\}$ such that for all $V > 0$,

$$\lambda_{\mathbf{c}}(V) \in \sigma\left(H_{\mathbf{c},V}(\theta_{\mathbf{c}})\right), \ \lambda_{[\mathbf{c},m]}(V) \in \sigma\left(H^{\times n}_{[\mathbf{c},m],V}(\theta_{[\mathbf{c},m]})\right), \ \lambda_{[\mathbf{c},m,n]}(V) \in \sigma\left(H_{[\mathbf{c},m,n],V}(\theta_{[\mathbf{c},m,n]})\right).$$

Remark 4.4. We emphasize here that $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}$ are admissible if the triple has an even number of π's. In particular,

$$\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]} \text{ are not admissible} \quad \Leftrightarrow \quad \theta_{\mathbf{c}}, \pi - \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]} \text{ are admissible.}$$

We further note that the maps $\lambda_{\tilde{\mathbf{c}}} : V \mapsto \lambda_{\tilde{\mathbf{c}}}(V), V > 0$, appearing in the definition are Lipschitz continuous by Proposition 2.3. Moreover, Lemma 5.4 implies that for each such map – left or right end point of a fixed spectral band $I_{\tilde{\mathbf{c}}}$ – there exists a unique $\theta_{\tilde{\mathbf{c}}} \in \{0, \pi\}$ such that $\lambda_{\tilde{\mathbf{c}}}(V) \in \sigma\left(H_{\tilde{\mathbf{c}},V}(\theta_{\tilde{\mathbf{c}}})\right)$ for all $V > 0$. In particular, admissibility of $\lambda_{\mathbf{c}}, \lambda_{[\mathbf{c},m]}, \lambda_{[\mathbf{c},m,n]}$ is independent of $V > 0$.

[1]Note that simplicity may depend on $V > 0$.

4.2. **Transfer matrices and their traces .** We present the well-known formalism for transfer matrices, see e.g. [DF22, Ch. 2]. For $V \in \mathbb{R}$, define

$$M_{[0]}(E,V) := \begin{pmatrix} 1 & -V \\ 0 & 1 \end{pmatrix}, \qquad M_{[0,0]}(E,V) := \begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix}$$

and recursively define the transfer matrices for $\mathbf{c} = [0,0,c_1,\ldots,c_k] \in \mathcal{C}$ (where $k \in \mathbb{N}$) by

$$M_{\mathbf{c}}(E,V) := M_{[0,0,c_1\ldots,c_{k-2}]}(E,V) M_{[0,0,c_1,\ldots,c_{k-1}]}(E,V)^{c_k}.$$

Denote the traces of the transfer matrices by

$$t_{\mathbf{c}}(E,V) := tr(M_{\mathbf{c}}(E,V)). \tag{4.4}$$

Our description only slightly deviates from the conventional one, by referring to all the elements of $\mathcal{C}$ (within the literature above we take a route which is the closest to [Ray95]).

Denote by $\chi_{H_{\mathbf{c},V}(\theta)}$ the characteristic polynomial of the Floquet-Bloch matrix $H_{\mathbf{c},V}(\theta)$. Then we have (see e.g., [Hoc75, Eq. (23)], [Sim11, Thm. 5.4.1,(iii)] or [BBB$^+$26, Lem. II.2]) that

$$\chi_{H_{\mathbf{c},V}(\theta)}(E) = t_{\mathbf{c}}(E,V) - 2\cos(\theta), \qquad \theta \in [0,2\pi]\,. \tag{4.5}$$

This leads to the following well-known result, see e.g. [DF25, Thm. 7.2.7] and [Sim11, Sec. 5.4].

Lemma 4.5. *For all $\boldsymbol{c}, \widetilde{\boldsymbol{c}} \in \mathcal{C}$ with $\varphi(\widetilde{\boldsymbol{c}}) = \varphi(\boldsymbol{c})$, we have $t_{\widetilde{\boldsymbol{c}}}(E,V) = t_{\boldsymbol{c}}(E,V)$ for all $E, V \in \mathbb{R}$. Furthermore,*

$$\sigma_{\boldsymbol{c}}(V) = \{E \in \mathbb{R} \,:\, |t_{\boldsymbol{c}}(E,V)| \leq 2\}\,, \qquad \boldsymbol{c} \in \mathcal{C},\, V \in \mathbb{R}.$$

Proof. This is an immediate consequence of (4.1) and (4.5). □

Lemma 4.6. *Let $V \in \mathbb{R} \setminus \{0\}$, $\boldsymbol{c} \in \mathcal{C}$ with $\varphi(\boldsymbol{c}) \notin \{-1, \infty\}$. Then the following statements hold.*

(a) For $E \in \mathbb{R}$, we have $|t_{\boldsymbol{c}}(E,V)| = 2$, if and only if $E \in \{L(I_{\boldsymbol{c}}(V)), R(I_{\boldsymbol{c}}(V))\}$ for some spectral band $I_{\boldsymbol{c}}(V)$ *in $\sigma_{\boldsymbol{c}}(V)$.*

(b) If a spectral band $I_{\boldsymbol{c}}$ in $\sigma_{\boldsymbol{c}}$ is
- *of backward type A, then $|t_{[\boldsymbol{c},0]}(E,V)| \leq 2$ for all $E \in I_{\boldsymbol{c}}$. The estimate is strict if $\varphi(\boldsymbol{c}) \in (0,1)$.*
- *of backward type B, then $|t_{[\boldsymbol{c},-1]}(E,V)| \leq 2$ for all $E \in I_{\boldsymbol{c}}$. The estimate is strict if $\varphi(\boldsymbol{c}) \in (0,1)$.*

(c) For $m \geq 0$, we have $t_{[\boldsymbol{c},m+1]} = t_{\boldsymbol{c}} t_{[\boldsymbol{c},m]} - t_{[\boldsymbol{c},m-1]}$.

Proof. Let $\varphi(\mathbf{c}) = \frac{p}{q}$ be such that p, q are coprime.
(a) This is an immediate consequence of [Sim11, Thm. 5.4.2] and that $\sigma_{\mathbf{c}}(V)$ consists of exactly q spectral bands, see Proposition 2.1.
(b) This follows from Definition 2.6, Lemma 4.5 and (a).
(c) This well-known identity is proven in [Ray95]. The reader is also referred to Appendix II.2 for related results and more references, see also [BBB$^+$26, Lem. 3.8]. □

Remark. The first statement (a) of the lemma says that the traces attain the values ± 2 exactly at the spectral band edges. This does not hold for $\mathbf{c} = [0]$ where $t_{\mathbf{c}}(E,V) = 2$ and $\sigma_{\mathbf{c}}(V) = \mathbb{R}$.

The next statement is based on standard techniques of transfer matrix traces and its proof is included in the Appendix II.2.

Lemma 4.7. *Let $V \in \mathbb{R}$, $m \in \mathbb{N}$, $\boldsymbol{c} \in \mathcal{C}$ be such that $\varphi(\boldsymbol{c}) \notin \{-1, \infty\}$ and $[\boldsymbol{c}, m] \in \mathcal{C}$. Let $I(V)$ be a spectral band in $\sigma_{\boldsymbol{c}}(V)$ of backward type A or backward type B. Then for $E \in \{L(I(V)), R(I(V))\}$ and $n \in \mathbb{N}$, the following holds.*

(a) $|t_{[\boldsymbol{c},m]}(E,V)| \geq 2 \quad \Rightarrow \quad |t_{[\boldsymbol{c},m,n]}(E,V)| \geq 2.$
(b) $|t_{[\boldsymbol{c},m]}(E,V)| > 2 \quad \Rightarrow \quad |t_{[\boldsymbol{c},m,n]}(E,V)| > 2.$

(c) $\varphi(\boldsymbol{c}) \in (0,1)$ *and* $|t_{[\boldsymbol{c},m]}(E,V)| \geq 2 \quad \Rightarrow \quad |t_{[\boldsymbol{c},m,n]}(E,V)| > 2$.

The concept of admissible eigenvalues (Definition 4.3) can also be characterized in terms of the traces of these eigenvalues, which is a central tool towards the solution of the dry ten Martini problem [BBL24].

Proposition 4.8. *Let* $m \in \mathbb{N}$ *and* $\boldsymbol{c} \in \mathcal{C}$ *be such that* $\varphi(\boldsymbol{c}) \notin \{-1, \infty\}$ *and* $[\boldsymbol{c}, m] \in \mathcal{C}$. *For each* $\tilde{\mathbf{c}} \in \{\boldsymbol{c}, [\boldsymbol{c},m], [\boldsymbol{c},m,1]\}$, *let* $I_{\tilde{\mathbf{c}}} : V \mapsto I_{\tilde{\mathbf{c}}}(V), V > 0$, *be a spectral band in* $\sigma_{\tilde{\mathbf{c}}}$ *and* $\lambda_{\tilde{\mathbf{c}}} \in \{L(I_{\tilde{\mathbf{c}}}), R(I_{\tilde{\mathbf{c}}})\}$. *Then* $\lambda_{\mathbf{c}}, \lambda_{[\mathbf{c},m]}, \lambda_{[\mathbf{c},m,1]}$ *are admissible if and only if*

$$\operatorname{sign}\left(t_{\boldsymbol{c}}\left(\lambda_{\boldsymbol{c}}(V)\right) \cdot t_{[\boldsymbol{c},m]}\left(\lambda_{[\boldsymbol{c},m]}(V)\right) \cdot t_{[\boldsymbol{c},m,1]}\left(\lambda_{[\boldsymbol{c},m,1]}(V)\right)\right) = +1 \quad \textit{for all } V > 0.$$

Proof. Let $\tilde{\mathbf{c}} \in \{\mathbf{c}, [\mathbf{c},m], [\mathbf{c},m,1]\}$. By Lemma 4.6 (a) and (4.5), $\lambda_{\tilde{\mathbf{c}}} \in \{L(I_{\tilde{\mathbf{c}}}), R(I_{\tilde{\mathbf{c}}})\}$ if and only if there exists $\theta_{\tilde{\mathbf{c}}} \in \{0, \pi\}$ such that $\lambda_{\tilde{\mathbf{c}}}(V) \in \sigma\left(H_{\tilde{\mathbf{c}},V}(\theta_{\tilde{\mathbf{c}}})\right)$ for all $V > 0$. Moreover, (4.5) gives $t_{\tilde{\mathbf{c}}}\left(\lambda_{\tilde{\mathbf{c}}}(V)\right) = 2\cos(\theta_{\tilde{\mathbf{c}}})$ for all $V \in \mathbb{R}$. Hence, this value equals 2 if $\theta_{\tilde{\mathbf{c}}} = 0$ and -2 if $\theta_{\tilde{\mathbf{c}}} = \pi$.

By Definition 4.3, the triple $\lambda_{\mathbf{c}}, \lambda_{[\mathbf{c},m]}, \lambda_{[\mathbf{c},m,1]}$ is admissible if and only if the triple $\theta_{\mathbf{c}}$, $\theta_{[\mathbf{c},m]}$, $\theta_{[\mathbf{c},m,1]}$ contains an even number of π's. Equivalently, the triple of trace values $t_{\mathbf{c}}\left(\lambda_{\mathbf{c}}(V)\right)$, $t_{[\mathbf{c},m]}\left(\lambda_{[\mathbf{c},m]}(V)\right)$, $t_{[\mathbf{c},m,1]}\left(\lambda_{[\mathbf{c},m,1]}(V)\right)$ contains an even number of entries equal to -2 for all $V > 0$, namely their product has a positive sign. □

5. Tools towards proving forward type

This section is devoted to various technical tools needed for the induction base of the proof of Theorem 1.1 (Section 7) and for proving Proposition 3.1 - that backward type implies forward type (Section 6).

Let us provide a short overview of this section. In Subsection 5.1, we introduce an eigenvalue counting function, which later plays a crucial role in application of the interlacing theorem (Theorem 4.2). Since eigenvalue admissibility is a necessary condition in the interlacing theorem, we give a useful characterization of it in Subsection 5.2. With this at hand, in Subsection 5.3 we provide Lemma 5.7 which is a manifestation of the interlacing theorem (Theorem 4.2). In effect, it is this lemma which is going to be directly applied, rather than Theorem 4.2. In Subsection 5.4, we develop index relations which are needed whenever we apply Lemma 5.7. Then, Subsection 5.5 applies the various index relations, eigenvalue estimates and trace estimates to prove that the spectral bands $I^i_{[\mathbf{c},m]}$ and $I^j_{[\mathbf{c},m,n]}$ maintain certain properties from Definition 2.9 when V decreases to zero.

Throughout this section we use the notational conventions of Definition 2.12 without pointing them out all the time.

5.1. Counting spectral bands and eigenvalues.

In this subsection we consider two types of counting functions: for the spectral bands in $\sigma_{\mathbf{c}}$ and for the eigenvalues of the matrices $H_{\mathbf{c},V}(\theta)$, $H^{\times n}_{\mathbf{c},V}(\theta)$ and relate both types of functions.

First, we recall that $\sigma_{\mathbf{c}}(V)$ consists of exactly q intervals for $\varphi(\mathbf{c}) = \frac{p}{q}$ (see Proposition 2.1 and Lemma 4.5) and that we consider each spectral band as a Lipschitz continuous map, $V \mapsto I_{\mathbf{c}}(V)$, for $V > 0$ (Definition 2.2 and Proposition 2.3). This justifies the following.

Definition 5.1. [Index of a spectral band] Let $I_{\mathbf{c}}$ be a spectral band of $\sigma_{\mathbf{c}}$. The *index* of $I_{\mathbf{c}}$ (in $\sigma_{\mathbf{c}}$) is defined by

$$\operatorname{ind}(I_{\mathbf{c}}) := \left|\{I \text{ is a spectral band of } \sigma_{\mathbf{c}} : I \prec I_{\mathbf{c}}\}\right|.$$

Remark 5.2. Note that the index counting starts from zero, namely $0 \leq \operatorname{ind}(I_{\mathbf{c}}) \leq q-1$ where $\varphi(\mathbf{c}) = \frac{p}{q}$ with p, q coprime. Moreover, we emphasize that $\operatorname{ind}(I_{\mathbf{c}})$ is independent of $V > 0$, allowing us to assume $V > 4$ in some instances and use Theorem 2.11.

In order to apply the interlacing theorem (Theorem 4.2), we need to count eigenvalues. Let $\{\lambda_i(H)\}_{i=0}^{n-1}$ be the eigenvalues (increasingly arranged and counted with multiplicity) of an $n \times n$ matrix H, as in (4.3).

Definition 5.3. [Counting function] For an $n\times n$ hermitian matrix H, the *eigenvalue counting function* is defined by

$$N(\lambda; H) := |\{0 \leq i \leq n-1 \, : \, \lambda_i(H) < \lambda\}| \, .$$

Remark. Note that $N(\lambda;\ H)$ may attain the value zero and also that $N(\lambda_i(H);\ H) = i$ for each $0 \leq i \leq n-1$ where $\lambda_i(H)$ is simple.

We will be in particular interested in evaluating the counting function for an eigenvalue which is also an edge of a certain spectral band. The index of that spectral band is then related to the counting of its edge point, as follows.

Lemma 5.4. *Let $V > 0$, $\boldsymbol{c} \in \mathcal{C}$ and $\{-1, \infty\} \not\ni \varphi(\boldsymbol{c}) = \frac{p}{q}$ with p, q coprime. Let $I_{\boldsymbol{c}}$ be a spectral band of $\sigma_{\boldsymbol{c}}$ and $\theta \in \{0, \pi\}$.*

(a) We have

$$ind(I_{\boldsymbol{c}}) - q \equiv \frac{1}{\pi}\theta \mod 2 \quad \Leftrightarrow \quad L(I_{\boldsymbol{c}}(V)) \in \sigma\left(H_{\boldsymbol{c},V}(\theta)\right)$$

and

$$ind(I_{\boldsymbol{c}}) + 1 - q \equiv \frac{1}{\pi}\theta \mod 2 \quad \Leftrightarrow \quad R(I_{\boldsymbol{c}}(V)) \in \sigma\left(H_{\boldsymbol{c},V}(\theta)\right).$$

(b) If $L(I_{\boldsymbol{c}}(V)) \in \sigma\left(H_{\boldsymbol{c},V}(\theta)\right)$, then

$$ind(I_{\boldsymbol{c}}) = N\left(L(I_{\boldsymbol{c}}(V)); H_{\boldsymbol{c},V}(\theta)\right).$$

(c) If $R(I_{\boldsymbol{c}}(V)) \in \sigma\left(H_{\boldsymbol{c},V}(\theta)\right)$, then

$$ind(I_{\boldsymbol{c}}) = N\left(R(I_{\boldsymbol{c}}(V)); H_{\boldsymbol{c},V}(\theta)\right).$$

Proof. This follows from the next lemma and the fact that $H_{\mathbf{c},V}(\theta) = H^{\times n}_{\mathbf{c},V}(\theta)$ if $n = 1$. ☐

Lemma 5.4 can be generalized as follows.

Lemma 5.5. *Let $V > 0$, $\boldsymbol{c} \in \mathcal{C}$ and $\{-1, \infty\} \not\ni \varphi(\boldsymbol{c}) = \frac{p}{q}$ with p, q coprime. Let $I_{\boldsymbol{c}}$ be a spectral band of $\sigma_{\boldsymbol{c}}$ and $\theta \in \{0, \pi\}$. Then the following holds for $n \in \mathbb{N}$.*

(a) If $n \in \mathbb{N}$ is even, then $L(I_{\boldsymbol{c}}(V)) \in \sigma\left(H^{\times n}_{\boldsymbol{c},V}(0)\right)$ and $R(I_{\boldsymbol{c}}(V)) \in \sigma\left(H^{\times n}_{\boldsymbol{c},V}(0)\right)$.

(b) If $n \in \mathbb{N}$ is odd, then

$$ind(I_{\boldsymbol{c}}) - q \equiv \frac{1}{\pi}\theta \mod 2 \quad \Leftrightarrow \quad L(I_{\boldsymbol{c}}(V)) \in \sigma\left(H^{\times n}_{\boldsymbol{c},V}(\theta)\right)$$

and

$$ind(I_{\boldsymbol{c}}) + 1 - q \equiv \frac{1}{\pi}\theta \mod 2 \quad \Leftrightarrow \quad R(I_{\boldsymbol{c}}(V)) \in \sigma\left(H^{\times n}_{\boldsymbol{c},V}(\theta)\right).$$

(c) If $L(I_{\boldsymbol{c}}(V)) \in \sigma\left(H^{\times n}_{\boldsymbol{c},V}(\theta)\right)$, then

$$n \cdot ind(I_{\boldsymbol{c}}) = N\left(L(I_{\boldsymbol{c}}(V)); H^{\times n}_{\boldsymbol{c},V}(\theta)\right)$$

and there exists $\lambda \in \sigma\left(H^{\times n}_{\boldsymbol{c},V}(\pi - \theta)\right)$ such that

$$L(I_{\boldsymbol{c}}(V)) < \lambda \leq R(I_{\boldsymbol{c}}(V)) \quad \text{and} \quad N\left(\lambda; H^{\times n}_{\boldsymbol{c},V}(\pi - \theta)\right) = N\left(L(I_{\boldsymbol{c}}(V)); H^{\times n}_{\boldsymbol{c},V}(\theta)\right).$$

(d) *If* $R(I_{\boldsymbol{c}}(V)) \in \sigma\left(H^{\times n}_{\boldsymbol{c},V}(\theta)\right)$, *then*

$$n \cdot (ind(I_{\boldsymbol{c}}) + 1) - 1 = N\left(R(I_{\boldsymbol{c}}(V)); H^{\times n}_{\boldsymbol{c},V}(\theta)\right)$$

and for $n \geq 2$, *there exists* $\lambda \in \sigma\left(H^{\times n}_{\boldsymbol{c},V}(\pi - \theta)\right)$ *such that*

$$L(I_{\boldsymbol{c}}(V)) \leq \lambda < R(I_{\boldsymbol{c}}(V)) \quad \text{and} \quad N\left(\lambda; H^{\times n}_{\boldsymbol{c},V}(\pi - \theta)\right) = N\left(R(I_{\boldsymbol{c}}(V)); H^{\times n}_{\boldsymbol{c},V}(\theta)\right) - 1.$$

(e) *We have* $\left|\left\{\lambda \in \sigma\left(H^{\times n}_{\boldsymbol{c},V}(\theta)\right) \cap I_{\boldsymbol{c}}(V)\right\}\right| = n$. *If* $\lambda \in \sigma\left(H^{\times n}_{\boldsymbol{c},V}(\theta)\right) \cap \{L(I_{\boldsymbol{c}}(V)), R(I_{\boldsymbol{c}}(V))\}$, *then* λ *is a simple eigenvalue of* $H^{\times n}_{\boldsymbol{c},V}(\theta)$.

Proof. Recall from (4.2) that the spectrum $\sigma_{\mathbf{c}}(V)$ is given as the union of the eigenvalues of $H^{\times n}_{\mathbf{c},V}(\theta)$ over all $\theta \in [0, \pi]$. Denote by $\lambda_j^{(\theta)} := \lambda_j\left(H^{\times n}_{\mathbf{c},V}(\theta)\right)$ for $0 \leq j \leq nq - 1$ the eigenvalues of $H^{\times n}_{\mathbf{c},V}(\theta)$ in increasing order counting multiplicities, see (4.3). These eigenvalues for $\theta \in \{0, \pi\}$ are arranged as follows,

$$\ldots \leq \lambda^{(\pi)}_{nq-4} \leq \lambda^{(\pi)}_{nq-3} < \lambda^{(0)}_{nq-3} \leq \lambda^{(0)}_{nq-2} < \lambda^{(\pi)}_{nq-2} \leq \lambda^{(\pi)}_{nq-1} < \lambda^{(0)}_{nq-1}, \tag{5.1}$$

noting that the strict inequalities above appear whenever we compare eigenvalues with different θ values (see e.g. [Hoc75, Eq. (25)]). We use these eigenvalues to recursively define the following intervals

$$\ldots,\ J_l := [\lambda_l^{(\theta_l)}, \lambda_l^{(\pi-\theta_l)}],\ \ldots,\ J_{nq-2} := [\lambda^{(0)}_{nq-2}, \lambda^{(\pi)}_{nq-2}],\ J_{nq-1} := [\lambda^{(\pi)}_{nq-1}, \lambda^{(0)}_{nq-1}],$$

for appropriately chosen $\theta_l \in \{0, \pi\}$. We note that these intervals are ordered, i.e. $J_l \prec J_{l+1}$ for all $0 \leq l \leq nq - 2$.

We now make a connection between these intervals, and the spectral bands $I_{\mathbf{c}}$ of $\sigma_{\mathbf{c}}$. By Proposition 2.1 and Lemma 4.5, $\sigma_{\mathbf{c}}(V)$ consists of exactly q disjoint intervals - called spectral bands. For each such spectral band $I_{\mathbf{c}}$ of $\sigma_{\mathbf{c}}$, set $j = \text{ind}(I_{\mathbf{c}})$ and $I_j := I_{\mathbf{c}}(V)$ for the given $V > 0$.

We show a few auxiliary claims, and then use them to prove the statements in the lemma.

(1) For all $0 \leq l \leq nq - 1$, the endpoints $L(J_l)$ and $R(J_l)$ correspond to eigenvalues with different θ values. Moreover, $R(J_l)$ and $L(J_{l+1})$ correspond to the same value of $\theta \in \{0, \pi\}$ for all $0 \leq l \leq nq - 2$.

(2) The equalities

$$\sigma_{\mathbf{c}}(V) = \bigcup_{j=0}^{q-1} I_j = \bigcup_{l=0}^{nq-1} J_l \quad \text{and} \quad I_{q-1-j} = \bigcup_{l=0}^{n-1} J_{nq-1-nj-l} \ \text{ for all } 0 \leq j \leq q - 1$$

hold.

(3) For $\theta \in \{0, \pi\}$, each I_j contains exactly n eigenvalues of $\sigma\left(H^{\times n}_{\mathbf{c},V}(\theta)\right)$.

(4) We have $R(I_{q-1}) = \lambda^{(0)}_{nq-1}$.

Claim (1) is immediate from the definition of the intervals J_l. The first equality in (2) follows from (4.1) and (4.2). Thus, each I_j is the union of some of the consecutive intervals J_l. By [Hoc84, Theorem 1] each n consecutive J_l bands touch (so that their union is a single connected component) and this inductively implies the second equality in (2). This also implies (3). To deduce (4) we combine $I_{q-1} = \bigcup_{l=0}^{n-1} J_{nq-1-l}$ (which follows from (2)) with $J_{nq-1} := [\lambda^{(\pi)}_{nq-1}, \lambda^{(0)}_{nq-1}]$.

We now use the claims above to prove the different statements of the lemma.

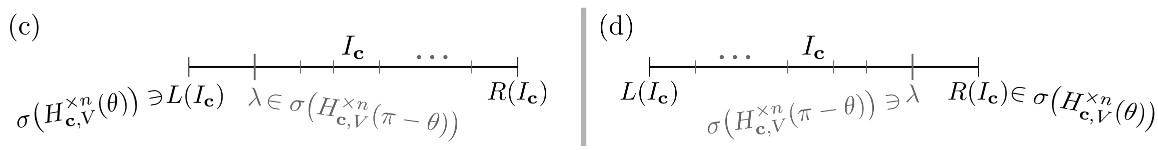

FIGURE 5.1. A sketch for the proof of (c) and (d) in Lemma 5.5.

(a): The claims (1) and (2) for even $n \in \mathbb{N}$ imply that the left and right spectral edges of $I_{\mathbf{c}}(V)$ correspond to the same value $\theta \in \{0, \pi\}$. Combining this with claim (4) implies that all spectral edges of $I_{\mathbf{c}}$ correspond to the value $\theta = 0$.

(b): The claims (1) and (2) for odd $n \in \mathbb{N}$ imply that the left and right spectral edge of $I_{\mathbf{c}}(V)$ correspond to a different value of $\theta \in \{0, \pi\}$. Hence, the value of $\theta \in \{0, \pi\}$ which corresponds to $L(I_j)$ alternates with j (and it also alternates for $R(I_j)$). Combining this with claim (4) yields the statement in (b).

(c) and (d): The first equality in (c) and (d) follows from claim (3). Note that for (d) the quantity $N\left(R(I_{\mathbf{c}}(V)); H^{\times n}_{\mathbf{c},V}(\theta)\right)$ counts $n-1$ eigenvalues in the spectral band $I_{\mathbf{c}}$(V) and n eigenvalues for each spectral band $I(V) \prec I_{\mathbf{c}}(V)$ (which are $\operatorname{ind}(I_{\mathbf{c}})$ many).

We turn to prove the second claim in (c). It follows from claim (2) that there exists $\theta \in \{0, \pi\}$ such that $L(I_{\mathbf{c}}(V)) \in \sigma\left(H^{\times n}_{\mathbf{c},V}(\theta)\right)$. Using the notation for the eigenvalues of $H^{\times n}_{\mathbf{c},V}(\theta)$ introduced in the beginning of the proof, we can write $\lambda^{(\theta)}_l := L(I_{\mathbf{c}}(V))$, for some $0 \leq l \leq nq-1$. Now, we define

$$\lambda := \lambda^{(\pi-\theta)}_l = \min\left\{\tilde{\lambda} \,:\, \tilde{\lambda} \in \sigma\left(H^{\times n}_{\mathbf{c},V}(\pi-\theta)\right) \quad \text{and} \quad \tilde{\lambda} > \lambda^{(\theta)}_l := L(I_{\mathbf{c}}(V))\right\}$$

(as sketched in Figure 5.1,(c)) and show that this is the desired $\lambda \in \sigma\left(H^{\times n}_{\mathbf{c},V}(\pi-\theta)\right)$ in the statement of (c). By the construction in the beginning of the proof we get that $L(I_{\mathbf{c}}(V)) < \lambda$ and $J_l = [L(I_{\mathbf{c}}(V)), \lambda]$. Furthermore, J_l is the left-most sub-interval within $I_{\mathbf{c}}(V)$, as in the decomposition of claim (2). Hence, $L(I_{\mathbf{c}}(V)) < \lambda \leq R(I_{\mathbf{c}}(V))$, as stated in (c). To complete the proof of (c) we just note that $N\left(L(I_{\mathbf{c}}(V)); H^{\times n}_{\mathbf{c},V}(\theta)\right) = l$, just by the choice of $0 \leq l \leq nq-1$ and similarly $N\left(\lambda; H^{\times n}_{\mathbf{c},V}(\pi-\theta)\right) = l$. Hence, $N\left(\lambda; H^{\times n}_{\mathbf{c},V}(\pi-\theta)\right) = N\left(L(I_{\mathbf{c}}(V)); H^{\times n}_{\mathbf{c},V}(\theta)\right)$.

It is left to prove the second claim in (d). This follows similarly as in (c). First, there exists $\theta \in \{0, \pi\}$ such that $R(I_{\mathbf{c}}(V)) \in \sigma\left(H^{\times n}_{\mathbf{c},V}(\theta)\right)$; we write $\lambda^{(\theta)}_l := R(I_{\mathbf{c}}(V))$, for some $0 \leq l \leq nq-1$; we define

$$\lambda := \lambda^{(\pi-\theta)}_l = \max\left\{\tilde{\lambda} \,:\, \tilde{\lambda} \in \sigma\left(H^{\times n}_{\mathbf{c},V}(\pi-\theta)\right) \quad \text{and} \quad \tilde{\lambda} < \lambda^{(\theta)}_l = R(I_{\mathbf{c}}(V))\right\}$$

(as sketched in Figure 5.1). Then $L(I_{\mathbf{c}}(V)) \leq \lambda < R(I_{\mathbf{c}}(V))$ holds. If $n \geq 2$, then λ is in the interior of $I_{\mathbf{c}}(V)$ and so the eigenvalue $\lambda \in \sigma\left(H^{\times n}_{\mathbf{c},V}(\pi-\theta)\right)$ has multiplicity two by (5.1) and claim (2). Thus, $N\left(\lambda; H^{\times n}_{\mathbf{c},V}(\theta)\right)$ counts $n-2$ eigenvalues in $I_{\mathbf{c}}(V)$ while $N\left(R(I_{\mathbf{c}}(V)); H^{\times n}_{\mathbf{c},V}(\theta)\right)$ counts $n-1$ eigenvalues in $I_{\mathbf{c}}(V)$. Hence, $N\left(\lambda; H^{\times n}_{\mathbf{c},V}(\pi-\theta)\right) = N\left(R(I_{\mathbf{c}}(V)); H^{\times n}_{\mathbf{c},V}(\theta)\right) - 1$ follows proving (d).

(e) This is an immediate consequence of claim (2) and (5.1). □

5.2. **A characterization of admissibility.** We recall the definition of admissibility (Definition 4.3) for a triple of eigenvalues. We now use the lemmata of the previous subsection in order to provide an equivalent condition for admissibility. Since the definition of admissibility is independent of $V > 0$ (as is also mentioned within Definition 4.3), we omit the V-dependence from the notation in this subsection. For example, we write $I_{\mathbf{c}}, \lambda_{\mathbf{c}}$ and $H_{\mathbf{c}}^{\times n}$ instead of writing $I_{\mathbf{c}}(V), \lambda_{\mathbf{c}}(V)$ and $H_{\mathbf{c},V}^{\times n}$.

Lemma 5.6. *Let $m, n \in \mathbb{N}$, and $\boldsymbol{c} \in \mathcal{C}$ be such that $\varphi(\boldsymbol{c}) \notin \{-1, \infty\}$ and $[\boldsymbol{c}, m] \in \mathcal{C}$. For each $\widetilde{\boldsymbol{c}} \in \{\boldsymbol{c},\ [\boldsymbol{c}, m],\ [\boldsymbol{c}, m, n]\}$, let $I_{\widetilde{\boldsymbol{c}}}$ be a spectral band in $\sigma_{\widetilde{\boldsymbol{c}}}$ and $\lambda_{\widetilde{\boldsymbol{c}}} \in \{L(I_{\widetilde{\boldsymbol{c}}}), R(I_{\widetilde{\boldsymbol{c}}})\}$ and denote*

$$\delta_R(\lambda_{\widetilde{\boldsymbol{c}}}) := \begin{cases} 0, & \lambda_{\widetilde{\boldsymbol{c}}} = L(I_{\widetilde{\boldsymbol{c}}}), \\ 1, & \lambda_{\widetilde{\boldsymbol{c}}} = R(I_{\widetilde{\boldsymbol{c}}}). \end{cases}$$

Then $\lambda_{\boldsymbol{c}}, \lambda_{[\boldsymbol{c},m]}, \lambda_{[\boldsymbol{c},m,n]}$ are admissible if and only if

$$ind(I_{\boldsymbol{c}}) + n \cdot ind(I_{[\boldsymbol{c},m]}) + ind(I_{[\boldsymbol{c},m,n]}) \equiv \delta_R(\lambda_{\boldsymbol{c}}) + n \cdot \delta_R(\lambda_{[\boldsymbol{c},m]}) + \delta_R(\lambda_{[\boldsymbol{c},m,n]}) \mod 2.$$

Proof. Let $\widetilde{\mathbf{c}} \in \mathcal{C}$ be such that $\{-1, \infty\} \not\ni \varphi(\widetilde{\mathbf{c}}) = \frac{p_{\widetilde{\mathbf{c}}}}{q_{\widetilde{\mathbf{c}}}}$ with $p_{\widetilde{\mathbf{c}}}, q_{\widetilde{\mathbf{c}}}$ coprime. Let $I_{\widetilde{\mathbf{c}}}$ be a spectral band of $\sigma_{\widetilde{\mathbf{c}}}$ and $\lambda_{\widetilde{\mathbf{c}}}$ an edge (left or right) of $I_{\widetilde{\mathbf{c}}}$. In particular, by Lemma 5.4 (a) $\lambda_{\widetilde{\mathbf{c}}}$ is an eigenvalue in $H_{\widetilde{\mathbf{c}}}(\theta_{\widetilde{\mathbf{c}}})$ for some $\theta_{\widetilde{\mathbf{c}}} \in \{0, \pi\}$ and

$$\text{ind}(I_{\widetilde{\mathbf{c}}}) + \delta_R(\lambda_{\widetilde{\mathbf{c}}}) - q_{\widetilde{\mathbf{c}}} \equiv \frac{1}{\pi}\theta_{\widetilde{\mathbf{c}}} \mod 2. \tag{5.2}$$

We will apply (5.2) in the following for both $\widetilde{\mathbf{c}} = \mathbf{c}$ and $\widetilde{\mathbf{c}} = [\mathbf{c}, m, n]$. However, recall from the admissibility definition (Definition 4.3) that we need to consider $\lambda_{[\mathbf{c},m]}$ as an eigenvalue of the matrix $H_{[\mathbf{c},m]}^{\times n}(\theta_{[\mathbf{c},m]})$ (rather than the matrix $H_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})$). Therefore, we need to develop an alternative identity to (5.2). This is done with the aid of Lemma 5.5 (a) and (b) from which we conclude that

$$n \cdot \left(\text{ind}(I_{[\mathbf{c},m]}) + \delta_R(\lambda_{[\mathbf{c},m]}) - q_{[\mathbf{c},m]}\right) \equiv \frac{1}{\pi}\theta_{[\mathbf{c},m]} \mod 2, \tag{5.3}$$

for both even and odd values of $n \in \mathbb{N}$.

To conclude the proof we sum Equation (5.2) for $\widetilde{\mathbf{c}} = \mathbf{c}$ and for $\widetilde{\mathbf{c}} = [\mathbf{c}, m, n]$ and we add to it Equation (5.3). This yields

$$\begin{aligned} &(\text{ind}(I_{\mathbf{c}}) + \delta_R(\lambda_{\mathbf{c}})) + \left(\text{ind}(I_{[\mathbf{c},m,n]}) + \delta_R(\lambda_{[\mathbf{c},m,n]})\right) \\ &+ n\left(\text{ind}(I_{[\mathbf{c},m]}) + \delta_R(\lambda_{[\mathbf{c},m]})\right) - \left(q_{\mathbf{c}} + n\ q_{[\mathbf{c},m]} + q_{[\mathbf{c},m,n]}\right) \equiv \frac{1}{\pi}\left(\theta_{\mathbf{c}} + \theta_{[\mathbf{c},m]} + \theta_{[\mathbf{c},m,n]}\right) \mod 2. \end{aligned}$$

By definition, admissibility of $\lambda_{\mathbf{c}}, \lambda_{[\mathbf{c},m]}, \lambda_{[\mathbf{c},m,n]}$ is equivalent to admissibility of the values $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]} \in \{0, \pi\}$, which is equivalent to $\frac{1}{\pi}\left(\theta_{\mathbf{c}} + \theta_{[\mathbf{c},m]} + \theta_{[\mathbf{c},m,n]}\right) \equiv 0 \mod 2$. To end the proof, we just substitute above the equality $q_{\mathbf{c}} + n \cdot q_{[\mathbf{c},m]} = q_{[\mathbf{c},m,n]}$, which is standard in the theory of finite continued fraction expansions (see Lemma I.1, (b)). □

5.3. **Eigenvalue inequalities resulting from interlacing theorem.** Combining the interlacing theorem (Theorem 4.2) with Lemma 5.5 gives the following useful lemma, which is applied many times in the following subsections.

Lemma 5.7. *Let $V > 0$, $m, n \in \mathbb{N}$, $\boldsymbol{c} \in \mathcal{C}$ be such that $\varphi(\boldsymbol{c}) \notin \{-1, \infty\}$ and $[\boldsymbol{c}, m] \in \mathcal{C}$. Let $\theta_{\boldsymbol{c}}, \theta_{[\boldsymbol{c},m]}, \theta_{[\boldsymbol{c},m,n]} \in \{0, \pi\}$ and*

$$\lambda_{\mathbf{o}} \in \sigma\left(H_{[\boldsymbol{c},m],V}^{\times n}(\theta_{[\boldsymbol{c},m]}) \oplus H_{\boldsymbol{c},V}(\theta_{\boldsymbol{c}})\right) \quad \text{and} \quad \mu_{\mathbf{o}} \in \sigma\left(H_{[\boldsymbol{c},m,n],V}(\theta_{[\boldsymbol{c},m,n]})\right).$$

Define

$$N_{\boldsymbol{c}} := N\left(\lambda_{\mathbf{o}};\ H_{\boldsymbol{c},V}(\theta_{\boldsymbol{c}})\right), \quad N_{[\boldsymbol{c},m]} := N\left(\lambda_{\mathbf{o}};\ H_{[\boldsymbol{c},m],V}^{\times n}(\theta_{[\boldsymbol{c},m]})\right)$$

and

$$N_{[\boldsymbol{c},m,n]} := N\left(\mu_{\mathbf{o}};\ H_{[\boldsymbol{c},m,n],V}(\theta_{[\boldsymbol{c},m,n]})\right).$$

(a) Let $\mathcal{M}_{\lambda_{\mathbf{o}}}$ be the multiplicity of the eigenvalue $\lambda_{\mathbf{o}}$ of $H^{\times n}_{[\boldsymbol{c},m],V}(\theta_{[\boldsymbol{c},m]}) \oplus H_{\boldsymbol{c},V}(\theta_{\boldsymbol{c}})$. If $\theta_{\boldsymbol{c}}, \theta_{[\boldsymbol{c},m]}, \theta_{[\boldsymbol{c},m,n]}$ are admissible, then the following implications hold:

$$N_{\boldsymbol{c}} + N_{[\boldsymbol{c},m]} < N_{[\boldsymbol{c},m,n]} \quad \Rightarrow \quad \lambda_{\mathbf{o}} \leq \mu_{\mathbf{o}}, \tag{5.4}$$

$$N_{\boldsymbol{c}} + N_{[\boldsymbol{c},m]} + \mathcal{M}_{\lambda_{\mathbf{o}}} - 1 > N_{[\boldsymbol{c},m,n]} \quad \Rightarrow \quad \lambda_{\mathbf{o}} \geq \mu_{\mathbf{o}}. \tag{5.5}$$

If, additionally, $\lambda_{\mathbf{o}}$ is a simple eigenvalue of $H^{\times n}_{[\boldsymbol{c},m],V}(\theta_{[\boldsymbol{c},m]}) \oplus H_{\boldsymbol{c},V}(\theta_{\boldsymbol{c}})$ (i.e., $\mathcal{M}_{\lambda_{\mathbf{o}}} = 1$), then the two inequalities on the right hand sides of (5.4) and (5.5) are strict.

(b) If

- *$\theta_{\boldsymbol{c}}, \theta_{[\boldsymbol{c},m]}, \theta_{[\boldsymbol{c},m,n]}$ are not admissible and*
- *$I_{[\boldsymbol{c},m]}$ is a spectral band in $\sigma_{[\boldsymbol{c},m]}$ satisfying $\sigma\left(H_{\boldsymbol{c},V}(\theta_{\boldsymbol{c}})\right) \cap I_{[\boldsymbol{c},m]}(V) = \emptyset$,*

then the following implications hold:

$$\lambda_{\mathbf{o}} = L(I_{[\boldsymbol{c},m]}(V)), \quad N_{\boldsymbol{c}} + N_{[\boldsymbol{c},m]} < N_{[\boldsymbol{c},m,n]} \quad \Rightarrow \quad \lambda_{\mathbf{o}} < \mu_{\mathbf{o}} \tag{5.6}$$

and for $n \geq 2$,

$$\lambda_{\mathbf{o}} = R(I_{[\boldsymbol{c},m]}(V)), \quad N_{\boldsymbol{c}} + N_{[\boldsymbol{c},m]} - 1 > N_{[\boldsymbol{c},m,n]} \quad \Rightarrow \quad \lambda_{\mathbf{o}} > \mu_{\mathbf{o}}. \tag{5.7}$$

Remark. We emphasize that $\lambda_{\mathbf{o}}$ and $\mu_{\mathbf{o}}$ do depend on V, but the implications of the lemma do not.

Proof. We start by noting the following rather trivial counting relation

$$N\left(\lambda_{\mathbf{o}};\ H^{\times n}_{[\mathbf{c},m],V}(\theta_{[\mathbf{c},m]}) \oplus H_{\mathbf{c},V}(\theta_{\mathbf{c}})\right) = N_{\mathbf{c}} + N_{[\mathbf{c},m]}. \tag{5.8}$$

(a) Suppose that $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}$ are admissible. Both of the required implications (5.4) and (5.5) follow from Theorem 4.2, when keeping in mind the counting relation (5.8). Explicitly, denoting the eigenvalues of $H^{\times n}_{[\mathbf{c},m],V}(\theta_{[\mathbf{c},m]}) \oplus H_{\mathbf{c},V}(\theta_{\mathbf{c}})$ by $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$ in increasing order, we get that $\lambda_{\mathbf{o}} = \lambda_{N_{\mathbf{c}}+N_{[\mathbf{c},m]}} = \ldots = \lambda_{N_{\mathbf{c}}+N_{[\mathbf{c},m]}+M_{\lambda_{\mathbf{o}}}-1}$. Similarly $\mu_{\mathbf{o}} = \mu_{N_{[\mathbf{c},m,n]}}$, if the eigenvalues of $H_{[\mathbf{c},m,n],V}(\theta_{[\mathbf{c},m,n]})$ are denoted by $\mu_0 \leq \mu_1 \leq \mu_2 \leq \ldots$ in increasing order. Hence,

- (5.4) follows by applying Theorem 4.2 for $\lambda_{\mathbf{o}} = \lambda_{N_{\mathbf{c}}+N_{[\mathbf{c},m]}}$, $\mu_{\mathbf{o}} = \mu_{N_{[\mathbf{c},m,n]}}$, and
- (5.5) follows by applying Theorem 4.2 for $\lambda_{\mathbf{o}} = \lambda_{N_{\mathbf{c}}+N_{[\mathbf{c},m]}+\mathcal{M}_{\lambda_{\mathbf{o}}}-1}$, $\mu_{\mathbf{o}} = \mu_{N_{[\mathbf{c},m,n]}}$.

If, additionally, $\lambda_{\mathbf{o}}$ is a simple eigenvalue of $H^{\times n}_{[\mathbf{c},m],V}(\theta_{[\mathbf{c},m]}) \oplus H_{\mathbf{c},V}(\theta_{\mathbf{c}})$, then $\mathcal{M}_{\lambda_{\mathbf{o}}} = 1$ and the relevant statement within Theorem 4.2 yields the corresponding strict inequalities.

(b) Suppose that $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}$ are not admissible and let $I_{[\mathbf{c},m]}$ be a spectral band in $\sigma_{[\mathbf{c},m]}$ satisfying $\sigma\left(H_{\mathbf{c},V}(\theta_{\mathbf{c}})\right) \cap I_{[\mathbf{c},m]}(V) = \emptyset$.

In the first case (Equation (5.6)), we assume $\lambda_{\mathbf{o}} = L(I_{[\mathbf{c},m]}(V))$ and $N_{\mathbf{c}} + N_{[\mathbf{c},m]} < N_{[\mathbf{c},m,n]}$. We aim to apply Theorem 4.2 directly but $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}$ are not admissible. Thus, we change one of these values to attain an admissible triple. More precisely, $\theta_{\mathbf{c}}, \pi - \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}$ are admissible, see Remark 4.4. By Lemma 5.5 (c), there exists a $\lambda \in H^{\times n}_{[\mathbf{c},m],V}(\pi - \theta_{[\mathbf{c},m]})$ such that

$$\begin{aligned} \lambda_{\mathbf{o}} = L(I_{[\mathbf{c},m]}(V)) < \lambda \leq R(I_{[\mathbf{c},m]}(V)) \\ \text{and} \quad N\left(\lambda; H^{\times n}_{[\mathbf{c},m],V}(\pi - \theta_{[\mathbf{c},m]})\right) = N\left(\lambda_{\mathbf{o}}; H^{\times n}_{[\mathbf{c},m],V}(\theta_{[\mathbf{c},m]})\right). \end{aligned}$$

Thus, $\lambda \in I_{[\mathbf{c},m]}(V)$ and $\sigma\left(H_{\mathbf{c},V}(\theta_{\mathbf{c}})\right) \cap I_{[\mathbf{c},m]}(V) = \emptyset$ lead to the equation $N\left(\lambda;\ H_{\mathbf{c},V}(\theta_{\mathbf{c}})\right) = N\left(\lambda_{\mathbf{o}};\ H_{\mathbf{c},V}(\theta_{\mathbf{c}})\right)$. Therefore, (5.8) implies

$$N\left(\lambda;\ H^{\times n}_{[\mathbf{c},m],V}(\pi - \theta_{[\mathbf{c},m]}) \oplus H_{\mathbf{c},V}(\theta_{\mathbf{c}})\right) = N_{\mathbf{c}} + N_{[\mathbf{c},m]}.$$

Since $N_{\mathbf{c}} + N_{[\mathbf{c},m]} < N_{[\mathbf{c},m,n]}$ and $\theta_{\mathbf{c}}, \pi - \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}$ are admissible, Theorem 4.2 yields $\lambda \le \mu_{\mathbf{o}}$. Using $\lambda_{\mathbf{o}} < \lambda$, we conclude $\lambda_{\mathbf{o}} < \mu_{\mathbf{o}}$, as claimed.

The second case (Equation (5.7)) follows similar arguments, using Lemma 5.5 (d). □

5.4. **Index identities of the spectral bands.**

In order to apply Lemma 5.7, we need to be able to compare the spectral positions of $\lambda_{\mathbf{o}}$ and $\mu_{\mathbf{o}}$ ($N_{\mathbf{c}}$, $N_{[\mathbf{c},m]}$ and $N_{[\mathbf{c},m,n]}$) which appear in Lemma 5.7. We have already seen (Lemma 5.5) that such spectral positions are connected to spectral band indices. Hence, towards applying Lemma 5.7, we develop in this subsection some connections between indices of spectral bands (Lemma 5.9). For the upcoming statements and proofs, we introduce the following notations (see Figure 5.2 for a sketch).

Definition 5.8. Let $m \in \mathbb{N}_0$ and $\mathbf{c}, [\mathbf{c}, m] \in \mathcal{C}$ be such that $\varphi(\mathbf{c}) \notin \{-1, \infty\}$ and $\varphi([\mathbf{c}, m]) \notin \{-1, \infty\}$. For a spectral band $I_{\mathbf{c}}$ in $\sigma_{\mathbf{c}}$, define the *associated spectral bands* $J_{[\mathbf{c},m]}$, $K_{[\mathbf{c},m]}$ in $\sigma_{[\mathbf{c},m]}$ to be the unique spectral bands (if they exist) such that for $V > 4$,

- $J_{[\mathbf{c},m]}(V)$ is the right-most band of $\sigma_{[\mathbf{c},m]}(V)$ for which $J_{[\mathbf{c},m]}(V) \prec I_{\mathbf{c}}(V)$, and
- $K_{[\mathbf{c},m]}(V)$ is the left-most band of $\sigma_{[\mathbf{c},m]}(V)$ for which $I_{\mathbf{c}}(V) \prec K_{[\mathbf{c},m]}(V)$.

Remark. Note that it might be that some of the bands $J_{[\mathbf{c},m]}$ and $K_{[\mathbf{c},m]}$ do not exist. In such a case, this in an empty convention. Further note that $\varphi([\mathbf{c}, m]) \in \{-1, \infty\}$ for $m \in \mathbb{N}_0$ can only happen if $\mathbf{c} = [0, 0]$ and $m = 0$ in which case such spectral bands $J_{[\mathbf{c},m]}$ and $K_{[\mathbf{c},m]}$ do not exist.

The reason for including $V > 4$ in the definition above is explained in the beginning of the proof of Lemma 5.9.

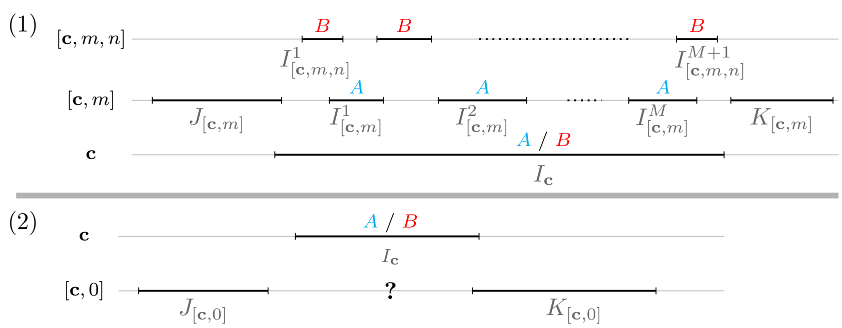

FIGURE 5.2. A sketch for Definition 5.8 and Lemma 5.9. (1) For a fixed spectral band $I_{\mathbf{c}}$ and $m \ge 1$, the associated bands $J_{[\mathbf{c},m]}$ and $K_{[\mathbf{c},m]}$ (from Definition 5.8) and the associated bands $\{I^i_{[\mathbf{c},m]}\}_{i=1}^M$ and $\{I^j_{[\mathbf{c},m,n]}\}_{j=1}^{M+1}$ (from Definition 2.12) are drawn. (2) For a fixed spectral band $I_{\mathbf{c}}$ and $m = 0$, the associated bands $J_{[\mathbf{c},0]}$ and $K_{[\mathbf{c},0]}$ (from Definition 5.8) are drawn. If $I_{\mathbf{c}}$ is of backward type A for $V > 4$, there is a spectral band between $J_{[\mathbf{c},0]}$ and $K_{[\mathbf{c},0]}$. Otherwise, there is no spectral band between them, namely $\mathrm{ind}(K_{[\mathbf{c},0]}) = \mathrm{ind}(J_{[\mathbf{c},0]}) + 1$ (indicated by the question mark).

Lemma 5.9. *Let $m, n \in \mathbb{N}$, and $\mathbf{c} \in \mathcal{C}$ be such that $\varphi(\mathbf{c}) \notin \{-1, \infty\}$ and $[\mathbf{c}, m] \in \mathcal{C}$. Consider a spectral band $I_{\mathbf{c}}$ in $\sigma_{\mathbf{c}}$ with associated spectral bands $\{I^i_{[\mathbf{c},m]}\}_{i=1}^M$ and $\{I^j_{[\mathbf{c},m,n]}\}_{j=1}^{M+1}$ introduced in Definition 2.12. If $M \ge 1$, then for all $1 \le i \le M$*

$$ind(I^i_{[\mathbf{c},m,n]}) = n \cdot ind(I^i_{[\mathbf{c},m]}) + ind(I_{\mathbf{c}}), \tag{5.9}$$

and

$$ind(I^{i+1}_{[\mathbf{c},m,n]}) = n \cdot (ind(I^i_{[\mathbf{c},m]}) + 1) + ind(I_{\mathbf{c}}). \tag{5.10}$$

Whenever the spectral bands $J_{[\mathbf{c},m]}$ *or* $K_{[\mathbf{c},m]}$ *associated with* $I_{\mathbf{c}}$ *exist, then the following hold. If* $M \geq 0$*, then*

$$ind(I^1_{[\mathbf{c},m,n]}) = n \cdot (ind(J_{[\mathbf{c},m]}) + 1) + ind(I_{\mathbf{c}}) \tag{5.11}$$

and

$$ind\left(I^{M+1}_{[\mathbf{c},m,n]}\right) = n \cdot ind(K_{[\mathbf{c},m]}) + ind(I_{\mathbf{c}}) \tag{5.12}$$

If $I_{\mathbf{c}}(V)$ *is of type* B *for* $V > 4$*, then*

$$ind(I^1_{[\mathbf{c},1]}) = ind(I_{\mathbf{c}}) + ind(J_{[\mathbf{c},0]}) + 1 = ind(I_{\mathbf{c}}) + ind(K_{[\mathbf{c},0]}). \tag{5.13}$$

Proof. We start by noting that the index of a spectral band is independent of $V > 0$ (Remark 5.2) allowing us to restrict to the case $V > 4$ where all spectral bands are either of type A or of type B by Theorem 2.11. Therefore, within this proof we allow ourselves to assume $V > 4$ and abuse notation, writing just I (meaning interval and not a map) instead of writing $I(V)$ for some $V > 4$. Namely, when writing within this proof sentences such as "I is a spectral band of type A (or B) and belongs to $\sigma_{\mathbf{c}}$", we actually mean that for some value of $V > 4$, $I(V)$ is of type A (or B) and belongs to $\sigma_{\mathbf{c}}(V)$.

We introduce the following extra notations for the band indices:

$$\text{ind}_A(I_{\mathbf{c}}) := \left|\{I \text{ is of type } A \text{ and it belongs to } \sigma_{\mathbf{c}} \,:\, I \prec I_{\mathbf{c}}\}\right|,$$
$$\text{ind}_B(I_{\mathbf{c}}) := \left|\{I \text{ is of type } B \text{ and it belongs to } \sigma_{\mathbf{c}} \,:\, I \prec I_{\mathbf{c}}\}\right|.$$

Clearly, $\text{ind}(I_{\mathbf{c}}) = \text{ind}_A(I_{\mathbf{c}}) + \text{ind}_B(I_{\mathbf{c}})$ for all $I_{\mathbf{c}}$, see Definition 5.1.

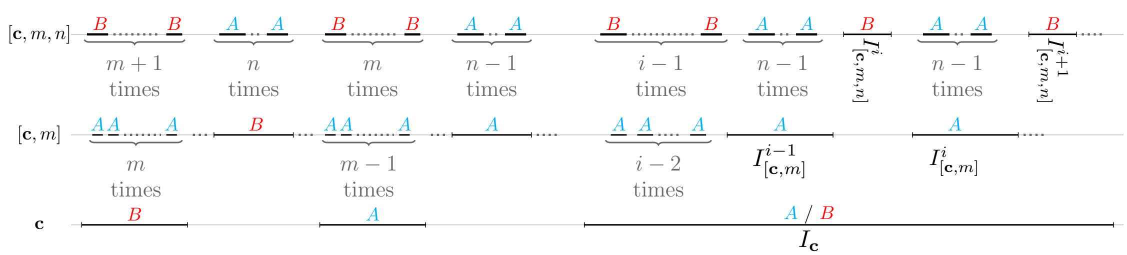

FIGURE 5.3. A sketch for the proof of (5.9) and (5.10) in Lemma 5.9.

We first assume that $M \geq 1$. Start by examining $I^i_{[\mathbf{c},m,n]}$ and evaluating $\text{ind}_B(I^i_{[\mathbf{c},m,n]})$ and $\text{ind}_A(I^i_{[\mathbf{c},m,n]})$. The spectral band $I^i_{[\mathbf{c},m,n]}$ is of type B and belongs to $\sigma_{[\mathbf{c},m,n]}$. We know that $I^i_{[\mathbf{c},m,n]}$ is included in $I_{\mathbf{c}}$ of $\sigma_{\mathbf{c}}$. There are additional $i-1$ spectral bands of type B in $\sigma_{[\mathbf{c},m,n]}$, which are to the left of $I^i_{[\mathbf{c},m,n]}$ and included in $I_{\mathbf{c}}$. All other spectral bands of type B to the left of $I^i_{[\mathbf{c},m,n]}$ come in groups of either m or $m+1$ and each such group is included in some spectral band I in $\sigma_{\mathbf{c}}$ that is to the left of $I_{\mathbf{c}}$, see Figure 5.3. The group is of size m if I is of type A and it is of size $m+1$ if I is of type B. This discussion may be summarized in the following identity

$$\text{ind}_B(I^i_{[\mathbf{c},m,n]}) = m \cdot \text{ind}_A(I_{\mathbf{c}}) + (m+1) \cdot \text{ind}_B(I_{\mathbf{c}}) + i - 1. \tag{5.14}$$

We now evaluate $\text{ind}_A(I^i_{[\mathbf{c},m,n]})$. We note that all the spectral bands of type A to the left of $I^i_{[\mathbf{c},m,n]}$ come in groups of either $n-1$ or n and each such group is included in some spectral band I in $\sigma_{[\mathbf{c},m]}$ that is to the left of $I^i_{[\mathbf{c},m]}$, see Figure 5.3. The group is of size $n-1$ if I is of type A and it is of size n if I is of type B. This discussion may be summarized in the following identity

$$\text{ind}_A(I^i_{[\mathbf{c},m,n]}) = (n-1) \cdot \text{ind}_A(I^i_{[\mathbf{c},m]}) + n \cdot \text{ind}_B(I^i_{[\mathbf{c},m]}). \tag{5.15}$$

We now evaluate $\mathrm{ind}_A(I^i_{[\mathbf{c},m]})$. We note that there are $i-1$ spectral bands of type A to the left of $I^i_{[\mathbf{c},m]}$ which are included in $I_{\mathbf{c}}$. Every other spectral band in $\sigma_{[\mathbf{c},m]}$ of type A to the left of $I^i_{[\mathbf{c},m]}$ is included in a spectral band of $\sigma_{\mathbf{c}}$ to the left of $I_{\mathbf{c}}$. Specifically, they come in groups of either $m-1$ or m and each group is included in a spectral band I in $\sigma_{\mathbf{c}}$ to the left of $I_{\mathbf{c}}$. The group is of size $m-1$ if I is of type A and it is of size m if I is of type B, see Figure 5.3. This discussion may be summarized in the following identity

$$\mathrm{ind}_A(I^i_{[\mathbf{c},m]}) = (m-1)\cdot \mathrm{ind}_A(I_{\mathbf{c}}) + m \cdot \mathrm{ind}_B(I_{\mathbf{c}}) + i - 1. \tag{5.16}$$

Combining the Equations (5.14) and (5.15) together with the identity $\mathrm{ind}(I) = \mathrm{ind}_A(I) + \mathrm{ind}_B(I)$, which holds for all I, gives

$$\begin{aligned}\mathrm{ind}(I^i_{[\mathbf{c},m,n]}) &= \mathrm{ind}_A(I^i_{[\mathbf{c},m,n]}) + \mathrm{ind}_B(I^i_{[\mathbf{c},m,n]})\\ &= \Big(n\cdot \mathrm{ind}(I^i_{[\mathbf{c},m]}) - \mathrm{ind}_A(I^i_{[\mathbf{c},m]})\Big) + (m\cdot \mathrm{ind}_A(I_{\mathbf{c}}) + (m+1)\cdot \mathrm{ind}_B(I_{\mathbf{c}}) + i - 1)\\ &= n\cdot \mathrm{ind}(I^i_{[\mathbf{c},m]}) + ind(I_{\mathbf{c}}),\end{aligned}$$

where in the last line we used (5.16). This proves Equation (5.9).

To prove (5.10), we observe that between $I^i_{[\mathbf{c},m,n]}$ and $I^{i+1}_{[\mathbf{c},m,n]}$ there are $n-1$ spectral bands of type A (the bands which are contained in $I^i_{[\mathbf{c},m]}$) and no spectral bands of type B, see Figure 5.3. We therefore get

$$\mathrm{ind}(I^{i+1}_{[\mathbf{c},m,n]}) = \Big(\mathrm{ind}(I^i_{[\mathbf{c},m,n]}) + 1\Big) + (n-1) = n\cdot(\mathrm{ind}(I^i_{[\mathbf{c},m]}) + 1) + ind(I_{\mathbf{c}}),$$

which proves Equation (5.10).

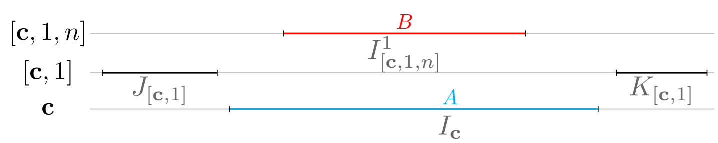

FIGURE 5.4. A sketch for the proof of (5.11) in Lemma 5.9. We have $\mathrm{ind}(K_{[\mathbf{c},m]}) = \mathrm{ind}(J_{[\mathbf{c},m]}) + 1$.

For $M \geq 1$, Equation (5.11) follows from Equation (5.9) for $i = 1$ and $\mathrm{ind}(I^1_{[\mathbf{c},m]}) = \mathrm{ind}(J_{[\mathbf{c},m]}) + 1$. Similarly, Equation (5.12) follows for $M \geq 1$ from Equation (5.10) for $i = M$ and $\mathrm{ind}(K_{[\mathbf{c},m]}) = \mathrm{ind}(I^M_{[\mathbf{c},m]}) + 1$.

For $M = 0$, (5.11) and (5.12) follow similar arguments as (5.9) and (5.10) using $\mathrm{ind}(K_{[\mathbf{c},m]}) = \mathrm{ind}(J_{[\mathbf{c},m]}) + 1$ if $M = 0$.

To prove (5.13) for the index of $I^1_{[\mathbf{c},1]}$ we note the following. There is a bijection between bands of type A in $\sigma_{[\mathbf{c},1]}$ and bands of type B in $\sigma_{\mathbf{c}}$: a spectral band $I_{\mathbf{c}}$ in $\sigma_{\mathbf{c}}$ of type A does not contain any spectral band in $\sigma_{[\mathbf{c},1]}$ but if $I_{\mathbf{c}}$ in $\sigma_{\mathbf{c}}$ is of type B, then it contains exactly (using uniqueness of these bands for $V > 4$, see Theorem 2.11) one band in $\sigma_{[\mathbf{c},1]}$ of type A. Thus,

$$\mathrm{ind}_A(I^1_{[\mathbf{c},1]}) = \mathrm{ind}_B(I_{\mathbf{c}})$$

follows. We denote by c_k the last digit in $\mathbf{c}$, namely, $\mathbf{c} := [0, 0, c_1, \ldots, c_k]$. Similar counting arguments as for (5.14) lead to

$$\mathrm{ind}_B(I^1_{[\mathbf{c},1]}) = c_k \cdot \mathrm{ind}_A(J_{[\mathbf{c},0]}) + (c_k+1)\cdot \mathrm{ind}_B(J_{[\mathbf{c},0]}) + c_k + \begin{cases} 0 & J_{[\mathbf{c},0]} \text{ is of type } A,\\ 1 & J_{[\mathbf{c},0]} \text{ is of type } B,\end{cases} \tag{5.17}$$

Moreover, similar counting arguments as in (5.16) imply

$$\begin{aligned}\operatorname{ind}_A(I_{\mathbf{c}}) &= (c_k-1)\cdot\operatorname{ind}_A(J_{[\mathbf{c},0]}) + c_k\cdot\operatorname{ind}_B(J_{[\mathbf{c},0]}) + c_k - 1 + \begin{cases}0 & J_{[\mathbf{c},0]} \text{ is of type } A,\\ 1 & J_{[\mathbf{c},0]} \text{ is of type } B,\end{cases}\\ &= \operatorname{ind}_B(I^1_{[\mathbf{c},1]}) - \operatorname{ind}(J_{[\mathbf{c},0]}) - 1,\end{aligned}$$

where in the last line we used (5.17). Hence, we arrive at

$$\begin{aligned}\operatorname{ind}(I^1_{[\mathbf{c},1]}) = \operatorname{ind}_B(I^1_{[\mathbf{c},1]}) + \operatorname{ind}_A(I^1_{[\mathbf{c},1]}) &= \big(\operatorname{ind}_A(I_{\mathbf{c}}) + \operatorname{ind}(J_{[\mathbf{c},0]}) + 1\big) + \operatorname{ind}_B(I_{\mathbf{c}})\\ &= \operatorname{ind}(I_{\mathbf{c}}) + \operatorname{ind}(J_{[\mathbf{c},0]}) + 1\\ &= \operatorname{ind}(I_{\mathbf{c}}) + \operatorname{ind}(K_{[\mathbf{c},0]})\end{aligned}$$

using $\operatorname{ind}(K_{[\mathbf{c},0]}) = \operatorname{ind}(J_{[\mathbf{c},0]}) + 1$, which holds by definition. Thus, (5.13) is proven. □

5.5. **Sufficient conditions for the forward type properties.** In this subsection, we provide various lemmata and corollaries which allow us to prove the various properties (A1), (A2), (B1), (B2), and (I) of the forward type as in Definition 2.9. These will be used in the next section, where we prove that backward type implies forward type (Proposition 3.1).

We start with proving that the interlacing property (I) holds under some conditions.

Lemma 5.10. *Let $V_1 > 0$, $m, n \in \mathbb{N}$, $\boldsymbol{c} \in \mathcal{C}$ be such that $\varphi(\boldsymbol{c}) \notin \{-1, \infty\}$ and $[\boldsymbol{c}, m] \in \mathcal{C}$. Consider a spectral band $I_{\boldsymbol{c}}$ in $\sigma_{\boldsymbol{c}}$ with the associated spectral bands $\{I^i_{[\boldsymbol{c},m]}\}_{i=1}^M$ and $\{I^j_{[\boldsymbol{c},m,n]}\}_{j=1}^{M+1}$ introduced in Definition 2.12. If $1 \le i \le M$ and $I^i_{[\boldsymbol{c},m]}(V_1) \subseteq_{\mathrm{str}} I_{\boldsymbol{c}}(V_1)$, then*

$$I^i_{[\boldsymbol{c},m,n]}(V_1) \prec I^i_{[\boldsymbol{c},m]}(V_1) \prec I^{i+1}_{[\boldsymbol{c},m,n]}(V_1).$$

Proof. Let $1 \le i \le M$ and $V_1 > 0$. We need to show the following inequalities

(a) $L\left(I^i_{[\mathbf{c},m,n]}(V_1)\right) < L\left(I^i_{[\mathbf{c},m]}(V_1)\right)$,
(b) $R\left(I^i_{[\mathbf{c},m]}(V_1)\right) < R\left(I^{i+1}_{[\mathbf{c},m,n]}(V_1)\right)$,
(c) $R\left(I^i_{[\mathbf{c},m,n]}(V_1)\right) < R\left(I^i_{[\mathbf{c},m]}(V_1)\right)$,
(d) $L\left(I^i_{[\mathbf{c},m]}(V_1)\right) < L\left(I^{i+1}_{[\mathbf{c},m,n]}(V_1)\right)$.

We proceed proving the inequalities above one at a time via an appropriate application of Lemma 5.7. Although the inequalities above and the assumption $I^i_{[\mathbf{c},m]}(V_1) \subseteq_{\mathrm{str}} I_{\mathbf{c}}(V_1)$ depend on the fixed $V_1 > 0$, we will abbreviate notation, for the sake of easier reading, and omit the V_1 dependence in most parts of this proof.

(a) We aim to apply Lemma 5.7 for $\lambda_{\mathbf{o}} = L(I^i_{[\mathbf{c},m]})$, $\mu_{\mathbf{o}} = L(I^i_{[\mathbf{c},m,n]})$. Let $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]} \in \{0,\pi\}$ be such that

$$L(I_{\mathbf{c}}) \in \sigma\left(H_{\mathbf{c}}(\theta_{\mathbf{c}})\right), \quad L(I^i_{[\mathbf{c},m]}) \in \sigma\left(H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) \quad \text{and} \quad L(I^i_{[\mathbf{c},m,n]}) \in \sigma\left(H_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m,n]})\right).$$

These spectral edges, respectively $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}$, are admissible, as can be verified by using the index relation (5.9) of Lemma 5.9 in the characterization of admissibility from Lemma 5.6. Furthermore, Lemma 5.5 (c) applied to $[\mathbf{c},m] \in \mathcal{C}$ implies

$$N_{[\mathbf{c},m]} := N\left(L(I^i_{[\mathbf{c},m]});\ H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) = n \cdot \operatorname{ind}(I^i_{[\mathbf{c},m]}).$$

Apply Lemma 5.4 (b) to $[\mathbf{c},m,n] \in \mathcal{C}$ and the index relation (5.9) of Lemma 5.9 to conclude

$$N_{[\mathbf{c},m,n]} := N\left(L(I^i_{[\mathbf{c},m,n]});\ H_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m,n]})\right) = \operatorname{ind}(I^i_{[\mathbf{c},m,n]}) = \operatorname{ind}(I_{\mathbf{c}}) + n\cdot\operatorname{ind}(I^i_{[\mathbf{c},m]}).$$

Since $I^i_{[\mathbf{c},m]}(V_1) \subseteq_{\text{str}} I_{\mathbf{c}}(V_1)$, we infer $L(I_{\mathbf{c}}(V_1)) < L(I^i_{[\mathbf{c},m]}(V_1))$ and $\sigma\left(H_{\mathbf{c},V_1}(\theta_{\mathbf{c}})\right) \cap I^i_{[\mathbf{c},m]}(V_1) = \emptyset$. Hence, $L(I_{\mathbf{c}}) \in \sigma\left(H_{\mathbf{c}}(\theta_{\mathbf{c}})\right)$ and Lemma 5.4 (b) applied to $\mathbf{c} \in \mathcal{C}$ lead to

$$N_{\mathbf{c}} := N\left(L(I^i_{[\mathbf{c},m]});\ H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) = N(L(I_{\mathbf{c}});\ H_{\mathbf{c}}(\theta_{\mathbf{c}})) + 1 = \operatorname{ind}(I_{\mathbf{c}}) + 1.$$

Summing up, we obtained $N_{\mathbf{c}} + N_{[\mathbf{c},m]} > N_{[\mathbf{c},m,n]}$. Moreover, $\sigma\left(H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) \cap I^i_{[\mathbf{c},m]} = \emptyset$ and Lemma 5.5 (e) that $\lambda_{\mathbf{o}} = L(I^i_{[\mathbf{c},m]})$ is a simple eigenvalue of $H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]}) \oplus H_{\mathbf{c}}(\theta_{\mathbf{c}})$. Using admissibility, Lemma 5.7 (a) yields the required inequality $\lambda_{\mathbf{o}} = L(I^i_{[\mathbf{c},m]}) > L(I^i_{[\mathbf{c},m,n]}) = \mu_{\mathbf{o}}$.

(b) We aim to apply Lemma 5.7 for $\lambda_{\mathbf{o}} = R(I^i_{[\mathbf{c},m]})$, $\mu_{\mathbf{o}} = R(I^{i+1}_{[\mathbf{c},m,n]})$. Let $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]} \in \{0,\pi\}$ be such that

$$R(I_{\mathbf{c}}) \in \sigma\left(H_{\mathbf{c}}(\theta_{\mathbf{c}})\right),\ R(I^i_{[\mathbf{c},m]}) \in \sigma\left(H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) \text{ and } R(I^{i+1}_{[\mathbf{c},m,n]}) \in \sigma\left(H_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m,n]})\right).$$

Then these spectral edges, respectively $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}$, are admissible by inserting the index relation (5.10) of Lemma 5.9 into the characterization of admissibility from Lemma 5.6. Furthermore, Lemma 5.5 (d) applied to $[\mathbf{c},m] \in \mathcal{C}$ implies

$$N_{[\mathbf{c},m]} := N\left(R(I^i_{[\mathbf{c},m]});\ H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) = n \cdot \left(\operatorname{ind}(I^i_{[\mathbf{c},m]}) + 1\right) - 1.$$

Apply Lemma 5.4 (c) to $[\mathbf{c},m,n] \in \mathcal{C}$ and the index relation (5.10) of Lemma 5.9 to conclude

$$\begin{aligned} N_{[\mathbf{c},m,n]} := N\left(R(I^{i+1}_{[\mathbf{c},m,n]});\ H_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m,n]})\right) &= \operatorname{ind}(I^{i+1}_{[\mathbf{c},m,n]}) \\ &= n \cdot \left(\operatorname{ind}(I^i_{[\mathbf{c},m]}) + 1\right) + \operatorname{ind}(I_{\mathbf{c}}). \end{aligned}$$

Since $I^i_{[\mathbf{c},m]}(V_1) \subseteq_{\text{str}} I_{\mathbf{c}}(V_1)$, we infer $R(I^i_{[\mathbf{c},m]}(V_1)) < R(I_{\mathbf{c}}(V_1))$ and $\sigma\left(H_{\mathbf{c},V_1}(\theta_{\mathbf{c}})\right) \cap I^i_{[\mathbf{c},m]}(V_1) = \emptyset$. Hence, Lemma 5.4 (c) applied to $\mathbf{c} \in \mathcal{C}$ leads to

$$N_{\mathbf{c}} := N\left(R(I^i_{[\mathbf{c},m]});\ H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) = N(R(I_{\mathbf{c}});\ H_{\mathbf{c}}(\theta_{\mathbf{c}})) = \operatorname{ind}(I_{\mathbf{c}}).$$

Summing up, we obtained $N_{\mathbf{c}} + N_{[\mathbf{c},m]} < N_{[\mathbf{c},m,n]}$. Moreover, $\sigma\left(H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) \cap I^i_{[\mathbf{c},m]} = \emptyset$ and Lemma 5.5 (e) that $\lambda_{\mathbf{o}} = R(I^i_{[\mathbf{c},m]})$ is a simple eigenvalue of $H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]}) \oplus H_{\mathbf{c}}(\theta_{\mathbf{c}})$. Using admissibility, Lemma 5.7 (a) yields the required inequality $\lambda_{\mathbf{o}} = R(I^i_{[\mathbf{c},m]}) < R(I^{i+1}_{[\mathbf{c},m,n]}) = \mu_{\mathbf{o}}$.

(c) We aim to apply Lemma 5.7 for $\lambda_{\mathbf{o}} = R(I^i_{[\mathbf{c},m]})$, $\mu_{\mathbf{o}} = R(I^i_{[\mathbf{c},m,n]})$. Let $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]} \in \{0,\pi\}$ be such that

$$L(I_{\mathbf{c}}) \in \sigma\left(H_{\mathbf{c}}(\theta_{\mathbf{c}})\right),\ R(I^i_{[\mathbf{c},m]}) \in \sigma\left(H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) \text{ and } R(I^{i}_{[\mathbf{c},m,n]}) \in \sigma\left(H_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m,n]})\right).$$

Lemma 5.5 (d) applied to $[\mathbf{c},m] \in \mathcal{C}$ implies

$$N_{[\mathbf{c},m]} := N\left(R(I^i_{[\mathbf{c},m]});\ H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) = n \cdot \left(\operatorname{ind}(I^i_{[\mathbf{c},m]}) + 1\right) - 1.$$

Apply Lemma 5.4 (c) to $[\mathbf{c},m,n] \in \mathcal{C}$ and the index relation (5.9) of Lemma 5.9 to conclude

$$\begin{aligned} N_{[\mathbf{c},m,n]} := N\left(R(I^{i}_{[\mathbf{c},m,n]});\ H_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m,n]})\right) &= \operatorname{ind}(I^{i}_{[\mathbf{c},m,n]}) \\ &= n \cdot \operatorname{ind}(I^i_{[c,m]}) + \operatorname{ind}(I_c). \end{aligned}$$

Since $I^i_{[\mathbf{c},m]}(V_1) \subseteq_{\text{str}} I_{\mathbf{c}}(V_1)$, we infer $L(I_{\mathbf{c}}(V_1)) < R(I^i_{[\mathbf{c},m]}(V_1)) < R(I_{\mathbf{c}}(V_1))$ and $\sigma\left(H_{\mathbf{c},V_1}(\theta_{\mathbf{c}})\right) \cap I^i_{[\mathbf{c},m]}(V_1) = \emptyset$. Hence, $L(I_{\mathbf{c}}) \in \sigma\left(H_{\mathbf{c}}(\theta_{\mathbf{c}})\right)$ and Lemma 5.4 (b) applied to $\mathbf{c} \in \mathcal{C}$ lead to

$$N_{\mathbf{c}} := N\left(R(I^i_{[\mathbf{c},m]});\ H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) = N\left(L(I_{\mathbf{c}});\ H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) + 1 = \operatorname{ind}(I_{\mathbf{c}}) + 1.$$

Thus, $N_{\mathbf{c}} + N_{[\mathbf{c},m]} = N_{[\mathbf{c},m,n]} + n > N_{[\mathbf{c},m,n]}$ follows. Moreover, $\sigma\left(H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) \cap I^i_{[\mathbf{c},m]} = \emptyset$ and Lemma 5.5 (e) that $\lambda_{\mathbf{o}} = R(I^i_{[\mathbf{c},m]})$ is a simple eigenvalue of $H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]}) \oplus H_{\mathbf{c}}(\theta_{\mathbf{c}})$. Observe that $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}$ are admissible, if and only if n is odd by inserting the index relation (5.9) of Lemma 5.9 into the characterization of admissibility from Lemma 5.6. Thus, if n is odd, the previous considerations with Lemma 5.7 (a) yield $\lambda_{\mathbf{o}} = R(I^i_{[\mathbf{c},m]}(V)) > R(I^i_{[\mathbf{c},m,n]}) = \mu_{\mathbf{o}}$.

If n is even, then $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}$ are not admissible. Moreover, $N_{\mathbf{c}} + N_{[\mathbf{c},m]} - 1 > N_{[\mathbf{c},m,n]}$ follows since $n \geq 2$ if n is even. Thus, Lemma 5.7 (b) with $\sigma\left(H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) \cap I^i_{[\mathbf{c},m]} = \emptyset$implies $\lambda_{\mathbf{o}} = R(I^i_{[\mathbf{c},m]}) > R(I^i_{[\mathbf{c},m,n]}) = \mu_{\mathbf{o}}$.

(d) We aim to apply Lemma 5.7 for $\lambda_{\mathbf{o}} = L(I^i_{[\mathbf{c},m]})$, $\mu_{\mathbf{o}} = L(I^{i+1}_{[\mathbf{c},m,n]})$. Let $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]} \in \{0, \pi\}$ be such that

$$R(I_{\mathbf{c}}) \in \sigma\left(H_{\mathbf{c}}(\theta_{\mathbf{c}})\right), \quad L(I^i_{[\mathbf{c},m]}) \in \sigma\left(H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) \quad \text{and} \quad L(I^{i+1}_{[\mathbf{c},m,n]}) \in \sigma\left(H_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m,n]})\right).$$

Lemma 5.5 (c) applied to $[\mathbf{c}, m] \in \mathcal{C}$ implies

$$N_{[\mathbf{c},m]} := N\left(L(I^i_{[\mathbf{c},m]});\ H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) = n \cdot \operatorname{ind}(I^i_{[\mathbf{c},m]}).$$

Apply Lemma 5.4 (b) to $[\mathbf{c}, m, n] \in \mathcal{C}$ and the index relation (5.10) of Lemma 5.9 to conclude

$$N_{[\mathbf{c},m,n]} := N\left(L(I^{i+1}_{[\mathbf{c},m,n]});\ H_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m,n]})\right) = \operatorname{ind}(I^{i+1}_{[\mathbf{c},m,n]}) = n \cdot \left(\operatorname{ind}(I^i_{[c,m]}) + 1\right) + \operatorname{ind}(I_c).$$

Since $I^i_{[\mathbf{c},m]}(V_1) \subseteq_{\text{str}} I_{\mathbf{c}}(V_1)$, we infer $L(I_{\mathbf{c}}(V_1)) < L(I^i_{[\mathbf{c},m]}(V_1)) < R(I_{\mathbf{c}}(V_1))$ and $\sigma\left(H_{\mathbf{c},V_1}(\theta_{\mathbf{c}})\right) \cap I^i_{[\mathbf{c},m]}(V_1) = \emptyset$. Hence, Lemma 5.4 (c) applied to $\mathbf{c} \in \mathcal{C}$ leads to

$$N_{\mathbf{c}} := N\left(L(I^i_{[\mathbf{c},m]});\ H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) = N(R(I_{\mathbf{c}});\ H_{\mathbf{c}}(\theta_{\mathbf{c}})) = \operatorname{ind}(I_{\mathbf{c}}).$$

Thus, $N_{\mathbf{c}} + N_{[\mathbf{c},m]} < N_{[\mathbf{c},m,n]}$ follows. Moreover, $\sigma\left(H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) \cap I^i_{[\mathbf{c},m]} = \emptyset$ and Lemma 5.5 (e) that $\lambda_{\mathbf{o}} = L(I^i_{[\mathbf{c},m]})$ is a simple eigenvalue of $H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]}) \oplus H_{\mathbf{c}}(\theta_{\mathbf{c}})$. Observe that $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}$ are admissible, if and only if $n \in \mathbb{N}$ is odd by inserting the index relation (5.10) of Lemma 5.9 into the characterization of admissibility from Lemma 5.6. Thus, if $n \in \mathbb{N}$ is odd, the previous considerations with Lemma 5.7 (b) yield $\lambda_{\mathbf{o}} = L(I^i_{[\mathbf{c},m]}) < L(I^{i+1}_{[\mathbf{c},m,n]}) = \mu_{\mathbf{o}}$ as claimed.

If $n \in \mathbb{N}$ is even, then $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}$ are not admissible. Thus, Lemma 5.7 (b) with $\sigma\left(H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) \cap I^i_{[\mathbf{c},m]} = \emptyset$ and $N_{\mathbf{c}} + N_{[\mathbf{c},m]} < N_{[\mathbf{c},m,n]}$ imply $\lambda_{\mathbf{o}} = L(I^i_{[\mathbf{c},m]}) < L(I^{i+1}_{[\mathbf{c},m,n]}) = \mu_{\mathbf{o}}$. $\square$

The next lemma is tailored towards proving property (B2).

Lemma 5.11. *Let $V_1 > 0$, $m, n \in \mathbb{N}$ and $\boldsymbol{c} \in \mathcal{C}$ be such that $\varphi(\boldsymbol{c}) \notin \{-1, \infty\}$ and $[\boldsymbol{c}, m] \in \mathcal{C}$. Consider a spectral band $I_{\boldsymbol{c}}$ in $\sigma_{\boldsymbol{c}}$ with the associated spectral bands $\{I^i_{[\mathbf{c},m]}\}_{i=1}^M$ and $\{I^j_{[\mathbf{c},m,n]}\}_{j=1}^{M+1}$ introduced in Definition 2.12. Let $J_{[\mathbf{c},m]}$ and $K_{[\boldsymbol{c},m]}$ be the spectral bands associated with $I_{\boldsymbol{c}}$ as defined in Definition 5.8. If*

$$I^1_{[\boldsymbol{c},m,n]}(V_1) \subseteq_{\text{str}} I_{\boldsymbol{c}}(V_1) \quad \textit{and} \quad I^{M+1}_{[\boldsymbol{c},m,n]}(V_1) \subseteq_{\text{str}} I_{\boldsymbol{c}}(V_1),$$

then

$$R(J_{[\boldsymbol{c},m]}(V_1)) < R(I^1_{[\boldsymbol{c},m,n]}(V_1)) \quad \textit{and} \quad L(I^{M+1}_{[\mathbf{c},m,n]}(V_1)) < L(K_{[\mathbf{c},m]}(V_1)).$$

Remark. It might be that either $J_{[\mathbf{c},m]}$ or $K_{[\mathbf{c},m]}$ as defined in Definition 5.8 do not exist. In such a case, part of the statement is empty.

Combining Lemma 5.10 and Lemma 5.11, we get the following corollary which shows that properties (A1), (B2) and (I) hold under some conditions.

Corollary 5.12. *Let $V_1 > 0$, $m, n \in \mathbb{N}$ and $\boldsymbol{c} \in \mathcal{C}$ be such that $\varphi(\boldsymbol{c}) \notin \{-1, \infty\}$ and $[\boldsymbol{c}, m] \in \mathcal{C}$. Consider a spectral band $I_{\boldsymbol{c}}$ in $\sigma_{\boldsymbol{c}}$ with associated spectral bands $\{I^i_{[\boldsymbol{c},m]}\}_{i=1}^M$ and $\{I^j_{[\boldsymbol{c},m,n]}\}_{j=1}^{M+1}$ introduced in Definition 2.12. If*

$$I^1_{[\mathbf{c},m]}(V_1),\ I^M_{[\mathbf{c},m]}(V_1) \subseteq_{\text{str}} I_{\boldsymbol{c}}(V_1) \qquad \textit{and} \qquad I^1_{[\mathbf{c},m,n]}(V_1),\ I^{M+1}_{[\mathbf{c},m,n]}(V_1) \subseteq_{\text{str}} I_{\boldsymbol{c}}(V_1),$$

then $I_{\boldsymbol{c}}(V_1)$ satisfies the properties (A1), (B2) and (I).

Proof of Corollary 5.12. First, we note that the condition in the corollary is equivalent to $I^j_{[\mathbf{c},m,n]}(V_1) \subseteq_{\rm str} I_{\mathbf{c}}(V_1)$ for all $1 \le j \le M+1$ and $I^i_{[\mathbf{c},m]}(V_1) \subseteq_{\rm str} I_{\mathbf{c}}(V_1)$ for all $1 \le i \le M$, since

$$I^i_{[\mathbf{c},m]} \prec I^{i+1}_{[\mathbf{c},m]} \qquad \text{and} \qquad I^j_{[\mathbf{c},m,n]} \prec I^{j+1}_{[\mathbf{c},m,n]}.$$

The assumption that $I^i_{[\mathbf{c},m]}(V_1) \subseteq_{\rm str} I_{\mathbf{c}}(V_1)$ for all $1 \le i \le M$ is exactly property (A1) of $I_{\mathbf{c}}(V_1)$. Moreover, this assumption allows us to apply Lemma 5.10 and obtain

$$I^i_{[\mathbf{c},m,n]}(V_1) \prec I^i_{[\mathbf{c},m]}(V_1) \prec I^{i+1}_{[\mathbf{c},m,n]}(V_1) \qquad \text{for all } 1 \le i \le M.$$

Thus, $I_{\mathbf{c}}(V_1)$ satisfies property (I). Furthermore, these relations imply that each of the bands $\left\{ I^j_{[\mathbf{c},m,n]}(V_1) \right\}_{j=1}^{M+1}$ is not contained in any of the bands $\left\{ I^i_{[\mathbf{c},m]}(V_1) \right\}_{i=1}^{M}$, which is useful towards proving property (B2). Recall (Definition 5.8) the notation of the spectral bands $J_{[\mathbf{c},m]}$ and $K_{[\mathbf{c},m]}$ associated with $I_{\mathbf{c}}$. In order to prove (B2), it is enough to prove that $I^1_{[\mathbf{c},m,n]}(V_1)$ is not contained $J_{[\mathbf{c},m]}(V_1)$ and $I^{M+1}_{[\mathbf{c},m,n]}(V_1)$ is not contained in $K_{[\mathbf{c},m]}(V_1)$. This follows from Lemma 5.11. □

Proof of Lemma 5.11. (a) We prove that $R(J_{[\mathbf{c},m]}(V_1)) < R(I^1_{[\mathbf{c},m,n]}(V_1))$. First we note that this inequality immediately holds if $R(J_{[\mathbf{c},m]}(V_1)) \le L(I_{\mathbf{c}}(V_1))$, because $I^1_{[\mathbf{c},m,n]}(V_1) \subseteq_{\rm str} I_{\mathbf{c}}(V_1)$ by assumption. Therefore, we assume from now on that $R(J_{[\mathbf{c},m]}(V_1)) > L(I_{\mathbf{c}}(V_1))$. Although the assumptions and the conclusions of the lemma depend on the fixed $V_1 > 0$, we will abbreviate notation, for the sake of easier reading, and omit the V_1 dependence in most parts of this proof. Let $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]} \in \{0, \pi\}$ be such that

$$R(I_{\mathbf{c}}) \in \sigma\left(H_{\mathbf{c}}(\theta_{\mathbf{c}})\right), \;\; R(J_{[\mathbf{c},m]}) \in \sigma\left(H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) \;\text{ and }\; R(I^1_{[\mathbf{c},m,n]}) \in \sigma\left(H_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m,n]})\right).$$

Then these spectral edges, respectively $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}$, are admissible by inserting the index relation (5.11) of Lemma 5.9 into the characterization of admissibility from Lemma 5.6. Furthermore, Lemma 5.4 (c) for $[\mathbf{c},m,n] \in \mathcal{C}$ and $\mathbf{c} \in \mathcal{C}$, the index relation (5.11) of Lemma 5.9 and Lemma 5.5 (d) for the spectral band $J_{[\mathbf{c},m]}$ imply

$$\begin{aligned} N\left(R(I^1_{[\mathbf{c},m,n]});\; H_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m,n]})\right) &= \operatorname{ind}(I^1_{[\mathbf{c},m,n]}) \\ &= \operatorname{ind}(I_{\mathbf{c}}) + n \cdot (\operatorname{ind}(J_{[\mathbf{c},m]}) + 1) \\ &= N\left(R(I_{\mathbf{c}});\; H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) + N\left(R(J_{[\mathbf{c},m]});\; H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) + 1. \end{aligned} \tag{5.18}$$

In order to proceed, we first show that $R(J_{[\mathbf{c},m]}) < R(I_{\mathbf{c}})$. Assume by contradiction this is not the case, namely $R(J_{[\mathbf{c},m]}) \ge R(I_c)$. We aim to apply Lemma 5.7 for $\lambda_{\mathbf{o}} = R(I_{\mathbf{c}})$ and $\mu_{\mathbf{o}} = R(I^1_{[\mathbf{c},m,n]})$. With (5.18) and $R(J_{[\mathbf{c},m]}) \ge R(I_c)$ at hand, we conclude

$$\begin{aligned} N\left(R(I^1_{[\mathbf{c},m,n]});\; H_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m,n]})\right) &= N\left(R(I_{\mathbf{c}});\; H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) + N\left(R(J_{[\mathbf{c},m]});\; H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) + 1 \\ &\ge N\left(R(I_{\mathbf{c}});\; H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) + N\left(R(I_{\mathbf{c}});\; H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) + 1. \end{aligned}$$

Using the notation from Lemma 5.7, the latter reads $N_{[\mathbf{c},m,n]} \ge N_{\mathbf{c}} + N_{[\mathbf{c},m]} + 1$. Thus, Lemma 5.7 yields $\mu_{\mathbf{o}} \ge \lambda_{\mathbf{o}}$ as $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}$ are admissible. On the other hand, $I^1_{[\mathbf{c},m,n]} \subseteq_{\rm str} I_{\mathbf{c}}$ implies $R(I^1_{[\mathbf{c},m,n]}) = \mu_{\mathbf{o}} < \lambda_{\mathbf{o}} = R(I_{\mathbf{c}})$, a contradiction. Hence, $R(J_{[\mathbf{c},m]}) < R(I_{\mathbf{c}})$ follows as claimed.

With this at hand, we continue applying once again Lemma 5.7, but this time for $\lambda_{\mathbf{o}} = R(J_{[\mathbf{c},m]})$ and $\mu_{\mathbf{o}} = R(I^1_{[\mathbf{c},m,n]})$. Using (5.18) and $R(J_{[\mathbf{c},m]}) < R(I_{\mathbf{c}})$, we infer

$$N\left(R(I^1_{[\mathbf{c},m,n]}); H_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m,n]})\right) = N\left(R(I_{\mathbf{c}}); H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) + N\left(R(J_{[\mathbf{c},m]}); H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) + 1$$

$$\geq N\left(R(J_{[\mathbf{c},m]});\ H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) + N\left(R(J_{[\mathbf{c},m]});\ H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) + 1.$$

Using the notation from Lemma 5.7, the latter reads $N_{[\mathbf{c},m,n]} \geq N_{\mathbf{c}} + N_{[\mathbf{c},m]} + 1$. Recall that we showed in the beginning of the proof, $L(I_{\mathbf{c}}(V_1)) < R(J_{[\mathbf{c},m]}(V_1))$. This, together with $R(J_{[\mathbf{c},m]}(V_1)) < R(I_{\mathbf{c}}(V_1))$, implies that $R(J_{[\mathbf{c},m]}(V_1))$ is not an eigenvalue of $H_{\mathbf{c},V_1}(\theta_{\mathbf{c}})$. Hence, $\lambda_{\mathbf{o}} = R(J_{[\mathbf{c},m]}(V_1))$ is a simple eigenvalue of $H_{\mathbf{c},V_1}(\theta_{\mathbf{c}}) \oplus H^{\times n}_{[\mathbf{c},m],V_1}(\theta_{[\mathbf{c},m]})$ using Lemma 5.5 (e). Thus, Lemma 5.7 (a) applied with $N_{[\mathbf{c},m,n]} \geq N_{\mathbf{c}} + N_{[\mathbf{c},m]} + 1$ yields that $\mu_{\mathbf{o}} > \lambda_{\mathbf{o}}$, i.e., $R(I^1_{[\mathbf{c},m,n]}) > R(J_{[\mathbf{c},m]})$, as required.

(b) We prove that $L(I^{M+1}_{[\mathbf{c},m,n]}(V_1)) < L(K_{[\mathbf{c},m]}(V_1))$. First we note that this inequality immediately holds if $R(I_{\mathbf{c}}(V_1)) \leq L(K_{[\mathbf{c},m]}(V_1))$, because $I^{M+1}_{[\mathbf{c},m,n]}(V_1) \subseteq_{\text{str}} I_{\mathbf{c}}(V_1)$ by assumption. Therefore, we assume from now on that $L(K_{[\mathbf{c},m]}(V_1)) < R(I_{\mathbf{c}}(V_1))$. In order to simplify the notation, we will omit the dependence on V_1 in the following unless we want to emphasize its dependence. Let $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]} \in \{0, \pi\}$ be such that

$$L(I_{\mathbf{c}}) \in \sigma\left(H_{\mathbf{c}}(\theta_{\mathbf{c}})\right),\ L(K_{[\mathbf{c},m]}) \in \sigma\left(H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) \text{ and } L(I^{M+1}_{[\mathbf{c},m,n]}) \in \sigma\left(H_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m,n]})\right).$$

Then these spectral edges, respectively $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}$, are admissible by inserting the index relation (5.12) of Lemma 5.9 into the characterization of admissibility from Lemma 5.6. By Definition 5.8, we have $\operatorname{ind}(K_{[\mathbf{c},m]}) = \operatorname{ind}(I^M_{[\mathbf{c},m]}) + 1$. With this at hand, Lemma 5.4 (b) for $[\mathbf{c}, m, n] \in \mathcal{C}$ and $\mathbf{c} \in \mathcal{C}$, the index relation (5.12) of Lemma 5.9 and Lemma 5.5 (c) for the spectral band $K_{[\mathbf{c},m]}$ imply

$$\begin{aligned} N\left(L(I^{M+1}_{[\mathbf{c},m,n]});\ H_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m,n]})\right) &= \operatorname{ind}(I^{M+1}_{[\mathbf{c},m,n]}) \\ &= \operatorname{ind}(I_{\mathbf{c}}) + n \cdot \operatorname{ind}(K_{[\mathbf{c},m]}) \\ &= N\left(L(I_{\mathbf{c}});\ H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) + N\left(L(K_{[\mathbf{c},m]});\ H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right). \end{aligned} \tag{5.19}$$

In order to proceed, we first show that $L(K_{[\mathbf{c},m]}) > L(I_{\mathbf{c}})$. Assume by contradiction this is not the case, namely $L(K_{[\mathbf{c},m]}) \leq L(I_c)$. We aim to apply Lemma 5.7 for $\lambda_{\mathbf{o}} = L(I_{\mathbf{c}})$ and $\mu_{\mathbf{o}} = L(I^{M+1}_{[\mathbf{c},m,n]})$. If $L(K_{[\mathbf{c},m]}) = L(I_c)$, then $\lambda_{\mathbf{o}}$ has multiplicity $\mathcal{M}_{\lambda_{\mathbf{o}}} = 2$. Thus, the previous identity (5.19) leads to

$$N\left(L(I^{M+1}_{[\mathbf{c},m,n]});\ H_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m,n]})\right) < N\left(L(I_{\mathbf{c}});\ H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) + N\left(L(I_{\mathbf{c}});\ H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) + \mathcal{M}_{\lambda_{\mathbf{o}}} - 1.$$

If $L(K_{[\mathbf{c},m]}) < L(I_c)$, then $N\left(L(K_{[\mathbf{c},m]});\ H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) \leq N\left(L(I_{\mathbf{c}});\ H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) - 1$ follows and the multiplicity of $\lambda_{\mathbf{o}}$ satisfies $\mathcal{M}_{\lambda_{\mathbf{o}}} \geq 1$. Combing these with the Equation (5.19) and $L(K_{[\mathbf{c},m]}) \in \sigma\left(H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right)$, we conclude

$$\begin{aligned} N\left(L(I^{M+1}_{[\mathbf{c},m,n]});\ H_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m,n]})\right) &= N\left(L(I_{\mathbf{c}});\ H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) + N\left(L(K_{[\mathbf{c},m]});\ H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) \\ &< N\left(L(I_{\mathbf{c}});\ H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) + N\left(L(I_{\mathbf{c}});\ H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) + \mathcal{M}_{\lambda_{\mathbf{o}}} - 1. \end{aligned}$$

Using the notation from Lemma 5.7, the latter reads $N_{[\mathbf{c},m,n]} < N_{\mathbf{c}} + N_{[\mathbf{c},m]} + \mathcal{M}_{\lambda_{\mathbf{o}}} - 1$ whenever $L(K_{[\mathbf{c},m]}) \leq L(I_c)$. Thus, Lemma 5.7 (a) yields $\mu_{\mathbf{o}} \leq \lambda_{\mathbf{o}}$ as $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}$ are admissible. On the other hand, $I^{M+1}_{[\mathbf{c},m,n]} \subseteq_{\text{str}} I_{\mathbf{c}}$ implies $L(I^{M+1}_{[\mathbf{c},m,n]}) = \mu_{\mathbf{o}} > \lambda_{\mathbf{o}} = L(I_{\mathbf{c}})$, a contradiction. Hence, $L(K_{[\mathbf{c},m]}) > L(I_{\mathbf{c}})$ follows as claimed.

With this at hand, we continue applying once again Lemma 5.7, but this time for $\lambda_{\mathbf{o}} = L(K_{[\mathbf{c},m]})$ and $\mu_{\mathbf{o}} = L(I^{M+1}_{[\mathbf{c},m,n]})$. Using (5.19), the inequality $L(K_{[\mathbf{c},m]}) > L(I_{\mathbf{c}})$ and $L(I_{\mathbf{c}}) \in \sigma(H_{\mathbf{c}}(\theta_{\mathbf{c}})$, we infer

$$N\left(L(I^{M+1}_{[\mathbf{c},m,n]});\ H_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m,n]})\right) = N\left(L(I_{\mathbf{c}});\ H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) + N\left(L(K_{[\mathbf{c},m]});\ H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right)$$

$$< N\left(L(K_{[\mathbf{c},m]});\ H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) + N\left(L(K_{[\mathbf{c},m]});\ H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right).$$

Using the notation from Lemma 5.7, the latter reads $N_{[\mathbf{c},m,n]} < N_{\mathbf{c}} + N_{[\mathbf{c},m]}$. Recall that we showed in the beginning of the proof, $L(K_{[\mathbf{c},m]}(V_1)) < R(I_{\mathbf{c}}(V_1))$. This, together with $L(I_{\mathbf{c}}(V_1)) < L(K_{[\mathbf{c},m]}(V_1))$, implies that $L(K_{[\mathbf{c},m]}(V_1))$ is not an eigenvalue of $H_{\mathbf{c},V_1}(\theta_{\mathbf{c}})$. Hence, $\lambda_{\mathbf{o}} = L(K_{[\mathbf{c},m]}(V_1))$ is a simple eigenvalue of $H_{\mathbf{c},V_1}(\theta_{\mathbf{c}}) \oplus H^{\times n}_{[\mathbf{c},m],V_1}(\theta_{[\mathbf{c},m]})$ using Lemma 5.5 (e). Thus, Lemma 5.7 applied with $N_{[\mathbf{c},m,n]} < N_{\mathbf{c}} + N_{[\mathbf{c},m]}$ leads to $\mu_{\mathbf{o}} < \lambda_{\mathbf{o}}$, i.e., $L(I^{M+1}_{[\mathbf{c},m,n]}) < L(K_{[\mathbf{c},m]})$. □

Next we show that the assumptions in the previous Corollary 5.12 are satisfied whenever the spectral band is of backward type A or B. The proofs of the previous lemmata in this subsection were mainly based on applications of Lemma 5.7. This lemma will keep being applied in the next proofs, but we will also need to make use of some trace identities, as appear in Lemma 4.6 and Lemma 4.7.

Lemma 5.13. *Let $m \in \mathbb{N}$ and $\boldsymbol{c} \in \mathcal{C}$ be such that $\varphi(\boldsymbol{c}) \in (0,1)$. Let $V_1 > 0$ and $I_{\boldsymbol{c}}$ be a spectral band in $\sigma_{\boldsymbol{c}}$ with associated spectral bands $\{I^i_{[\boldsymbol{c},m]}\}_{i=1}^M$ and $\{I^j_{[\boldsymbol{c},m,1]}\}_{j=1}^{M+1}$ introduced in Definition 2.12. If either*

- $I_{\mathbf{c}}(V)$ is of backward type A for all $V \geq V_1$ and $M := m-1$, or
- $I_{\mathbf{c}}(V)$ is of backward type B for all $V \geq V_1$ and $M := m$,

then for all $V \geq V_1$,

$$I^1_{[\mathbf{c},m,1]}(V),\ I^{M+1}_{[\mathbf{c},m,1]}(V) \subseteq_{\mathrm{str}} I_{\mathbf{c}}(V) \qquad \text{and} \qquad I^1_{[\mathbf{c},m]}(V),\ I^M_{[\mathbf{c},m]}(V) \subseteq_{\mathrm{str}} I_{\mathbf{c}}(V).$$

Remark. We have to exclude the cases $\varphi(\mathbf{c}) \in \{0, \pm 1, \infty\}$ so that we can apply Lemma 4.7 (c).

Proof. The claim follows once we show that for all $V \geq V_1$,

$$L(I_{\mathbf{c}}(V)) < \min\left\{L(I^1_{[\mathbf{c},m,1]}(V)),\ L(I^1_{[\mathbf{c},m]}(V))\right\},$$
$$\max\left\{R(I^M_{[\mathbf{c},m]}(V)),\ R(I^{M+1}_{[\mathbf{c},m,1]}(V))\right\} < R(I_{\mathbf{c}}(V)). \tag{5.20}$$

Assume by contradiction that (5.20) does not hold for some $V \geq V_1$. Due to Theorem 2.11, these strict inequalities in (5.20) hold for $V > 4$. Thus, the continuity of the spectral band edges in $V > 0$ (Corollary 2.3) implies that the maximum

$$V_2 := \max\{V \geq V_1 \ :\ (5.20) \text{ does not hold}\}$$

exists and $V_2 \in [V_1, 4]$. Due to Lemma 5.10, the strict inclusions $I^1_{[\mathbf{c},m]}(V) \subseteq_{\mathrm{str}} I_{\mathbf{c}}(V)$ and $I^M_{[\mathbf{c},m]}(V) \subseteq_{\mathrm{str}} I_{\mathbf{c}}(V)$ for $V > V_2$ yield

$$L(I^1_{[\mathbf{c},m,1]}(V)) < L(I^1_{[\mathbf{c},m]}(V)) \qquad \text{and} \qquad R(I^M_{[\mathbf{c},m]}(V)) < R(I^{M+1}_{[\mathbf{c},m,1]}(V)) \qquad \text{for } V > V_2.$$

Let $J_{[\mathbf{c},m]}$ and $K_{[\mathbf{c},m]}$ be the spectral bands associated with $I_{\mathbf{c}}$ (Definition 5.8). Since the strict inclusions $I^1_{[\mathbf{c},m,1]}(V) \subseteq_{\mathrm{str}} I_{\mathbf{c}}(V)$ and $I^{M+1}_{[\mathbf{c},m,1]}(V) \subseteq_{\mathrm{str}} I_{\mathbf{c}}(V)$ hold for $V > V_2$, Lemma 5.11 asserts for $V > V_2$,

$$R(J_{[\mathbf{c},m]}(V)) < R(I^1_{[\mathbf{c},m,1]}(V)) \quad \text{and} \quad L(I^{M+1}_{[\mathbf{c},m,1]}(V)) < L(K_{[\mathbf{c},m]}(V)).$$

Note that we have $R(I^1_{[\mathbf{c},m,1]}(V)) \leq R(I^{M+1}_{[\mathbf{c},m,1]}(V))$ and $L(I^1_{[\mathbf{c},m,1]}(V)) \leq L(I^{M+1}_{[\mathbf{c},m,1]}(V))$ for $V > 0$. Hence, the continuity of the spectral band edges in $V > 0$ (Corollary 2.3) leads to

$$L(I^1_{[\mathbf{c},m,1]}(V_2)) \leq \min\left\{L(I^1_{[\mathbf{c},m]}(V_2)), L(K_{[\mathbf{c},m]}(V_2))\right\},$$
$$\max\left\{R(I^M_{[\mathbf{c},m]}(V_2)), R(J_{[\mathbf{c},m]}(V_2))\right\} \leq R(I^{M+1}_{[\mathbf{c},m,1]}(V_2)). \tag{5.21}$$

Note that the spectral bands $J_{[\mathbf{c},m]}$ and $K_{[\mathbf{c},m]}$ may not exist simplifying our considerations below. This in particular implies

$$V_2 = \max\left\{V \geq V_1 \,:\, L(I_{\mathbf{c}}(V)) = L(I^1_{[\mathbf{c},m,1]}(V)) \text{ or } R(I_{\mathbf{c}}(V)) = R(I^{M+1}_{[\mathbf{c},m,1]}(V))\right\}.$$

We continue proving that this leads to a contradiction.

<u>Case 1:</u> We show that $L(I_{\mathbf{c}}(V_2)) = L(I^1_{[\mathbf{c},m,1]}(V_2))$ yields a contradiction. Set $E := L(I_{\mathbf{c}}(V_2))$. Thus, $\left|t_{[\mathbf{c},m,1]}(E;\ V_2)\right| = 2$ follows from Lemma 4.6 (a). Since $I_{\mathbf{c}}(V_2)$ is of backward type A or B (using $V_2 \geq V_1$) and $\varphi(\mathbf{c}) \in (0,1)$, Lemma 4.7 (c) yields $\left|t_{[\mathbf{c},m]}(E;\ V_2)\right| < 2$. Hence, E must lie in the interior of a spectral band in $\sigma_{[\mathbf{c},m]}(V_2)$ by Lemma 4.6 (a). Thus, Equation (5.21) and $L(I_{\mathbf{c}}(V_2)) = L(I^1_{[\mathbf{c},m,1]}(V_2))$ lead to

$$E = L(I_{\mathbf{c}}(V_2)) < R(J_{[\mathbf{c},m]}(V_2)).$$

Note that if $J_{[\mathbf{c},m]}$ does not exist, then there is no spectra to the left of $L(I^1_{[\mathbf{c},m]}(V_2))$ contradicting $\left|t_{[\mathbf{c},m]}(E;\ V_2)\right| < 2$ and (5.21). Hence, we may continue assuming that $J_{[\mathbf{c},m]}$ exists. Next we aim to apply Lemma 5.7 for $\lambda_{\mathbf{o}} = L(I_{\mathbf{c}}(V_2))$ and $\mu_{\mathbf{o}} = L(I^1_{[\mathbf{c},m,1]}(V_2))$. For the sake of simplification, we drop the V_2 notation in the following. Let $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,1]} \in \{0,\pi\}$ be such that

$$L(I_{\mathbf{c}}) \in \sigma\left(H_{\mathbf{c}}(\theta_{\mathbf{c}})\right),\ R(J_{[\mathbf{c},m]}) \in \sigma\left(H^{\times 1}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) \text{ and } L(I^1_{[\mathbf{c},m,1]}) \in \sigma\left(H_{[\mathbf{c},m,1]}(\theta_{[\mathbf{c},m,1]})\right).$$

Then these spectral edges, respectively $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,1]}$, are admissible by inserting the index relation (5.11) into the characterization of admissibility from Lemma 5.6 for $n = 1$. With this at hand, and Lemma 5.4 (b) applied to $\mathbf{c}, [\mathbf{c},m,1] \in \mathcal{C}$ leads to

$$N_{\mathbf{c}} := N\left(L(I_{\mathbf{c}});\ H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) = \operatorname{ind}(I_{\mathbf{c}})$$

and using (5.11)

$$N_{[\mathbf{c},m,1]} := N\left(L(I^1_{[\mathbf{c},m,1]});\ H_{[\mathbf{c},m,1]}(\theta_{[\mathbf{c},m,1]})\right) = \operatorname{ind}(I^1_{[\mathbf{c},m,1]}) = \operatorname{ind}(J_{[\mathbf{c},m]}) + 1 + \operatorname{ind}(I_{\mathbf{c}}).$$

Furthermore, $L(I_{\mathbf{c}}) < R(J_{[\mathbf{c},m]})$ and Lemma 5.5 (d) applied to $[\mathbf{c},m] \in \mathcal{C}$ for $n = 1$ lead to

$$N_{[\mathbf{c},m]} := N\left(L(I_{\mathbf{c}});\ H^{\times 1}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) \leq N\left(R(J_{[\mathbf{c},m]});\ H^{\times 1}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) = \operatorname{ind}(J_{[\mathbf{c},m]}).$$

Thus, $N_{\mathbf{c}} + N_{[\mathbf{c},m]} < N_{[\mathbf{c},m,1]}$ follows. Since $\lambda_{\mathbf{o}} = E = L(I_{\mathbf{c}}(V_2))$ lies in the interior of a spectral band in $\sigma_{[\mathbf{c},m]}(V_2)$ and the eigenvalues of $H^{\times 1}_{[\mathbf{c},m],V_2}(\theta_{[\mathbf{c},m]})$ are contained in the spectral band edges of $\sigma_{[\mathbf{c},m]}(V_2)$ (by Lemma 5.5), we conclude that $\lambda_{\mathbf{o}}$ is not an eigenvalue of $H^{\times 1}_{[\mathbf{c},m],V_2}(\theta_{[\mathbf{c},m]})$. Thus, $\lambda_{\mathbf{o}}$ is a simple eigenvalue of $H^{\times 1}_{[\mathbf{c},m],V_2}(\theta_{[\mathbf{c},m]}) \oplus H_{\mathbf{c},V_2}(\theta_{\mathbf{c}})$ using Lemma 5.5 (e). Hence, Lemma 5.7 (a) yields

$$\lambda_{\mathbf{o}} = L(I_{\mathbf{c}}(V_2)) < L(I^1_{[\mathbf{c},m,1]}(V_2)) = \mu_{\mathbf{o}},$$

contradicting that these two values are equal by the initial assumption of the considered case.

<u>Case 2:</u> Similarly as in Case 1, $R(I^{M+1}_{[\mathbf{c},m,1]}(V_2)) = R(I_{\mathbf{c}}(V_2))$ yields a contradiction. □

We have seen that Corollary 5.12 is set towards proving the forward properties (A1), (B2), (I). Next, we aim to prove the forward property (B1), (also called the *tower* property).

Corollary 5.14. *Let $V_1 > 0$, $m \in \mathbb{N}$ and $\boldsymbol{c} \in \mathcal{C}$ be such that $\varphi(\boldsymbol{c}) \notin \{-1,\infty\}$ and $[\boldsymbol{c},m] \in \mathcal{C}$. Consider a spectral band $I_{\boldsymbol{c}}$ in $\sigma_{\boldsymbol{c}}$ with associated spectral bands $\{I^i_{[\boldsymbol{c},m]}\}_{i=1}^M$ and $\{I^j_{[\boldsymbol{c},m,n]}\}_{j=1}^{M+1}$ introduced in Definition 2.12. If $1 \leq j \leq M+1$ and $I^j_{[\boldsymbol{c},m,n]}(V) \subseteq_{\text{str}} I_{\boldsymbol{c}}(V)$ holds for all $V \geq V_1$ and all $n \in \mathbb{N}$, then*

$$I^j_{[\boldsymbol{c},m,n]}(V) \subseteq_{\text{str}} I^j_{[\boldsymbol{c},m,n-1]}(V)$$

holds for all $n \in \mathbb{N}$ and $V \geq V_1$ where $I^j_{[\boldsymbol{c},m,0]}(V) = I_{\boldsymbol{c}}(V)$.

Proof. The proof is by induction over $n \in \mathbb{N}$. The induction base ($n = 1$) holds trivially since $I^j_{[\mathbf{c},m,1]}(V) \subseteq_{\text{str}} I_{\mathbf{c}}(V)$ for all $V \geq V_1$ and $\sigma_{[\mathbf{c},m,n-1]}(V) = \sigma_{\mathbf{c}}(V)$ if $n = 1$ by Proposition II.2 (a).

For the induction step, suppose $I^j_{[\mathbf{c},m,n]}(V) \subseteq_{\text{str}} I^j_{[\mathbf{c},m,n-1]}(V)$ holds for all $V \geq V_1$. We show that $I^j_{[\mathbf{c},m,n+1]}(V) \subseteq_{\text{str}} I^j_{[\mathbf{c},m,n]}(V)$ holds for all $V \geq V_1$. Due to Proposition II.2 (a), we have $\sigma_{[\mathbf{c},m,n+1]}(V) = \sigma_{[\mathbf{c},m,n,1]}(V)$. Furthermore, $I^j_{[\mathbf{c},m,n+1]}(V) \subseteq_{\text{str}} I^j_{[\mathbf{c},m,n]}(V)$ holds for $V > 4$ since $I_{\mathbf{c}}(V)$ is either of type A or B for $V > 4$ by Theorem 2.11. Thus, $I^j_{[\mathbf{c},m,n+1]}(V)$ equals to the unique spectral band $I^1_{[\mathbf{c},m,n,1]}(V)$ of type A that is strictly contained in $I^j_{[\mathbf{c},m,n]}(V)$ for $V > 4$. Hence, it suffices to prove $I^1_{[\mathbf{c},m,n,1]}(V) \subseteq_{\text{str}} I^j_{[\mathbf{c},m,n]}(V)$ for all $V \geq V_1$.

Let $V \geq V_1$. By induction hypothesis, we have $I^j_{[\mathbf{c},m,n]}(V) \subseteq_{\text{str}} I^j_{[\mathbf{c},m,n-1]}(V)$ for all $V \geq V_1$, namely $I^j_{[\mathbf{c},m,n]}(V)$ is of backward type B for all $V \geq V_1$. Furthermore, $\varphi([\mathbf{c},m,n]) \in (0,1)$ holds as $m, n \in \mathbb{N}$. Thus, Lemma 5.13 applied to $[\mathbf{c},m,n]$ implies $I^1_{[\mathbf{c},m,n,1]}(V) \subseteq_{\text{str}} I^j_{[\mathbf{c},m,n]}(V)$ for all $V \geq V_1$. □

The next lemma is the crucial ingredient to prove the forward property (A2).

Lemma 5.15. *Let $V_1 > 0$ and $\mathbf{c} \in \mathcal{C}$ be such that $\varphi(\mathbf{c}) \in (0,1)$. Consider a spectral band $V \mapsto I_{\mathbf{c}}(V)$ in $\sigma_{\mathbf{c}}(V)$ which is of backward type B for all $V \geq V_1$ and $I^1_{[\mathbf{c},1]}$ is the associated spectral band introduced in Definition 2.12. Then for all $V \geq V_1$, $I^1_{[\mathbf{c},1]}(V) \subseteq_{\text{str}} I_{\mathbf{c}}(V)$ (namely $I^1_{[\mathbf{c},1]}(V)$ is of backward type A) and $I^1_{[\mathbf{c},1]}(V)$ is not of weak backward type B.*

Proof. Since $I_{\mathbf{c}}(V)$ is of backward type B for all $V \geq V_1$, it follows that $I_{\mathbf{c}}(V)$ is of type B for all $V > 4$ by Theorem 2.11. Thus, there is a unique spectral band $I^1_{[\mathbf{c},1]}$ in $\sigma_{[\mathbf{c},1]}$ such that $I^1_{[\mathbf{c},1]}(V) \subseteq_{\text{str}} I_{\mathbf{c}}(V)$ for all $V > 4$. By Theorem 2.11, the lemma holds for all $V_1 > 4$, and so we can assume in the proof that $V_1 \leq 4$.

Consider the spectral bands $J_{[\mathbf{c},0]}$ and $K_{[\mathbf{c},0]}$ associated with $I_{\mathbf{c}}$ (see Definition 5.8). Since $I_{\mathbf{c}}(V)$is of backward type B for $V > 4$, we have $\operatorname{ind}(K_{[\mathbf{c},0]}) = \operatorname{ind}(J_{[\mathbf{c},0]}) + 1$ (i.e., there is no other spectral band between those two) and

$$\forall V > 4, \qquad J_{[\mathbf{c},0]}(V) \prec I_{\mathbf{c}}(V) \prec K_{[\mathbf{c},0]}(V).$$

Lemma 5.13 implies $I^1_{[\mathbf{c},1]}(V) \subseteq_{\text{str}} I_{\mathbf{c}}(V)$ for all $V \geq V_1$. It is left to prove that for $V \geq V_1$, $I^1_{[\mathbf{c},1]}(V)$ is not contained in any spectral band of $\sigma_{[\mathbf{c},1,-1]}(V) = \sigma_{[\mathbf{c},0]}(V)$ (where the last equality follows from Proposition II.2). Actually, it suffices to prove that for all $V \geq V_1$,

$$R(J_{[\mathbf{c},0]}(V)) < R(I^1_{[\mathbf{c},1]}(V)) \quad \text{and} \quad L(I^1_{[\mathbf{c},1]}(V)) < L(K_{[\mathbf{c},0]}(V)). \tag{5.22}$$

Assume by contradiction that (5.22) does not hold for some $V \geq V_1$. By Theorem 2.11, (5.22) holds for $V > 4$. Thus, the continuity of the spectral band edges in $V > 0$ (Corollary 2.3) implies that the maximum

$$V_2 := \max\left\{ V \geq V_1 \,:\, R(J_{[\mathbf{c},0]}(V)) = R(I^1_{[\mathbf{c},1]}(V)) \text{ or } L(I^1_{[\mathbf{c},1]}(V)) = L(K_{[\mathbf{c},0]}(V)) \right\}$$

exists and $V_2 \in [V_1, 4]$. We split into cases according to the nature of failure of (5.22) at $V = V_2$, and show a contradiction for each of these cases. First note that Equation (5.13) of Lemma 5.9 implies

$$\operatorname{ind}(I^1_{[\mathbf{c},1]}) = \operatorname{ind}(J_{[\mathbf{c},0]}) + 1 + \operatorname{ind}(I_{\mathbf{c}}) = \operatorname{ind}(K_{[\mathbf{c},0]}) + \operatorname{ind}(I_{\mathbf{c}}). \tag{5.23}$$

Since $\varphi(\mathbf{c}) \in (0,1)$, there is a $k \in \mathbb{N}$ such that $\mathbf{c} = [0, c_0, \dots, c_k]$. In the following we apply Lemma 5.7 to $\widetilde{\mathbf{c}}, [\widetilde{\mathbf{c}}, m], [\widetilde{\mathbf{c}}, m, n] \in \mathcal{C}$ where $\widetilde{\mathbf{c}} = [0, c_0, \dots, c_{k-1}]$, $m = c_k$ and $n = 1$. Note that $\varphi(\widetilde{\mathbf{c}}) = \varphi([\mathbf{c}, 0])$, $\varphi([\widetilde{\mathbf{c}}, m]) = \varphi(\mathbf{c})$ and $\varphi([\widetilde{\mathbf{c}}, m, n]) = \varphi([\mathbf{c}, 1])$. Thus, in effect it is as if we apply Lemma 5.7 to $[\mathbf{c},0],\mathbf{c},[\mathbf{c},1]\in\mathcal{C}$ (rather than to $\widetilde{\mathbf{c}},[\widetilde{\mathbf{c}},m],[\widetilde{\mathbf{c}},m,n]\in\mathcal{C}$). We use this convention until the end of the current proof.

Case 1: We show that $R(J_{[\mathbf{c},0]}(V_2))=R(I^1_{[\mathbf{c},1]}(V_2))$ yields a contradiction. We aim to apply Lemma 5.7 for $\lambda_{\mathbf{o}}=R(J_{[\mathbf{c},0]}(V_2))$ and $\mu_{\mathbf{o}}=R(I^1_{[\mathbf{c},1]}(V_2))$. For the sake of simplification, we drop the V_2 notation in the following unless we want to emphasize its dependence. Let $\theta_{[\mathbf{c},0]},\theta_{\mathbf{c}},\theta_{[\mathbf{c},1]}\in\{0,\pi\}$ be such that

$$R(J_{[\mathbf{c},0]})\in\sigma\left(H_{[\mathbf{c},0]}(\theta_{[\mathbf{c},0]})\right),\ R(I_{\mathbf{c}})\in\sigma\left(H^{\times 1}_{\mathbf{c}}(\theta_{\mathbf{c}})\right)\ \text{and}\ R(I^1_{[\mathbf{c},1]})\in\sigma\left(H_{[\mathbf{c},1]}(\theta_{[\mathbf{c},1]})\right).$$

Then these spectral edges, respectively $\theta_{[\mathbf{c},0]},\theta_{\mathbf{c}},\theta_{[\mathbf{c},1]}$, are admissible by inserting the index relation (5.23) into the characterization of admissibility from Lemma 5.6 for $n=1$. With this at hand, Lemma 5.4 (c) applied to $[\mathbf{c},0]\in\mathcal{C}$ and $[\mathbf{c},1]\in\mathcal{C}$ leads to

$$N_{[\mathbf{c},0]}:=N\left(R(J_{[\mathbf{c},0]});\ H_{[\mathbf{c},0]}(\theta_{[\mathbf{c},0]})\right)=\operatorname{ind}(J_{[\mathbf{c},0]})$$

and

$$N_{[\mathbf{c},1]}:=N\left(R(I^1_{[\mathbf{c},1]});\ H_{[\mathbf{c},1]}(\theta_{[\mathbf{c},1]})\right)=\operatorname{ind}(I^1_{[\mathbf{c},1]}).$$

Furthermore, $I^1_{[\mathbf{c},1]}(V_2)\subseteq_{\text{str}} I_{\mathbf{c}}(V_2)$ and the assumption $R(J_{[\mathbf{c},0]}(V_2))=R(I^1_{[\mathbf{c},1]}(V_2))$ imply $R(J_{[\mathbf{c},0]}(V_2))<R(I_{\mathbf{c}}(V_2))\in\sigma\left(H^{\times 1}_{\mathbf{c},V_2}(\theta_{\mathbf{c}})\right)$. Thus, Lemma 5.5 (d) applied to $n=1$ and $\mathbf{c}\in\mathcal{C}$ imply

$$N_{\mathbf{c}}:=N\left(R(J_{[\mathbf{c},0]});\ H^{\times 1}_{\mathbf{c}}(\theta_{\mathbf{c}})\right)\le N\left(R(I_{\mathbf{c}});\ H^{\times 1}_{\mathbf{c}}(\theta_{\mathbf{c}})\right)=\operatorname{ind}(I_{\mathbf{c}}).$$

Thus, (5.23) implies $N_{[\mathbf{c},1]}>N_{\mathbf{c}}+N_{[\mathbf{c},0]}$. If we prove that $\lambda_{\mathbf{o}}=R(J_{[\mathbf{c},0]}(V_2))$ is a simple eigenvalue of $H^{\times 1}_{\mathbf{c},V_2}(\theta_{\mathbf{c}})\oplus H_{[\mathbf{c},0],V_2}(\theta_{[\mathbf{c},0]})$, then Lemma 5.7 yields $\lambda_{\mathbf{o}}=R(J_{[\mathbf{c},0]}(V_2))<R(I^1_{[\mathbf{c},1]}(V_2))=\mu_{\mathbf{o}}$, a contradiction.

By Lemma 5.5 (e), simplicity of the eigenvalue $\lambda_{\mathbf{o}}$ holds if it is not an eigenvalue of $H^{\times 1}_{\mathbf{c},V_2}(\theta_{\mathbf{c}})=H_{\mathbf{c},V_2}(\theta_{\mathbf{c}})$. Using Lemma 5.5 (e), $R(I_{\mathbf{c}}(V_2))$ is the only eigenvalue of $H_{\mathbf{c},V_2}(\theta_{\mathbf{c}})$ in $I_{\mathbf{c}}(V_2)$. Thus, our working assumption, $R(J_{[\mathbf{c},0]}(V_2))=R(I^1_{[\mathbf{c},1]}(V_2))<R(I_{\mathbf{c}}(V_2))$ implies that $R(J_{[\mathbf{c},0]}(V_2))$ is not an eigenvalue of $H_{\mathbf{c},V_2}(\theta_{\mathbf{c}})$.

Case 2: We show that $L(I^1_{[\mathbf{c},1]}(V_2))=L(K_{[\mathbf{c},0]}(V_2))$ yields a contradiction. We aim to apply Lemma 5.7 for $\lambda_{\mathbf{o}}=L(K_{[\mathbf{c},0]}(V_2))$ and $\mu_{\mathbf{o}}=L(I^1_{[\mathbf{c},1]}(V_2))$. For the sake of simplification, we drop the V_2 notation in the following unless we want to emphasize its dependence. Let $\theta_{[\mathbf{c},0]},\theta_{\mathbf{c}},\theta_{[\mathbf{c},1]}\in\{0,\pi\}$ be such that

$$L(K_{[\mathbf{c},0]})\in\sigma\left(H_{[\mathbf{c},0]}(\theta_{[\mathbf{c},0]})\right),\ L(I_{\mathbf{c}})\in\sigma\left(H^{\times 1}_{\mathbf{c}}(\theta_{\mathbf{c}})\right)\ \text{and}\ L(I^1_{[\mathbf{c},1]})\in\sigma\left(H_{[\mathbf{c},1]}(\theta_{[\mathbf{c},1]})\right).$$

Then these spectral edges, respectively $\theta_{[\mathbf{c},0]},\theta_{\mathbf{c}},\theta_{[\mathbf{c},1]}$, are admissible by inserting the index relation (5.23) into the characterization of admissibility from Lemma 5.6 for $n=1$. With this at hand, Lemma 5.4 (b) applied to $[\mathbf{c},0]\in\mathcal{C}$ and $[\mathbf{c},1]\in\mathcal{C}$ leads to

$$N_{[\mathbf{c},0]}:=N\left(L(K_{[\mathbf{c},0]});\ H_{[\mathbf{c},0]}(\theta_{[\mathbf{c},0]})\right)=\operatorname{ind}(K_{[\mathbf{c},0]})$$

and

$$N_{[\mathbf{c},1]}:=N\left(L(I^1_{[\mathbf{c},1]});\ H_{[\mathbf{c},1]}(\theta_{[\mathbf{c},1]})\right)=\operatorname{ind}(I^1_{[\mathbf{c},1]}).$$

Furthermore, $I^1_{[\mathbf{c},1]}(V_2)\subseteq_{\text{str}} I_{\mathbf{c}}(V_2)$ and the assumption $L(I^1_{[\mathbf{c},1]}(V_2))=L(K_{[\mathbf{c},0]}(V_2))$ imply $\sigma\left(H^{\times 1}_{\mathbf{c},V_2}(\theta_{\mathbf{c}})\right)\ni L(I_{\mathbf{c}}(V_2))<L(K_{[\mathbf{c},0]}(V_2))$. Thus, Lemma 5.5 (c) applied to $n=1$ and $\mathbf{c}\in\mathcal{C}$ leads to

$$N_{\mathbf{c}}:=N\left(L(K_{[\mathbf{c},0]});\ H^{\times 1}_{\mathbf{c}}(\theta_{\mathbf{c}})\right)\ge N\left(L(I_{\mathbf{c}});\ H^{\times 1}_{\mathbf{c}}(\theta_{\mathbf{c}})\right)+1=\operatorname{ind}(I_{\mathbf{c}})+1.$$

Thus, (5.23) implies $N_{[\mathbf{c},1]}<N_{\mathbf{c}}+N_{[\mathbf{c},0]}$. If we prove that $L(K_{[\mathbf{c},0]}(V_2))$ is a simple eigenvalue of $H^{\times 1}_{\mathbf{c},V_2}(\theta_{\mathbf{c}})\oplus H_{[\mathbf{c},0],V_2}(\theta_{[\mathbf{c},0]})$, then Lemma 5.7 yields $\lambda_{\mathbf{o}}=L(K_{[\mathbf{c},0]}(V_2))>L(I^1_{[\mathbf{c},1]}(V_2))=\mu_{\mathbf{o}}$, a contradiction.

By Lemma 5.5 (e), simplicity of the eigenvalue $\lambda_{\mathbf{o}}$ holds if it is not an eigenvalue of $H^{\times 1}_{\mathbf{c},V_2}(\theta_{\mathbf{c}}) = H_{\mathbf{c},V_2}(\theta_{\mathbf{c}})$. Using Lemma 5.5 (e), $L(I_{\mathbf{c}}(V_2))$ is the only eigenvalue of $H_{\mathbf{c},V_2}(\theta_{\mathbf{c}})$ in $I_{\mathbf{c}}(V_2)$. Thus, our working assumption, $L(I_{\mathbf{c}}(V_2)) < L(I^1_{[\mathbf{c},1]}(V_2)) = L(K_{[\mathbf{c},0]}(V_2))$ implies that $L(K_{[\mathbf{c},0]}(V_2))$ is not an eigenvalue of $H_{\mathbf{c},V_2}(\theta_{\mathbf{c}})$. □

6. Proving that backward type implies forward type

In this section we prove Proposition 3.1, asserting that a fixed backward type implies a fixed forward type.

We recall (Definition 2.10) that a spectral band is of m-type A (respectively B) if it is both of backward type A (B) and of m-forward type A (B). We also recall from Definition 2.13 the notion of the (m, V)-property (or simply $P(m, V)$) for spectral bands. For $\mathbf{c} \in \mathcal{C}$ and $m \in \mathbb{N}$ such that $\varphi(\mathbf{c}) \notin \{-1, \infty\}$ and $[\mathbf{c}, m] \in \mathcal{C}$, the associated critical value is given by

$$V_{\text{crit}}([\mathbf{c}, m]) := \inf \left\{ V \geq 0 : \text{ each spectral band } I_{\mathbf{c}} \text{ in } \sigma_{\mathbf{c}} \text{ satisfies } P(m, V) \right\}$$

We first state two lemmata and two corollaries leading to Proposition 3.1. Before that, we slightly relax the notion of $P(m, V)$ and the definition of V_{crit}.

Definition 6.1. Let $m \in \mathbb{N}$, $\mathbf{c} \in \mathcal{C}$ be such that $\varphi(\mathbf{c}) \notin \{-1, \infty\}$ and $[\mathbf{c}, m] \in \mathcal{C}$. For $V > 0$ a spectral band $I_{\mathbf{c}}$ in $\sigma_{\mathbf{c}}$ satisfies $P^{quasi}(m, V)$ (the *quasi (m, V)-property*) if

(a) either
  - for all $V' \geq V$, $I_{\mathbf{c}}(V')$ is of backward type A with $M = m - 1$,

  or
  - for all $V' \geq V$, $I_{\mathbf{c}}(V')$ is of backward type B with $M = m$,

(b) for all $V' \geq V$ and all $n \in \mathbb{N}$, the unique spectral bands $\left\{ I^i_{[\mathbf{c},m]}(V') \right\}_{i=1}^{M}$ of $\sigma_{[\mathbf{c},m]}$ and the unique spectral bands $\left\{ I^j_{[\mathbf{c},m,n]}(V') \right\}_{j=1}^{M+1}$ of $\sigma_{[\mathbf{c},m,n]}$ satisfy (A1), (B) and (I) of Definition 2.9.

With this notion at hand, define

$$V^{quasi}_{\text{crit}}([\mathbf{c}, m]) := \inf \left\{ V \geq 0 : \text{ each spectral band } I_{\mathbf{c}} \text{ in } \sigma_{\mathbf{c}} \text{ satisfies } P^{quasi}(m, V) \right\}.$$

We note that the only difference compared to Definition 2.13 is that here we do not require the property (A2) to hold for the spectral bands $\left\{ I^i_{[\mathbf{c},m]} \right\}_{i=1}^{M}$.

Noting that

$$V^{quasi}_{\text{crit}}([\mathbf{c}, m]) \leq V_{\text{crit}}([\mathbf{c}, m]),$$

the strategy for proving Proposition 3.1 is to first show $V^{quasi}_{\text{crit}}([\mathbf{c}, m]) = 0$ and afterwards $V_{\text{crit}}([\mathbf{c}, m]) = 0$.

Lemma 6.2. *Let $m \in \mathbb{N}$, $\mathbf{c} \in \mathcal{C}$ be such $\varphi(\mathbf{c}) \notin \{-1, \infty\}$ and $[\mathbf{c}, m] \in \mathcal{C}$, and $V_0 \geq V^{quasi}_{\text{crit}}([\mathbf{c}, m])$. Let $I_{\mathbf{c}}$ be a spectral band in $\sigma_{\mathbf{c}}$ such that both of the following hold:*

(a) $I_{\mathbf{c}}(V)$ is either of backward type A for all $V > 0$ or of backward type B for all $V > 0$,
(b) $I^1_{[\mathbf{c},m,1]}(V)$, $I^{M+1}_{[\mathbf{c},m,1]}(V) \subseteq_{\text{str}} I_{\mathbf{c}}(V)$ holds for all $V \geq V_0$, where

$$M := \begin{cases} m - 1 & I_{\mathbf{c}}(V) \text{ is of backward type } A \text{ for all } V > 0, \\ m & I_{\mathbf{c}}(V) \text{ is of backward type } B \text{ for all } V > 0. \end{cases}$$

If $V_0 > 0$, then there is a $\delta > 0$ such that $I_{\mathbf{c}}$ satisfies $P^{quasi}(m, V_0 - \delta)$.

Remark. The δ in the statement of the lemma only depends on the values $\left|L\left(I^1_{[\mathbf{c},m,1]}(V_0)\right) - L\left(I_{\mathbf{c}}(V_0)\right)\right|$, $\left|R\left(I^{M+1}_{[\mathbf{c},m,1]}(V_0)\right) - R\left(I_{\mathbf{c}}(V_0)\right)\right|$ and V_0. Here we use that the Lipschitz continuity in Corollary 2.3 is independent in $\mathbf{c} \in \mathcal{C}$.

Proof. Since for $V > 4$, $I_{\mathbf{c}}(V)$ is either of m-type A or of m-type B (by Theorem 2.11), we may proceed assuming that $V_0 \leq 4$. Since $I_{\mathbf{c}}(V)$ is either of backward type A or backward type B for all $V > 0$, it is sufficient to prove the existence of a $\delta > 0$ such that

$$\text{all } I^i_{[\mathbf{c},m]} \text{ and } I^j_{[\mathbf{c},m,n]} \text{ satisfy properties (A1), (B1), (B2) and (I) for all } n \in \mathbb{N} \tag{6.1}$$

for all $V > V_0 - \delta$. Since by the assumptions of the lemma, $V_0 \geq V_{\text{crit}}^{quasi}$, we get that (6.1) holds for all $V > V_0$. In particular, we infer that

$$I^i_{[\mathbf{c},m]}(V), I^j_{[\mathbf{c},m,n]}(V) \subseteq_{\text{str}} \left[L\left(I^1_{[\mathbf{c},m,1]}(V)\right), R\left(I^{M+1}_{[\mathbf{c},m,1]}(V)\right)\right], \tag{6.2}$$

for all $V > V_0$, $1 \leq i \leq M$, $1 \leq j \leq M+1$ and $n \in \mathbb{N}$.

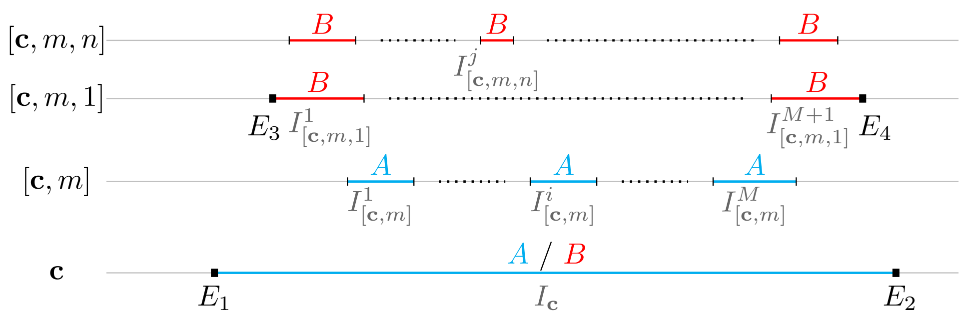

FIGURE 6.1. A sketch of the spectral bands considered in the proof of Lemma 6.2.

Define

$$E_1(V_0) := L\left(I_{\mathbf{c}}(V_0)\right), \qquad E_2(V_0) := R\left(I_{\mathbf{c}}(V_0)\right),$$

and

$$E_3(V_0) := L\left(I^1_{[\mathbf{c},m,1]}(V_0)\right), \qquad E_4(V_0) := R\left(I^{M+1}_{[\mathbf{c},m,1]}(V_0)\right),$$

confer Figure 6.1. Let $V \mapsto E(V)$ be a spectral band edge of $I^i_{[\mathbf{c},m]}(V)$ or $I^j_{[\mathbf{c},m,n]}(V)$ for some $1 \leq i \leq M$ or $1 \leq j \leq M+1$. By Corollary 2.3, the spectral band edges vary continuously in V, namely $V \mapsto E(V)$ is continuous, Thus, Equation (6.2) and assumption (b) yield

$$E_1(V_0) < E_3(V_0) \leq E(V_0) \leq E_4(V_0) < E_2(V_0).$$

Hence, $\min_{k\in\{1,2\}} |E(V_0) - E_k(V_0)| \geq 3\delta$ where

$$\delta := \frac{1}{3}\min\left\{|E_1(V_0) - E_3(V_0)|,\ |E_4(V_0) - E_2(V_0)|,\ V_0\right\} > 0.$$

Now, we use

$$\max\{|E(V) - E(V_0)|,\ |E_i(V) - E_i(V_0)|\} \leq |V - V_0|, \qquad i \in \{1,2,3,4\},$$

which holds by Corollary 2.3, to conclude

$$E_3(V) < E(V) < E_4(V), \quad V > V_0, \qquad \Longrightarrow \qquad E_1(V) < E(V) < E_2(V), \qquad V > V_0 - \delta.$$

We note that $V_0 - \delta > 0$ holds, by the definition of δ.

Since $E(V)$ was an arbitrary spectral edge of $I^j_{[\mathbf{c},m]}(V)$ or $I^j_{[\mathbf{c},m,n]}(V)$ for $n \in \mathbb{N}$, we deduce for all $V > V_0 - \delta$, $1 \leq i \leq M$, $1 \leq j \leq M+1$, and $n \in \mathbb{N}$,

$$I^j_{[\mathbf{c},m]}(V) \subseteq_{\text{str}} I_{\mathbf{c}}(V) \qquad \text{and} \qquad I^j_{[\mathbf{c},m,n]}(V) \subseteq_{\text{str}} I_{\mathbf{c}}(V).$$

Now, we apply Corollary 5.12 which implies that $I_{\mathbf{c}}(V)$ satisfies the forward properties (A1), (B2) and (I) for all $V > V_0-\delta$. Since $I^j_{[\mathbf{c},m,n]}(V) \subseteq_{\text{str}} I_{\mathbf{c}}(V)$ holds for all $n \in \mathbb{N}$ and $V > V_0-\delta$, Corollary 5.14 implies that $I_{\mathbf{c}}(V)$ satisfies (B1) for all $V > V_0 - \delta$. ☐

We now apply Lemma 6.2 for all spectral bands $I_{\mathbf{c}}$ in $\sigma_{\mathbf{c}}$. Using that the number of spectral bands in $\sigma_{\mathbf{c}}$ is finite and taking the minimum δ among all $I_{\mathbf{c}}$ in $\sigma_{\mathbf{c}}$ (the δ which is provided by Lemma 6.2), we get:

Corollary 6.3. *Let $V_1 > 0$, $m \in \mathbb{N}$ and $\boldsymbol{c} \in \mathcal{C}$ be such that $\varphi(\boldsymbol{c}) \notin \{-1,\infty\}$ and $[\boldsymbol{c}, m] \in \mathcal{C}$. Suppose that each spectral band $I_{\boldsymbol{c}}$ in $\sigma_{\boldsymbol{c}}$ satisfies both of the following:*

(a) $I_{\boldsymbol{c}}(V)$ is either of backward type A for all $V > 0$ or of backward type B for all $V > 0$,
(b) $I^1_{[\boldsymbol{c},m,1]}(V),\ I^{M+1}_{[\boldsymbol{c},m,1]}(V) \subseteq_{\text{str}} I_{\boldsymbol{c}}(V)$ holds for all $V \geq V_1$, where

$$M := \begin{cases} m-1 & I_{\boldsymbol{c}}(V) \text{ is of backward type } A \text{ for all } V > 0, \\ m & I_{\boldsymbol{c}}(V) \text{ is of backward type } B \text{ for all } V > 0. \end{cases}$$

Then $V^{quasi}_{\text{crit}}([\boldsymbol{c}, m]) < V_1$. In particular, if (b) holds for all $V_1 > 0$, then $V^{quasi}_{\text{crit}}([\boldsymbol{c}, m]) = 0$.

Proof. Set $V_0 := V^{quasi}_{\text{crit}}([\mathbf{c}, m])$. First of all note that $V_0 \leq 4$ by Theorem 2.11. We seek to prove $V_0 < V_1$. Assume by contradiction $V_0 \geq V_1$.

Let $I_{\mathbf{c}}(V)$ be a spectral band in $\sigma_{\mathbf{c}}(V)$, which by (a) is either of backward type A for all $V > 0$ or of backward type B for all $V > 0$. Since $V_0 \geq V_1$, (b) implies

$$I^1_{[\mathbf{c},m,1]}(V_0),\ I^{M+1}_{[\mathbf{c},m,1]}(V_0) \subseteq_{\text{str}} I_{\mathbf{c}}(V_0).$$

Due to Lemma 6.2 and $V_0 \geq V_1 > 0$, there exists a $\delta := \delta(I_{\mathbf{c}}(V)) > 0$ such that $I_{\mathbf{c}}$ satisfies $P^{quasi}(m, V_0 - \delta)$. Since there are at most finitely spectral bands in $\sigma_{\mathbf{c}}(V)$, we can take the minimum of all these $\delta(I_{\mathbf{c}}(V))$'s and denote it by $\delta' > 0$. Then every spectral band $I_{\mathbf{c}}$ in $\sigma_{\mathbf{c}}$ satisfies $P^{quasi}(m, V_0 - \delta')$. Hence, by definition of $V^{quasi}_{\text{crit}}([\mathbf{c}, m])$, we conclude

$$V_0 := V^{quasi}_{\text{crit}}([\mathbf{c}, m]) \leq V_0 - \delta',$$

a contradiction. ☐

By adding an additional condition to the assumption of Lemma 6.2 we may get a stronger implication, which is done in the following lemma.

Lemma 6.4. *Let $m \in \mathbb{N}$, $\boldsymbol{c} \in \mathcal{C}$ with $\varphi(\boldsymbol{c}) \notin \{-1,\infty\}$ and $[\boldsymbol{c}, m] \in \mathcal{C}$, and $V_0 \geq V_{\text{crit}}([\boldsymbol{c}, m])$. Let $I_{\boldsymbol{c}}$ be a spectral band in $\sigma_{\boldsymbol{c}}$ such that all of the following hold:*

(a) $I_{\boldsymbol{c}}(V)$ is either of backward type A for all $V > 0$ or of backward type B for all $V > 0$,
(b) $I^1_{[\boldsymbol{c},m,1]}(V),\ I^{M+1}_{[\boldsymbol{c},m,1]}(V) \subseteq_{\text{str}} I_{\boldsymbol{c}}(V)$ holds for all $V \geq V_0$, where

$$M := \begin{cases} m-1 & I_{\boldsymbol{c}}(V) \text{ is of backward type } A \text{ for all } V > 0, \\ m & I_{\boldsymbol{c}}(V) \text{ is of backward type } B \text{ for all } V > 0, \end{cases}$$

(c) if $m = 1$, then $\varphi(\boldsymbol{c}) \neq 1$,
if $m \geq 2$, then $V^{quasi}_{\text{crit}}([\boldsymbol{c}, m-1]) = 0$.

If $V_0 > 0$, then there is a $\delta > 0$ such that $I_{\boldsymbol{c}}$ satisfies $P(m, V_0 - \delta)$.

Proof. Applying Lemma 6.2, which is justified by assumptions (a) and (b) here, there exists $\delta > 0$ such that $I_{\mathbf{c}}$ satisfies $P^{quasi}(m, V_0 - \delta)$. Thus, we only have to show that $I_{\mathbf{c}}(V)$ also satisfies the forward property (A2) for all $V > V_0 - \delta$. We consider the following two cases:

Case 1: $(m = 1)$ If $m = 1$ and $I_{\mathbf{c}}(V)$ is of backward type A then it does not contain any spectral band of $\sigma_{[\mathbf{c},1]}(V)$ for all $V > 4$ by Theorem 2.11. Hence, there are no $I^i_{[\mathbf{c},1]}$ spectral bands (see Definition 2.12) and there is nothing to prove in this case. We need only to deal with the case $m = 1$ when $I_{\mathbf{c}}(V)$ is of backward type B. Towards doing this, notice that if $\varphi(\mathbf{c}) = 0$ then $\mathbf{c} = [0, 0]$ since $[\mathbf{c}, m] \in \mathcal{C}$ is assumed. But, $\sigma_{[0,0]}(V) = [-2, 2]$ only consists of a backward type A band, see Example 2.8. Hence, when checking the case that $I_{\mathbf{c}}(V)$ is of backward type B, we may further assume $\varphi(\mathbf{c}) \neq 0$.

Combining this with condition (c) of the lemma, we may now assume that $\varphi(\mathbf{c}) \in (0, 1)$ and $I_{\mathbf{c}}(V)$ is of backward type B. By Definition 2.12, since $m = 1$, we have exactly one spectral band $I^1_{[\mathbf{c},1]}(V)$ for which we need to show that it is not of weak backward type B for all $V > V_0 - \delta$. Indeed, Lemma 5.15 implies that for all $V > V_0 - \delta$, $I^1_{[\mathbf{c},1]}(V) \subseteq_{\text{str}} I_{\mathbf{c}}(V)$ and $I^1_{[\mathbf{c},1]}(V)$ is not of weak backward type B.

Case 2: $(m \geq 2)$ We need to show that $I^i_{[\mathbf{c},m]}(V)$ is not of weak backward type B for all $1 \leq i \leq M$ and $V > V_0 - \delta$. Let $1 \leq i \leq M$. We know by Theorem 2.11 that $I^i_{[\mathbf{c},m]}(V)$ is of backward type A in $\sigma_{[\mathbf{c},m]}(V)$ for $V > 4$. Denoting $m' := m - 1 \geq 1$, Proposition 2.7 implies that $I^i_{[\mathbf{c},m]}$ equals to the spectral band $I^i_{[\mathbf{c},m',1]}$ in $\sigma_{[\mathbf{c},m',1]}$, which is of backward type B for $V > 4$. Using Proposition 2.7 again, it suffices to show that $I^i_{[\mathbf{c},m',1]}(V)$ is not of weak backward type A in $\sigma_{[\mathbf{c},m',1]}(V)$ for all $V > V_0 - \delta$.

By assumption (c) for $m \geq 2$, we have $V^{quasi}_{\text{crit}}([\mathbf{c}, m']) = V^{quasi}_{\text{crit}}([\mathbf{c}, m - 1]) = 0$. Hence, for all $V > 0$, $I_{\mathbf{c}}$ satisfies $P^{quasi}(m - 1, V)$. This implies by (B2) that $I^i_{[\mathbf{c},m',1]}(V)$ is not of weak backward type A for all $V > 0$. ☐

We now apply Lemma 6.4 for all spectral bands $I_{\mathbf{c}}$ in $\sigma_{\mathbf{c}}$. Using that the number of spectral bands in $\sigma_{\mathbf{c}}$ is finite and taking the minimum δ among all $I_{\mathbf{c}}$ in $\sigma_{\mathbf{c}}$ (the δ which is provided by Lemma 6.4), we get:

Corollary 6.5. *Let $V_1 > 0$, $m \in \mathbb{N}$ and $\boldsymbol{c} \in \mathcal{C}$ be such that $\varphi(\boldsymbol{c}) \notin \{-1, \infty\}$ and $[\boldsymbol{c}, m] \in \mathcal{C}$. Suppose that each spectral band $I_{\mathbf{c}}$ in $\sigma_{\mathbf{c}}$ satisfies all of the following:*

(a) $I_{\boldsymbol{c}}(V)$ is either of backward type A for all $V > 0$ or of backward type B for all $V > 0$,
(b) $I^1_{[\boldsymbol{c},m,1]}(V)$, $I^{M+1}_{[\boldsymbol{c},m,1]}(V) \subseteq_{\text{str}} I_{\boldsymbol{c}}(V)$ holds for all $V \geq V_1$, where

$$M := \begin{cases} m - 1 & I_{\boldsymbol{c}}(V) \text{ is of backward type } A \text{ for all } V > 0, \\ m & I_{\boldsymbol{c}}(V) \text{ is of backward type } B \text{ for all } V > 0, \end{cases}$$

(c) if $m = 1$, then $\varphi(\boldsymbol{c}) \neq 1$,
if $m \geq 2$, then $V^{quasi}_{\text{crit}}([\boldsymbol{c}, m - 1]) = 0$.

Then $V_{\text{crit}}([\boldsymbol{c}, m]) < V_1$. In particular, if (b) holds for all $V_1 > 0$, then $V_{\text{crit}}([\boldsymbol{c}, m]) = 0$.

Proof. Similarly as in Corollary 6.3, this follows immediately from Lemma 6.4 and the fact that $\sigma_{\mathbf{c}}(V)$ consists only of finitely many spectral bands independent of $V > 0$. ☐

Finally, we are ready to prove Proposition 3.1.

Proof of Proposition 3.1. Since $\varphi(\mathbf{c}) \in (0, 1)$ and each spectral band $I_{\mathbf{c}}(V)$ in $\sigma_{\mathbf{c}}(V)$ is either of backward type A or B for all $V > 0$, Lemma 5.13 implies

$$I^1_{[\mathbf{c},m,1]}(V), I^{M+1}_{[\mathbf{c},m,1]}(V) \subseteq_{\text{str}} I_{\mathbf{c}}(V), \qquad \text{for all } m \in \mathbb{N},\ V > 0, \tag{6.3}$$

where $M = m - 1$ if $I_{\mathbf{c}}$ is of backward type A and $M = m$ if $I_{\mathbf{c}}$ is of backward type B. Now $V_{\text{crit}}([\mathbf{c}, m]) = 0$ is proven by induction over $m \in \mathbb{N}$.

For the induction base, let $m = 1$. Since Equation (6.3) holds for $m = 1$ and $\varphi(\mathbf{c}) \neq 1$, Corollary 6.5 (for $m = 1$) implies $V_{\text{crit}}([\mathbf{c}, 1]) = 0$.

For the induction step, let $m \in \mathbb{N}$ be such that $V_{\rm crit}([\mathbf{c}, m]) = 0$. Thus, $V_{\rm crit}^{quasi}([\mathbf{c}, m]) = 0$ follows as $0 \le V_{\rm crit}^{quasi}([\mathbf{c}, m]) \le V_{\rm crit}([\mathbf{c}, m])$. Since Equation (6.3) holds for $m+1$ and $V_{\rm crit}^{quasi}([\mathbf{c}, m]) = 0$, Corollary 6.5 (for $m+1 \ge 2$) implies $V_{\rm crit}([\mathbf{c}, m+1]) = 0$. □

7. The induction base of the main theorem

This section contains the proof of the induction base which is used in the proof of Theorem 1.1. Specifically, we show in this section that for all $V \neq 0$, the spectral bands in $\sigma_{[0,0]}(V)$ and $\sigma_{[0,0,1]}(V)$ are either of type A or B. For this proof we express the transfer matrices, $M_{\mathbf{c}}(E, V)$, and their traces, $t_{\mathbf{c}}(E, V)$, (see Section 4.2) using the dilated Chebyshev polynomials of the second kind $S_l : \mathbb{R} \to \mathbb{R}$, $l \in \mathbb{N}_0$. These polynomials are defined by

$$S_{-1}(x) := 0, \quad S_0(x) := 1, \quad S_l(x) := xS_{l-1}(x) - S_{l-2}(x),$$

see Appendix II for more details and properties of these polynomials.

Lemma 7.1. *For all $m \in \mathbb{N}$ and $V \in \mathbb{R}$, we have*

$$M_{[0,0]}^m(E, V) = \begin{pmatrix} S_m(E) & -S_{m-1}(E) \\ S_{m-1}(E) & -S_{m-2}(E) \end{pmatrix}, \qquad E \in \mathbb{R}.$$

Proof. We prove this by induction on m. The induction base ($m = 1$) follows just by definition as

$$M_{[0,0]}^1(E, V) = \begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} S_m(E) & -S_{m-1}(E) \\ S_{m-1}(E) & -S_{m-2}(E) \end{pmatrix}$$

using that $S_1(E) = S_0(E)E - S_{-1}(E) = E$. For the induction step, suppose the statement is true for m. Then

$$\begin{aligned} M_{[0,0]}^{m+1}(E, V) = M_{[0,0]}(E, V)M_{[0,0]}^m(E, V) &= \begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} S_m(E) & -S_{m-1}(E) \\ S_{m-1}(E) & -S_{m-2}(E) \end{pmatrix} \\ &= \begin{pmatrix} ES_m(E) - S_{m-1}(E) & -ES_{m-1}(E) + S_{m-2}(E) \\ S_m(E) & -S_{m-1}(E) \end{pmatrix} \\ &= \begin{pmatrix} S_{m+1}(E) & -S_m(E) \\ S_m(E) & -S_{m-1}(E) \end{pmatrix} \end{aligned}$$

proving the statement. □

Lemma 7.2. *For all $E, V \in \mathbb{R}$ and $m \in \mathbb{N}$ the following holds:*

(a) $t_{[0,0,m]}(E, V) = S_m(E) - VS_{m-1}(E) - S_{m-2}(E)$ *for all* $E \in \mathbb{R}$.
(b) $t_{[0,0,1,m]}(E, V) = ES_m(E - V) - 2S_{m-1}(E - V)$ *for all* $E \in \mathbb{R}$.
(c) $t_{[0,0,1,m,1]}(E, V) = ES_{m+1}(E - V) - 2S_m(E - V)$ *for all* $E \in \mathbb{R}$.

Proof. We recall (Section 4.2) that the transfer matrices are recursively defined by

$$M_{[0,0,c_1,\dots,c_k]}(E, V) := M_{[0,0,c_1\dots,c_{k-2}]}(E, V)M_{[0,0,c_1,\dots,c_{k-1}]}(E, V)^{c_k}.$$

(a) Using Lemma 7.1, we get

$$\begin{aligned} t_{[0,0,m]}(E, V) =& tr\left(M_{[0,0,m]}(E, V)\right) = tr\left(M_{[0]}(E, V)M_{[0,0]}^m(E, V)\right) \\ =& tr\left(\begin{pmatrix} 1 & -V \\ 0 & 1 \end{pmatrix} \begin{pmatrix} S_m(E) & -S_{m-1}(E) \\ S_{m-1}(E) & -S_{m-2}(E) \end{pmatrix}\right) \\ =& S_m(E) - VS_{m-1}(E) - S_{m-2}(E). \end{aligned}$$

(b) We first observe that

$$M_{[0,0,1]}(E, V) = M_{[0]}(E, V)M_{[0,0]}(E, V) = \begin{pmatrix} 1 & -V \\ 0 & 1 \end{pmatrix} \begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} E-V & -1 \\ 1 & 0 \end{pmatrix} = M_{[0,0]}(E-V, V).$$

Thus, Lemma 7.1 leads to

$$\begin{aligned} t_{[0,0,1,m]} =& tr\left(M_{[0,0]}(E,V)M_{[0,0,1]}(E,V)^m\right) \\ =& tr\left(M_{[0,0]}(E,V)M_{[0,0]}(E-V,V)^m\right) \\ =& tr\left(\begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} S_m(E-V) & -S_{m-1}(E-V) \\ S_{m-1}(E-V) & -S_{m-2}(E-V) \end{pmatrix}\right) \\ =& ES_m(E-V) - 2S_{m-1}(E-V). \end{aligned}$$

(c) This follows from (b) and Proposition II.2 (a) asserting $t_{[0,0,1,m,1]} = t_{[0,0,1,m+1]}$. □

Example 7.3. We explicitly write here the expressions of a few traces, which will turn to be useful in the sequel. We have

$$t_{[0,0,1,-1]}(E,V) = t_{[0]}(E,V) = t_{[0,0,0]}(E,V) = 2,$$

and

$$\begin{aligned} t_{[0,0]}(E,V) =& E, & t_{[0,0,-1]}(E,V) =& E+V, \\ t_{[0,0,1]}(E,V) =& E-V, & t_{[0,0,2]}(E,V) =& E^2 - EV - 2, \\ t_{[0,0,1,2]}(E,V) =& E^3 - 2E^2V + E(V^2-3) + 2V, & t_{[0,0,3]}(E,V) =& E^3 - E^2V - 3E + V. \end{aligned}$$

Next, we prove two lemmata. The first lemma states that the spectral band $I_{[0,0]}(V) := [-2,2]$ (in $\sigma_{[0,0]}(V)$) is of type A. The second lemma states that the spectral band $I_{[0,0,1]}(V) := [-2+V, 2+V]$ (in $\sigma_{[0,0]}(V)$) is of type B. Hence, both lemmata provide the induction base needed to prove Theorem 1.1.

Lemma 7.4. *Let $I_{[0,0]}(V) := [-2,2]$ be the unique spectral band of $\sigma_{[0,0]}(V)$ for $V > 0$. The following assertions hold for all $V > 0$.*

- *(a) $I_{[0,0]}(V)$ is of backward type A and not of weak backward type B,*
- *(b) For all $m \in \mathbb{N}$, $I_{[0,0]}(V)$ is of m-type A, namely $V_{\text{crit}}([0,0,m]) = 0$,*
- *(c) For all $m \in \mathbb{N}$, $\sigma_{[0,0,m]}(V)$ consists of m spectral bands satisfying*
  - *the left-most $m-1$ spectral bands are of backward type A and not of weak backward type B. These spectral bands are strictly contained in $I_{[0,0]}(V)$.*
  - *the right-most spectral band, which we denote $K_{[0,0,m]}(V)$, is of backward type B but not of weak backward type A. The spectral bands $K_{[0,0,m]}(V)$ (one for each $m \in \mathbb{N}$) satisfy*
  $$I_{[0,0]}(V) \prec K_{[0,0,m]}(V)$$
  and
  $$K_{[0,0,m]}(V) \subseteq_{\text{str}} K_{[0,0,m-1]}(V) \subseteq_{\text{str}} \ldots \subseteq_{\text{str}} K_{[0,0,1]}(V) \subseteq_{\text{str}} K_{[0]}(V),$$
  with the notational convention $K_{[0]}(V) := \mathbb{R} = \sigma_{[0]}(V)$.

For the convenience of the reader, we sketch in Figure 7.1 the spectral bands mentioned in the proof of Lemma 7.4 and their relations ($\prec$ and $\subseteq_{\text{str}}$).

Proof. Part (a) follows from $\sigma_{[0,0,0]}(V) = \sigma_{[0]}(V) = \mathbb{R}$ and $\sigma_{[0,0,-1]}(V) = [-2-V, 2-V]$.

Parts (b) and (c) of the lemma are proven together by induction over $m \in \mathbb{N}$. Towards proving them we denote $M := m-1$, since we are trying to prove that $I_{[0,0]}$ is of m-forward type A, confer Definition 2.9.

Induction base: The induction base consists of $m = 1$ and $m = 2$. We start with proving (b) and (c) for $m = 1$ and $m = 2$.

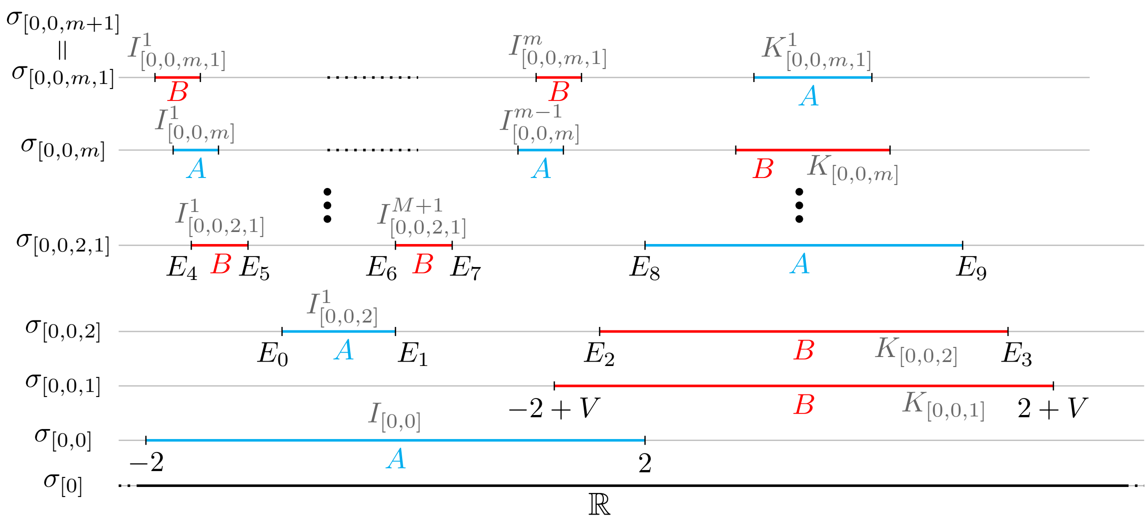

FIGURE 7.1. A sketch of the spectral bands considered in the proof of Lemma 7.4.

Let $m = 1$. Then $\sigma_{[0,0,1]}(V) = [-2+V, 2+V]$ and its unique spectral band $K_{[0,0,1]} := [-2+V, 2+V]$ is of backward type B but not of weak backward type A (as it is strictly contained in $\sigma_{[0]}(V) = \mathbb{R} = K_{[0]}(V)$ and $\sigma_{[0,0]}(V) = I_{[0,0]}(V) \prec K_{[0,0,1]}(V)$). This proves (c) for $m = 1$.

Let $m = 2$. We have $t_{[0,0,2]}(E, V) = E^2 - EV - 2$ (see Example 7.3) implying

$$t_{[0,0,2]}(E,V) = 2 \quad \Longleftrightarrow \quad E = \frac{V}{2} \pm \sqrt{\frac{V^2}{4} + 4},$$
$$t_{[0,0,2]}(E,V) = -2 \quad \Longleftrightarrow \quad E \in \{0, V\}.$$

This motivates to denote

$$E_0(V) := \frac{V}{2} - \sqrt{\frac{V^2}{4} + 4}, \quad E_1(V) := 0, \quad E_2(V) := V, \quad E_3(V) := \frac{V}{2} + \sqrt{\frac{V^2}{4} + 4},$$

so that $E_0(V) < E_1(V) < E_2(V) < E_3(V)$ holds for all $V > 0$. Thus, $I_{[0,0,2]}(V) := [E_0(V), E_1(V)]$ and $K_{[0,0,2]}(V) := [E_2(V), E_3(V)]$ are the two spectral bands in $\sigma_{[0,0,2]}(V)$. Clearly $I_{[0,0,2]}(V)$ is of backward type A for all $V > 0$ since $I_{[0,0,2]}(V) \subsetneq_{\text{str}} [-2, 2] = I_{[0,0]}(V)$. In addition $E_0(V) < -2 + V$ for $V > 0$ and so $I_{[0,0,2]}(V)$ is not contained in $K_{[0,0,1]}(V) = [-2+V, 2+V]$. Hence, $I_{[0,0,2]}(V)$ is not of weak backward type B for all $V > 0$. The spectral band $K_{[0,0,2]}(V)$ is of backward type B since $K_{[0,0,2]}(V) \subseteq_{\text{str}} [-2+V, 2+V] = K_{[0,0,1]}(V)$ for all $V > 0$. In addition, $E_3(V) \geq \frac{V}{2} + 2 > 2$ for all $V > 0$ leading to $I_{[0,0]}(V) \prec K_{[0,0,2]}(V)$. Thus, $K_{[0,0,2]}(V)$ is not of weak backward type A.

Summing up, we have proven (b) and (c), for $m = 1$ and $m = 2$. It is left to show (b) for $m = 1$ and $m = 2$, namely that $V_{\text{crit}}([0,0,1]) = 0$ and $V_{\text{crit}}([0,0,2]) = 0$.

$V_{\text{crit}}([0,0,1]) = 0$: We first aim to apply Corollary 6.3 for $\mathbf{c} = [0,0]$ and $m = 1$. Applying Corollary 6.3 would give that $V_{\text{crit}}^{quasi}([0,0,1]) = 0$ (see Definition 6.1) and then one needs only to show property (A2) in order to conclude $V_{\text{crit}}([0,0,1]) = 0$. But, in this case property (A2) is an empty statement since $M = m - 1 = 0$.

In order to apply Corollary 6.3 for $\mathbf{c} = [0,0]$ and $m = 1$ we note the following. The spectrum $\sigma_{\mathbf{c}}$ has only one spectral band $I_{[0,0]}$ that is of backward type A and not of weak backward type B for all $V > 0$, which is assumption (a) of Corollary 6.3. To check assumption (b) of Corollary 6.3 we need to prove that $I^1_{[0,0,1,1]}(V) \subseteq_{\text{str}} I_{[0,0]}(V)$ for all $V > 0$.

By Proposition 2.7, $\sigma_{[0,0,1,1]} = \sigma_{[0,0,2]}$. Consider the spectral band $I_{[0,0,2]}(V) = [E_0(V), E_1(V)]$, which we calculated above. In particular, we have seen above that $I_{[0,0,2]}(V) \subseteq_{\text{str}} I_{[0,0]}(V)$ holds for all $V > 0$. Therefore $I_{[0,0,2]}(V)$ equals to the unique spectral band $I^1_{[0,0,1,1]}(V)$ by Definition 2.12 . Thus, $I^1_{[0,0,1,1]}(V) \subseteq_{\text{str}} I_{[0,0]}(V)$ for all $V > 0$, which verifies all the assumptions of Corollary 6.3. As explained above, we conclude $V_{\text{crit}}([0,0,1]) = 0$.

$\underline{V_{\text{crit}}([0,0,2]) = 0}$: We aim to apply Corollary 6.5 for $\mathbf{c} = [0,0]$ and $m = 2$ in order to conclude $V_{\text{crit}}([0,0,2]) = 0$. Condition (a) of Corollary 6.5 was already verified above, as the spectral band $I_{[0,0]}$ that is of backward type A and not of weak backward type B for all $V > 0$. We have also proved above $V^{quasi}_{\text{crit}}([0,0,1]) = 0$, which verifies condition (c) of Corollary 6.5. We only have to check condition (b) for all $V > 0$. Specifically, it is sufficient to prove $I^1_{[0,0,2,1]}(V),\ I^{M+1}_{[0,0,2,1]}(V) \subseteq_{\text{str}} I_{[0,0]}(V)$ for all $V > 0$. Note that $M + 1 = m = 2$.

Using Proposition II.2 (a) and Lemma 7.2 we conclude

$$t_{[0,0,2,1]}(E,V) = t_{[0,0,3]}(E,V) = S_3(E) - VS_2(E) - S_1(E)$$

We use this to express all the E values for which $t_{[0,0,2,1]}(E,V) \in \{-2,2\}$:

$$E_4(V) := \frac{V-1}{2} - \frac{\sqrt{V^2+2V+9}}{2}, \qquad E_5(V) := -1,$$
$$E_6(V) := \frac{V+1}{2} - \frac{\sqrt{V^2-2V+9}}{2}, \qquad E_7(V) := 1,$$
$$E_8(V) := \frac{V-1}{2} + \frac{\sqrt{V^2+2V+9}}{2}, \qquad E_9(V) := \frac{V+1}{2} + \frac{\sqrt{V^2-2V+9}}{2},$$

where $E_4(V) < E_5(V) < E_6(V) < E_7(V) < E_8(V) < E_9(V)$. Now, it is straightforward to check that the three spectral bands in $\sigma_{[0,0,2,1]}$ are

$$I^1_{[0,0,2,1]}(V) = [E_4(V), E_5(V)], \qquad I^{M+1}_{[0,0,2,1]}(V) = [E_6(V), E_7(V)]$$

and

$$K_{[0,0,3]}(V) = [E_8(V), E_9(V)].$$

Furthermore, $I^1_{[0,0,2,1]}(V), I^{M+1}_{[0,0,2,1]}(V) \subseteq_{\text{str}} I_{[0,0]}(V)$ for all $V > 0$. Thus, Corollary 6.5 implies $V_{\text{crit}}([0,0,2]) = 0$, hence statement (b) of the current lemma holds for $m = 2$, and this finishes the proof of the induction base.

Induction step: (see Figure 7.1) Let $m \geq 2$ and suppose (induction hypothesis) that $V_{\text{crit}}([0,0,m]) = 0$ and $\sigma_{[0,0,m]}(V)$ satisfies (c) for all $V > 0$.

We have $\varphi([0,0,m+1]) = \frac{1}{m+1}$ and so $\sigma_{[0,0,m+1]}(V)$ consists of exactly $m+1$ spectral bands by Proposition 2.1 and Lemma 4.5. Since $K_{[0,0,m]}(V)$ is of backward type B (and not of weak backward type A) for all $V > 0$, we conclude that $K_{[0,0,m]}(V)$ is of type B for $V > 4$, see Theorem 2.11. Then property (A1) of $K_{[0,0,m]}$ implies that for $V > 4$, there is a spectral band $K_{[0,0,m,1]}(V)$ in $\sigma_{[0,0,m,1]}(V)$ of backward type A such that $K_{[0,0,m,1]}(V) \subseteq_{\text{str}} K_{[0,0,m]}(V)$.

Furthermore (referring again to Theorem 2.11), $I_{[0,0]}(V) = [-2,2]$ is of type A for $V > 4$ and so it strictly contains m spectral bands of type B in $\sigma_{[0,0,m,1]}(V)$ for $V > 4$. Since $\sigma_{[0,0,m,1]}(V)$ has $m+1$ spectral bands, the spectral band $K_{[0,0,m,1]}(V)$ mentioned above satisfies the following: for $V > 4$, it is the unique spectral band in $\sigma_{[0,0,m,1]}(V)$ of backward type A. In addition $K_{[0,0,m,1]}(V) \subseteq_{\text{str}} K_{[0,0,m]}(V)$ (as seen above) and $I_{[0,0]}(V) \prec K_{[0,0,m,1]}(V)$ for $V > 4$ (since $I_{[0,0]}(V) \prec K_{[0,0,m]}(V)$ by the induction hypothesis). Furthermore, the left-most m spectral bands in $\sigma_{[0,0,m,1]}(V)$ are strictly contained in $\sigma_{[0,0]}(V)$ for $V > 4$. By Proposition 2.7, $\sigma_{[0,0,m,1]}(V) = \sigma_{[0,0,m+1]}(V)$, and in particular, we can identify $K_{[0,0,m,1]}(V)$ with a spectral band $K_{[0,0,m+1]}(V)$ in $\sigma_{[0,0,m+1]}(V)$ and $K_{[0,0,m+1]}(V)$ is of backward type B for $V > 4$.

We will show that

(d) $K_{[0,0,m+1]}(V) \subseteq_{\text{str}} K_{[0,0,m]}(V)$ and $K_{[0,0,m+1]}(V) \not\subseteq I_{[0,0]}(V)$ for all $V > 0$,
(e) $I_{[0,0]}(V) \prec K_{[0,0,m+1]}(V)$ for all $V > 0$,
(f) $V_{\text{crit}}([0,0,m+1]) = 0$.

Observe that these statements imply that parts (b) and (c) of the lemma hold for $m+1$. These implications are rather straightforward, and one just needs to notice that to get the first bullet of (c) for $m+1$, one needs also to employ $V_{\text{crit}}([0,0,m+1]) = 0$, which provides the (A1) and (A2) properties of $I_{[0,0]}(V)$ for all $V > 0$.

Proof of (d): Since $m \geq 2$, we have $\varphi([0,0,m]) \in (0,1)$ and by induction hypothesis $K_{[0,0,m]}(V)$ is of backward type B for all $V > 0$. Proposition 2.7 implies $\sigma_{[0,0,m,1]} = \sigma_{[0,0,m+1]}$. Thus, Lemma 5.15 applied to the spectral band $K_{[0,0,m]}(V)$ implies $K_{[0,0,m+1]}(V) = K_{[0,0,m,1]}(V) \subseteq_{\text{str}} K_{[0,0,m]}(V)$ for all $V > 0$. Moreover, Lemma 5.15 asserts that $K_{[0,0,m,1]}(V)$ is not of weak backward type B for all $V > 0$, namely $K_{[0,0,m,1]}(V)$ is not contained in a spectral band of $\sigma_{[0,0,m,1,-1]} = \sigma_{[0,0]} = [-2,2]$. Since $K_{[0,0,m+1]}(V) = K_{[0,0,m,1]}(V)$ holds for all $V > 0$, we conclude $K_{[0,0,m+1]}(V) \not\subseteq [-2,2]$ for $V > 0$.

Proof of (e): For $V > 0$, (d) and $[-2,2] = I_{[0,0]}(V) \prec K_{[0,0,m]}(V)$ imply

$$-2 = L(I_{[0,0]}(V)) < L(K_{[0,0,m]}(V)) < L(K_{[0,0,m+1]}(V)).$$

Furthermore, (d) asserts $K_{[0,0,m+1]}(V) \not\subseteq [-2,2]$ for all $V > 0$ implying $R(I_{[0,0]}(V)) < R(K_{[0,0,m+1]}(V))$ for all $V > 0$. Thus, $I_{[0,0]}(V) \prec K_{[0,0,m+1]}(V)$ follows for $V > 0$.

Proof of (f): Since $\sigma_{[0,0]}(V)$ consists only of the spectral band $I_{[0,0]}(V)$ we need to show that $I_{[0,0]}(V)$ is of $(m+1)$-type A for all $V > 0$. Since $V_{\text{crit}}([0,0,m]) = 0$ and $I_{[0,0]}(V)$ if of backward type A but not of weak backward type B for all $V > 0$, Corollary 6.5 (for $m \geq 2$) asserts that we only have to prove

$$I^1_{[0,0,m+1,1]}(V), I^{M+1}_{[0,0,m+1,1]}(V) \subseteq_{\text{str}} I_{[0,0]}(V)$$

for all $V > 0$ where $M = (m+1) - 1 = m$. Lemma II.1 (b) asserts $S_l(\pm 2) = (\pm 1)^l (l+1)$ for $l \in \mathbb{N}$. Thus, Lemma 7.2 (a) leads to

$$\begin{aligned} \left|t_{[0,0,l+1]}(\pm 2, V)\right| &= \left|S_{l+1}(\pm 2) - V S_l(\pm 2) - S_{l-1}(\pm 2)\right| \\ &= \left|(\pm 1)^{l+1}(l+2) - (\pm 1)^l (l+1) V - (\pm 1)^{l-1} l\right| \\ &= \left|(\pm 1)^{l+1} 2 \mp (\pm 1)^{l+1}(l+1)V\right| \\ &= \left|2 \mp (l+1)V\right|. \end{aligned} \tag{7.1}$$

Hence, we conclude from Proposition II.2 (a) that

$$\left|t_{[0,0,m+1,1]}(-2,V)\right| = \left|t_{[0,0,m+2]}(-2,V)\right| = 2 + (m+2)V > 2, \qquad V > 0.$$

This means that for all $V > 0$, $E = -2$ is not a spectral edge of any spectral band in $\sigma_{[0,0,m+1,1]}(V)$, see Lemma 4.6 (a). Since $E = -2$ is a spectral band edge of $I_{[0,0]}(V)$, and since $I^j_{[0,0,m+1,1]}(V) \subseteq_{\text{str}} I_{[0,0]}(V)$ for $1 \leq j \leq M+1$ and $V > 4$ (by Theorem 2.11), we conclude that $I^j_{[0,0,m+1,1]}(V) \subseteq_{\text{str}} I_{[0,0]}(V)$ may be violated if and only if $R\left(I^j_{[0,0,m+1,1]}(V)\right) \geq 2$. Since $I^j_{[0,0,m+1,1]} \prec I^{M+1}_{[0,0,m+1,1]}$ holds by definition for all $1 \leq j < M+1$, it suffices to show that for all $V > 0$,

$$R\left(I^{M+1}_{[0,0,m+1,1]}(V)\right) < 2. \tag{7.2}$$

For $V \geq \frac{4}{m+1}$, Equation (7.1) and Proposition II.2 (a) lead to

$$\left|t_{[0,0,m+1,1]}(2,V)\right| = \left|t_{[0,0,m+2]}(2,V)\right| = \left|(m+2)V - 2\right| \geq 4\frac{m+2}{m+1} - 2 > 2.$$

Hence, (7.2) holds for all $V \geq \frac{4}{m+1}$, proving $I^1_{[0,0,m+1,1]}(V), I^{M+1}_{[0,0,m+1,1]}(V) \subseteq_{\text{str}} I_{[0,0]}(V)$ for all $V \geq \frac{4}{m+1}$. Recalling also the induction hypothesis, $V_{\text{crit}}([0,0,m]) = 0$, we apply Lemma 6.4 for $m+1 \geq 2$ and $\mathbf{c} = [0,0]$ and conclude that there exists $\delta > 0$ such that $I_{[0,0]}(V)$ is of $(m+1)$-type A for $V > \frac{4}{m+1} - \delta$. Since, $I_{[0,0]}$ is the only spectral band in $\sigma_{[0,0]}$ this implies $V_{\text{crit}}([0,0,m+1]) \leq \frac{4}{m+1} - \delta$. Thus, it is left to prove (7.2) for $0 < V \leq \frac{4}{m+1} - \delta$. Equation (7.1) together with $0 < V \leq \frac{4}{m+1} - \delta$ implies

$$\left|t_{[0,0,m+1]}(2,V)\right| = \left|2 - (m+1)V\right| < 2,$$

so that there is a spectral band of $\sigma_{[0,0,m+1]}(V)$ which contains $E = 2$, for $0 < V \leq \frac{4}{m+1} - \delta$. But, since $\sigma_{[0,0,m,1]} = \sigma_{[0,0,m+1]}$ (Proposition 2.7) and by the induction hypothesis, $I^m_{[0,0,m,1]}(V) \subseteq_{\text{str}} I_{[0,0]}(V)$ for $V > 0$, the only spectral band which can contain $E = 2$ is $K^1_{[0,0,m,1]}(V) = K_{[0,0,m+1]}(V)$, and so

$$L\left(K_{[0,0,m+1]}(V)\right) < R\left(I_{[0,0]}(V)\right) = 2 \qquad \text{for } 0 < V \leq \frac{4}{m+1} - \delta.$$

In order to conclude (7.2) for $0 < V \leq \frac{4}{m+1} - \delta$, we will apply Lemma 5.7 for $[0,0], [0,0,m+1], [0,0,m+1,1] \in \mathcal{C}$ and $\lambda_{\mathbf{o}} = R(I_{[0,0]}(V))$ and $\mu_{\mathbf{o}} = R(I^{M+1}_{[0,0,m+1,1]}(V))$. A direct computation invoking Lemma 5.4 and Lemma 5.5 yields

$$R(I_{[0,0]}(V)) \in \sigma\left(H_{[0,0],V}(0)\right), \; L(K_{[0,0,m+1]}(V)) \in \sigma\left(H^{\times 1}_{[0,0,m+1],V}(\pi)\right)$$

and

$$R(I^{M+1}_{[0,0,m+1,1]}(V)) \in \sigma\left(H_{[0,0,m+1,1],V}(\pi)\right).$$

Thus, $\theta_{[0,0]} = 0$, $\theta_{[0,0,m+1]} = \pi$, $\theta_{[0,0,m+1,1]} = \pi$ and these spectral edges are admissible (Definition 4.3). Moreover, we can directly compute the values of the counting function

$$N_{[0,0]} := N(R(I_{[0,0]}(V));\; H_{[0,0],V}(0)) = 0,$$
$$N_{[0,0,m+1,1]} := N(R(I^{M+1}_{[0,0,m+1,1]}(V);\; H_{[0,0,m+1,1],V}(\pi)) = m,$$

and using $L\left(K_{[0,0,m+1]}(V)\right) < R\left(I_{[0,0]}(V)\right)$for $0 < V \leq \frac{4}{m+1} - \delta$, we get

$$N_{[0,0,m+1]} := N(R(I_{[0,0]}(V));\; H^{\times 1}_{[0,0,m+1],V}(\pi)) = m+1.$$

Hence, $N_{[0,0]} + N_{[0,0,m+1]} > N_{[0,0,m+1,1]}$ follows. Moreover, $L\left(K_{[0,0,m+1]}(V)\right) < R\left(I_{[0,0]}(V)\right) = \lambda_{\mathbf{o}}$ for $0 < V \leq \frac{4}{m+1} - \delta$ implies that $\lambda_{\mathbf{o}}$ is a simple eigenvalue in $H^{\times 1}_{[0,0,m+1],V}(\pi) \oplus H_{[0,0],V}(0)$. Thus, Lemma 5.7 (a) yields

$$2 = R(I_{[0,0]}(V)) = \lambda_{\mathbf{o}} > R(I^{M+1}_{[0,0,m+1,1]}(V)) = \mu_{\mathbf{o}}.$$

Hence, we have proven (7.2) for all $0 < V \leq \frac{4}{m+1} - \delta$. Thus, Corollary 6.5 (for $m \geq 2$) implies $V_{\text{crit}}([0,0,m+1]) = 0$ proving (f). □

Next we prove that the unique spectral band $I_{[0,0,1]}(V) = [-2+V, 2+V]$ of $\sigma_{[0,0,1]}(V)$ is of type B for all $V > 0$. This is demonstrated in Figure 7.2.

Lemma 7.5. *Let $I_{[0,0,1]}(V) = [-2+V, 2+V]$ be the unique spectral band of $\sigma_{[0,0,1]}(V)$ for $V \in \mathbb{R}$. The following holds for all $V > 0$:*

(a) $I_{[0,0,1]}(V)$ is of backward type B but not of weak backward type A,
(b) For all $m \in \mathbb{N}$, $I_{[0,0,1]}(V)$ is of m-type B, i.e. $V_{\text{crit}}([0,0,1,m]) = 0$.

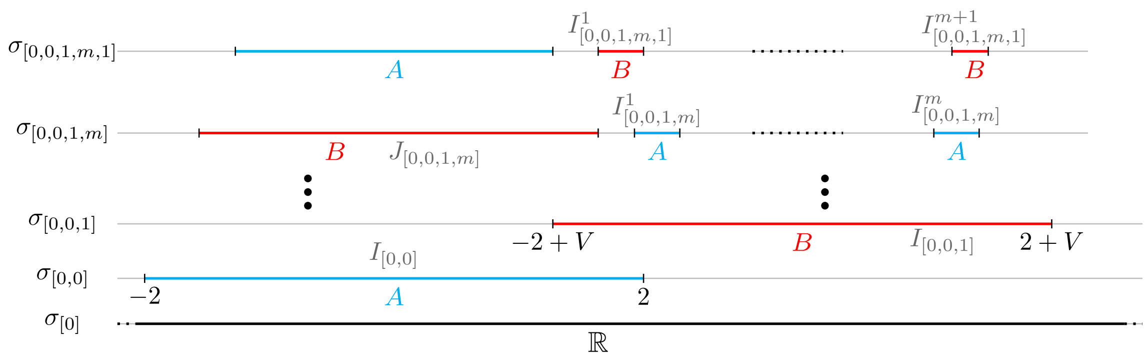

FIGURE 7.2. A sketch of the spectral bands considered in the proof of Lemma 7.5.

Proof. Statement (a) follows immediately from the equalities $\sigma_{[0,0,1,0]}(V) = \sigma_{[0,0]}(V) = [-2,2]$ and $\sigma_{[0,0,1,-1]}(V) = \sigma_{[0]}(V) = \mathbb{R}$ for all $V > 0$.

In order to prove (b), let $m \in \mathbb{N}$ and $M = m$ since we aim to show that $I_{[0,0,1]}$ is of m-type B, see Definition 2.9. This will be done in two steps: first by applying Lemma 6.2 to show $V_{\text{crit}}^{quasi}([0,0,1,m]) = 0$ for all $m \in \mathbb{N}$; then applying Corollary 6.5 to show $V_{\text{crit}}([0,0,1,m]) = 0$ for all $m \in \mathbb{N}$. In order to apply both corollaries we need to show that

$$I^1_{[0,0,1,m,1]}(V), I^{M+1}_{[0,0,1,m,1]}(V) \subseteq_{\text{str}} I_{[0,0,1]}(V), \qquad V > 0. \tag{7.3}$$

<u>Step 1:</u> We prove $V_{\text{crit}}^{quasi}([0,0,1,m]) = 0$. Let $V \geq \frac{4}{m+1}$. Lemma II.1 (b) asserts $S_l(\pm 2) = (\pm 1)^l(l+1)$. Thus, Lemma 7.2 (c) leads to

$$\begin{aligned} \left|t_{[0,0,1,m,1]}(\pm 2 + V, V)\right| &= \left|(\pm 2 + V)S_{m+1}(\pm 2) - 2S_m(\pm 2)\right| \\ &= \left|(\pm 2 + V)(m+2) \mp 2(m+1)\right| \\ &= \left|V(m+2) \pm 2\right|. \end{aligned}$$

Thus, $\left|t_{[0,0,1,m,1]}(\pm 2 + V, V)\right| > 2$ follows whenever $V \geq \frac{4}{m+1}$. The values $\pm 2 + V$, at which the trace is evaluated, are the spectral band edges of $I_{[0,0,1]}(V)$. By Theorem 2.11, the strict inclusions (7.3) hold for $V > 4$. Thus, by continuity of the spectral edges (Corollary 2.3) and $\left|t_{[0,0,1,m,1]}(\pm 2+V, V)\right| > 2$, (7.3) holds for all $V \geq \frac{4}{m+1}$. Hence, Lemma 6.2 implies that there is a $\delta > 0$ such that $I_{[0,0,1]}$ satisfies $P^{quasi}\left(m, \frac{4}{m+1} - \delta\right)$, namely $V_{\text{crit}}^{quasi}([0,0,1,m]) \leq \frac{4}{m+1} - \delta$.

Now let us consider the range $0 < V \leq \frac{4}{m+1} - \delta$. First, for all $V > 0$, the trace computation above gives

$$\left|t_{[0,0,1,m,1]}(2 + V, V)\right| = V(m+2) + 2 > 2.$$

Hence, $R(I^{M+1}_{[0,0,1,m,1]}(V)) < 2+V$ follows for all $V > 0$, since it holds for $V > 4$ by Theorem 2.11, and using continuity (Corollary 2.3). Since $I^1_{[0,0,1,m,1]}(V) \prec I^{M+1}_{[0,0,1,m,1]}(V)$ holds for all $V > 0$, it thus suffices to prove

$$-2 + V < L(I^1_{[0,0,1,m,1]}(V)) \qquad \text{for all } 0 < V \leq \tfrac{4}{m+1} - \delta. \tag{7.4}$$

Let $J_{[0,0,1,m]}$ be the spectral band associated with $I_{[0,0,1]}$ of Definition 5.8, see also Figure 7.2. Hence, it is the spectral band satisfying $\text{ind}(J_{[0,0,1,m]}) = \text{ind}(I^1_{[0,0,1,m]}) - 1$. Lemma 7.2 (b), Lemma II.1 (b) and $0 < V \leq \frac{4}{m+1} - \delta$ lead to

$$\begin{aligned} \left|t_{[0,0,1,m]}(-2 + V, V)\right| &= \left|(-2 + V)S_m(-2) - 2S_{m-1}(-2)\right| \\ &= \left|(-2 + V)(-1)^m(m+1) - 2(-1)^{m-1}m\right| \\ &= \left|(-2 + V)(m+1) + 2m\right| \end{aligned}$$

$$=\left|V(m+1)-2\right|<2.$$

By Lemma 4.5, we conclude that $-2+V$ is in the interior of a spectral band of $\sigma_{[0,0,1,m]}(V)$. On the other hand, $-2+V<L(I^1_{[0,0,1,m]}(V))$ holds for $V>\frac{4}{m+1}-\delta$. Thus, the continuity of the spectral edges in V implies $-2+V<R(J_{[0,0,1,m]})$ for $0\leq V\leq\frac{4}{m+1}-\delta$.

With this at hand, we prove (7.4) by applying Lemma 5.7 with $\mathbf{c}:=[0,0,1]$, $[\mathbf{c},m]=[0,0,1,m]$, $[\mathbf{c},m,n]=[0,0,1,m,1]$, $\lambda_{\mathbf{o}}=L(I_{[0,0,1]}(V))=-2+V$ and $\mu_{\mathbf{o}}=L(I^1_{[0,0,1,m,1]}(V))$. Let $\theta_{\mathbf{c}},\theta_{[\mathbf{c},m]},\theta_{[\mathbf{c},m,1]}\in\{0,\pi\}$ be such that

$$L(I_{[0,0,1]}(V))\in\sigma\left(H_{[0,0,1],V}(\theta_{\mathbf{c}})\right),\ R(J_{[0,0,1,m]}(V))\in\sigma\left(H^{\times 1}_{[0,0,1,m],V}(\theta_{[\mathbf{c},m]})\right)$$

and

$$L(I^1_{[0,0,1,m,1]}(V))\in\sigma\left(H_{[0,0,1,m,1],V}(\theta_{[\mathbf{c},m,1]})\right).$$

A direct computation yields $\operatorname{ind}(I_{[0,0,1]})=0$, $\operatorname{ind}(J_{[0,0,1,m]})=0$ and $\operatorname{ind}(I^1_{[0,0,1,m,1]})=1$, see Figure 7.2. Inserting these indices into the characterization of admissibility in Lemma 5.6, we conclude that these spectral edges are admissible. Furthermore, we can directly compute the values of the counting function

$$N_{\mathbf{c}}:=N(L(I_{[0,0,1]}(V));\ H_{[0,0,1],V}(\theta_{\mathbf{c}}))=0,$$
$$N_{[\mathbf{c},m,1]}:=N(L(I^1_{[0,0,1,m,1]}(V);\ H_{[0,0,1,m,1],V}(\theta_{[0,0,1,m,1]}))=1,$$

and using $L(I_{[0,0,1]}(V))=-2+V<R(J_{[0,0,1,m]})$ for $0<V\leq\frac{4}{m+1}-\delta$, we get

$$N_{[\mathbf{c},m]}:=N(L(I_{[0,0,1]}(V));\ H^{\times 1}_{[0,0,1,m],V}(\theta_{[\mathbf{c},m]}))=0.$$

Hence, $N_{\mathbf{c}}+N_{[\mathbf{c},m]}<N_{[\mathbf{c},m,1]}$ follows. Moreover, $\lambda_{\mathbf{o}}=L(I_{[0,0,1]}(V))<R(J_{[0,0,1,m]}(V))$ for $0<V\leq\frac{4}{m+1}-\delta$ implies that $\lambda_{\mathbf{o}}$ is a simple eigenvalue in $H^{\times 1}_{[0,0,1,m],V}(\theta_{[\mathbf{c},m]})\oplus H_{[0,0,1],V}(\theta_{\mathbf{c}})$. Thus, Lemma 5.7 yields $-2+V=\lambda_{\mathbf{o}}<\mu_{\mathbf{o}}=L(I^1_{[0,0,1,m,1]}(V))$ proving (7.4) for all $0<V\leq\frac{4}{m+1}-\delta$. Hence, $V^{quasi}_{\text{crit}}([0,0,1,m])=0$ follows.

Step 2: We prove $V_{\text{crit}}([0,0,1,m])=0$. Let $m\geq 2$. We aim to apply Corollary 6.5 and need to check its assumptions (a), (b) and (c). We have seen above that $\sigma_{[0,0,1]}(V)$ has exactly one spectral band $I_{[0,0,1]}$ which is of backward type B but not of weak backward type A, so that assumption (a) of Corollary 6.5 holds. By step 1 of the proof, we have $V^{quasi}_{\text{crit}}([0,0,1,m])=0$ and $V^{quasi}_{\text{crit}}([0,0,1,m-1])=0$. The first implies that assumption (b) in Corollary 6.5 holds and the second implies that assumption (c) in Corollary 6.5 holds using $m\geq 2$. Hence, $V_{\text{crit}}([0,0,1,m])=0$ follows for all $m\geq 2$.

Let $m=1$. Since $\varphi([0,0,1])=1$, assumption (c) in Corollary 6.5 does not hold for $m=1$, and we cannot apply that corollary. Instead, we directly verify that $V_{\text{crit}}([0,0,1,1])=0$. Recall $t_{[0,0,1,1]}=t_{[0,0,2]}=E^2-EV-2$, see Example 7.3. Thus, $\sigma_{[0,0,1,1]}(V)=[E_0(V),E_1(V)]\cup[E_2(V),E_3(V)]$ with

$$E_0(V):=\frac{V}{2}-\sqrt{\frac{V^2}{4}+4},\quad E_1(V):=0,\quad E_2(V):=V,\quad E_3(V):=\frac{V}{2}+\sqrt{\frac{V^2}{4}+4}.$$

Thus, $I_{[0,0,1,1]}(V)=[E_2(V),E_3(V)]\subseteq_{\text{str}}I_{[0,0,1]}(V)$ and $I_{[0,0,1,1]}(V)$ is the unique spectral band in $\sigma_{[0,0,1,1]}(V)$ of backward type A. Since $2<E_3(V)$ for all $V>0$, we conclude that $I_{[0,0,1,1]}(V)$ is not included in $I_{[0,0]}(V)$ and hence it is not of weak backward type B for all $V>0$. Thus, $I_{[0,0,1]}(V)$ satisfies (A2) for all $V>0$. Since $V^{quasi}_{\text{crit}}([0,0,1,1])=0$ by step 1, we conclude $V_{\text{crit}}([0,0,1,1])=0$. ☐

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References

[BBB+26] R. Band, S. Beckus, B. Biber, L. Raymond, and Y. Thomas, *A review of a work by L. Raymond: Sturmian Hamiltonians with a large coupling constant—periodic approximations and gap labels*, 1–83.

[BBL23] R. Band, S. Beckus, and R. Loewy, *Workshop: Aspects of Aperiodic Order*, Oberwolfach Rep. (2023), extended version in arXiv:2309.04351.

[BBL24] ———, *The dry ten Martini problem for Sturmian Hamiltonians*, arXiv:2402.16703 (2024).

[BBT25] S. Beckus, J. Bellissard, and Y. Thomas, *Spectral regularity and defects for the Kohmoto model*, Ann. Henri Poincaré (2025), doi:10.1007/s00023-025-01578-8.

[BIST89] J. Bellissard, B. Iochum, E. Scoppola, and D. Testard, *Spectral properties of one-dimensional quasi-crystals*, Comm. Math. Phys. **125** (1989), no. 3, 527–543.

[Cas86] M. Casdagli, *Symbolic dynamics for the renormalization map of a quasiperiodic Schrödinger equation*, Comm. Math. Phys. **107** (1986), no. 2, 295–318.

[CQ25] J. Cao and Y. Qu, *Almost sure dimensional properties for the spectrum and the density of states of Sturmian Hamiltonians*, Advances in Mathematics **478** (2025), 110387.

[DEGT08] D. Damanik, M. Embree, A. Gorodetski, and S. Tcheremchantsev, *The fractal dimension of the spectrum of the Fibonacci Hamiltonian*, Communications in Mathematical Physics 280 (2008).

[DF22] D. Damanik and J. Fillman, *One-dimensional ergodic Schrödinger operators—I. General theory*, vol. 221, American Mathematical Society, Providence, RI, 2022.

[DF25] ———, *One-dimensional ergodic Schrödinger operators—II. Specific classes*, American Mathematical Society, Providence, RI, 2025.

[DG15] D. Damanik and A. Gorodetski, *Almost sure frequency independence of the dimension of the spectrum of Sturmian Hamiltonians*, Comm. Math. Phys. **337** (2015), no. 3, 1241–1253.

[DLMF] *NIST Digital Library of Mathematical Functions*, `https://dlmf.nist.gov/`, Release 1.1.12 of 2023-12-15, F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.

[Fog02] N. Pytheas Fogg, *Substitutions in dynamics, arithmetics and combinatorics*, Lecture Notes in Mathematics, vol. 1794, Springer-Verlag, Berlin, 2002. MR 1970385

[HJ13] R. A. Horn and C. R. Johnson, *Matrix analysis*, second ed., Cambridge University Press, Cambridge, 2013.

[Hoc75] H. Hochstadt, *On the theory of Hill's matrices and related inverse spectral problems*, Linear Algebra Appl. **11** (1975), 41–52.

[Hoc84] ———, *An inverse spectral theorem for a hill's matrix*, Linear Algebra and its Applications **57** (1984), 21–30.

[Khi64] A. Ya. Khinchin, *Continued fractions*, University of Chicago Press, Chicago, Ill.-London, 1964.

[KKL03] R. Killip, A. Kiselev, and Y. Last, *Dynamical upper bounds on wavepacket spreading*, Amer. J. Math. **125** (2003), no. 5, 1165–1198.

[KKT83] M. Kohmoto, L. P. Kadanoff, and C. Tang, *Localization problem in one dimension: Mapping and escape*, Phys. Rev. Lett. **50** (1983), 1870–1872.

[Lot02] M. Lothaire, *Algebraic combinatorics on words*, vol. 90, Cambridge University Press, Cambridge, 2002.

[LQW14] Q.-H. Liu, Y.-H. Qu, and Z.-Y. Wen, *The fractal dimensions of the spectrum of Sturm Hamiltonian*, Adv. Math. **257** (2014), 285–336.

[OK85] S. Ostlund and S.-H. Kim, *Renormalization of quasiperiodic mappings*, Phys. Scripta **T9** (1985), 193–198.

[Ray] L. Raymond, *Constructive gap labelling for one-dimensional Schrödinger operators*, to appear in: Aperiodic Order, Vol. 4: Schrödinger Operators, eds. M. Baake, D. Damanik, and N. Mañibo, Cambridge University Press, in preparation.

[Ray95] ———, *A constructive gap labelling for the discrete Schrödinger operator on a quasiperiodic chain*, 1995, preprint.

[Sim11] B. Simon, *Szego's theorem and its descendants: Spectral theory for l2perturbations of orthogonal polynomials*, Princeton University Press, 2011.

[Tes00] G. Teschl, *Jacobi operators and completely integrable nonlinear lattices*, vol. 72, American Mathematical Society, Providence, RI, 2000.

Appendix I. Sturmian dynamical systems

This appendix contains a very short description of Sturmian dynamical systems. A thorough background may be found in the books [Fog02, Lot02, DF25]. In addition, we state a lemma summarizing some basic properties of Sturmian sequences and mechanical words, which are applied in this paper.

To define the Sturmian Hamiltonian we have defined the sequences

$$\omega_\alpha(n) := \chi_{[1-\alpha,1[}(n\alpha \mod 1), \quad n \in \mathbb{N},\ \alpha \in [0,1],$$

which are called *mechanical words* [Lot02, Sec. 2.1.2] . If $\alpha \notin \mathbb{Q}$, ω_α is also called a *Sturmian sequence*. These sequences naturally define a dynamical system as follows. Let $\mathcal{A} := \{0,1\}$ be equipped with the discrete topology and $\mathcal{A}^{\mathbb{Z}} := \{\omega : \mathbb{Z} \to \mathcal{A}\}$ be the compact metrizable space equipped with the product topology. Consider the shift $T : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$, $(T\omega)(n) := \omega(n-1)$, $n \in \mathbb{Z}$, being a homeomorphism. This induces a continuous group action $\mathbb{Z} \curvearrowright \mathcal{A}^{\mathbb{Z}}$ via $(n,\omega) \mapsto T^n\omega$. For $\alpha \in [0,1]$, we have $\omega_\alpha \in \{0,1\}^{\mathbb{Z}}$ and its associated orbit closure (in the product topology)

$$\Omega_\alpha := \overline{\mathrm{Orb}(\omega_\alpha)} := \overline{\{T^n\omega_\alpha : n \in \mathbb{Z}\}}$$

defines a dynamical system $\mathbb{Z} \curvearrowright \Omega_\alpha$. If $\alpha \in \mathbb{Q}$, then ω_α is periodic, i.e. there is a period $q \in \mathbb{N}$ such that $T^q\omega_\alpha = \omega_\alpha$. Note that in this case $\mathrm{Orb}(\omega_\alpha) = \Omega_\alpha$. There are various different representations of this dynamical system. For instance, the authors in [BIST89, Lem. 1] proved that

$$\omega_\alpha(n) = \lfloor \alpha(n+1) \rfloor - \lfloor \alpha\, n \rfloor, \qquad n \in \mathbb{N},\ \alpha \in [0,1] \setminus \mathbb{Q}.$$

A different approach to describe these words is via a recursive rule using the continued fraction expansion $(0, c_1, c_2, \dots)$ of $\alpha \in [0,1] \setminus \mathbb{Q}$ is described in [Lot02, Eq. .9]. The reader is also referred to [BBB$^+$26] for a more detailed discussion. The following lemma provides the properties of mechanical words which are useful in our paper.

Lemma I.1. *Let $\boldsymbol{c} = [0, 0, c_1, c_2, \dots, c_k] \in \mathcal{C}$ for $k \in \mathbb{N}$ and $\frac{p_k}{q_k} := \varphi(\boldsymbol{c})$, with co-prime p_k, q_k. Then the following holds.*

(a) The sequence $\omega_{\frac{p_k}{q_k}}$ is periodic with period length q_k. Let its period $W_k \in \{0,1\}^{q_k}$ be defined by
$$W_k(i) := \omega_{\alpha_k}(i), \qquad 0 \le i \le q_k - 1$$
(b) For $k \in \mathbb{N}$, we have $q_k = c_k \cdot q_{k-1} + q_{k-2}$ with $q_{-1} = 0$ and $q_0 = 1$.
(c) The period of $\omega_{\frac{p_k}{q_k}}$ satisfy the following $W_0 = 0, W_1 = 0^{c_1-1}1$ and if $k \ge 2$, then
$$W_k = \begin{cases} W_{k-2}W_{k-1}^{c_k}, & k \equiv 0 \mod 2,\\ W_{k-1}^{c_k}W_{k-2}, & k \equiv 1 \mod 2.\end{cases}$$
(d) For $k \ge 1$,
- If $k \equiv 0 \mod 2$ then $\omega_\alpha(i) = W_k(i)$ for all $0 \le i \le q_k - 1$.
- If $k \equiv 1 \mod 2$ then $\omega_\alpha(i) = W_k(i)$ for all $0 \le i \le q_k - 2$.

Proof. The first two parts of the lemma are basic. The third and fourth parts appears e.g. in [BBB+26, Lem. 2.4]. □

Appendix II. Chebyshev polynomials and trace identities

In this section, we provide several known identities of traces and their connection to Chebyshev polynomials, see e.g. [Cas86, Ray95, BIST89, BBB+26, Ray, DF25]. Moreover, we prove Lemma 4.7.

II.1. **Chebyshev polynomials.**

A crucial tool for studying the spectral theory of Sturmian Hamiltonians are the dilated Chebyshev polynomials of the second kind (see [DLMF, (18.1.3)]) $S_n : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}_{-1}$, defined by

$$S_{-1}(x) := 0, \quad S_0(x) := 1, \quad S_n(x) := xS_{n-1}(x) - S_{n-2}(x) \quad \text{for } x \in \mathbb{R}.$$

For $x \in \mathbb{R} \setminus \{0\}$, denote by $\operatorname{sign}(x) \in \{+1, -1\}$ the sign of x.

Lemma II.1. *Let $x \in \mathbb{R}$ and $n \in \mathbb{N}_0$. Then the following holds.*

(a) *We have $S_{n+1}S_{n-1} - S_n^2 = -1$.*
(b) *If $|x| = 2$, then $\operatorname{sign}(x)^{n-1}S_{n-1}(x) = n$.*
(c) *If $|x| \geq 2$, then $\operatorname{sign}(x)^n S_n(x) = |S_n(x)|$ and*

$$\operatorname{sign}(x)^n x S_{n-1}(x) \geq 2\big|S_{n-1}(x)\big|.$$

(d) *If $|x| \geq 2$, then*

$$\operatorname{sign}(x)^n \big(S_n(x) - \frac{x}{2}S_{n-1}(x)\big) \geq 1.$$

(e) *If $|x| > 2$ and $n \geq 1$, then*

$$\operatorname{sign}(x)^n \big(S_n(x) - \frac{x}{2}S_{n-1}(x)\big) > 1.$$

Proof. The proof follows by induction using the recursive relation, the details can be found for instance in [BBB+26, Lem. III.2.]. □

II.2. **Trace identities .**

This section is devoted to various trace identities and the proof of Lemma 4.7.

The following proposition is a collection of well-known identities of the traces, see e.g. [Cas86, Ray95, BIST89, DF22, BBB+26, Ray, DF25]. Recall that for $\mathbf{c} \in \mathcal{C}$, $t_{\mathbf{c}}$ is a function of $E, V \in \mathbb{R}$, but we abbreviate notation and suppress this dependencies in the following.

Proposition II.2 (trace maps)**.** *Let $m \in \mathbb{N}$, $n \in \mathbb{N}_0$ and $\boldsymbol{c} \in \mathcal{C}$ be such that $[\boldsymbol{c}, m] \in \mathcal{C}$. Then the following holds.*

(a) *We have $t_{[\boldsymbol{c},m,0]} = t_{\boldsymbol{c}}$, $t_{[\boldsymbol{c},m,1]} = t_{[\boldsymbol{c},m+1]}$ and $t_{[\boldsymbol{c},m,-1]} = t_{[\boldsymbol{c},m-1]}$.*
(b) *We have for all $V \in \mathbb{R}$,* (the Fricke–Vogt invariant)

$$V^2 + 4 = t_{[\boldsymbol{c},n+1]}^2 + t_{[\boldsymbol{c},n]}^2 + t_{\boldsymbol{c}}^2 - t_{[\boldsymbol{c},n+1]}t_{[\boldsymbol{c},n]}t_{\boldsymbol{c}}$$

(c) *For $-1 \leq l \leq n$, we have*

$$t_{[\boldsymbol{c},n+1]} = S_{l+1}(t_{\boldsymbol{c}})t_{[\boldsymbol{c},n-l]} - S_l(t_{\boldsymbol{c}})t_{[\boldsymbol{c},n-l-1]}.$$

In particular, we have $t_{[\boldsymbol{c},n+1]} = t_{\boldsymbol{c}}t_{[\boldsymbol{c},n]} - t_{[\boldsymbol{c},n-1]}$ (for $l = 0$).

We will continue with two auxiliary lemmata which are needed to prove Lemma 4.7. In order to treat certain cases of the backward type A ($\ell = 0$) bands or backward type B ($\ell = -1$) bands, we need the following identity.

Lemma II.3. *Let $m \in \mathbb{N}$ and $\boldsymbol{c} \in \mathcal{C}$ be such that $[\boldsymbol{c}, m] \in \mathcal{C}$. Let $V \in \mathbb{R}$ and $E \in \mathbb{R}$ be such that $|t_{\boldsymbol{c}}(E,V)| = 2$. Then for all $n \in \mathbb{N}$ and $\ell \in \{-1, 0\}$,*

$$t_{[\boldsymbol{c},\ell]}(E,V) S_n\big(t_{[\boldsymbol{c},m]}(E,V)\big) = z^{m-\ell}\Big(2(m-\ell) S_{n+1}\big(t_{[\boldsymbol{c},m]}(E,V)\big) - z(m-\ell) t_{[\boldsymbol{c},m,n]}(E,V) - (m-1-\ell) t_{[\boldsymbol{c},m]}(E,V) S_n\big(t_{[\boldsymbol{c},m]}(E,V)\big)\Big)$$

holds where $z := \operatorname{sign}(t_{\boldsymbol{c}}(E,V))$.

Remark. This lemma is closely related to [BIST89, Prop. 2].

Proof. For the sake of simplicity, we abbreviate the notation in the following and write $t_{\widetilde{\mathbf{c}}} = t_{\widetilde{\mathbf{c}}}(E,V)$ for $\widetilde{\mathbf{c}} \in \mathcal{C}$. As a direct consequence of Lemma II.1 (b), we conclude that $S_l(t_{\mathbf{c}}) \neq 0$ for all $l \geq 0$ since $|t_{\mathbf{c}}| = 2$. Proposition II.2 (c) (applied for $n = m-1$ and $l = m-2-\ell \geq -1$) leads to

$$t_{[\mathbf{c},m]} = S_{m-1-\ell}\big(t_{\mathbf{c}}\big) t_{[\mathbf{c},1+\ell]} - S_{m-2-\ell}\big(t_{\mathbf{c}}\big) t_{[\mathbf{c},\ell]}.$$

Since $S_{m-1-\ell}(t_{\mathbf{c}}) \neq 0$, we derive

$$t_{[\mathbf{c},1+\ell]} = \frac{1}{S_{m-1-\ell}\big(t_{\mathbf{c}}\big)}\Big(t_{[\mathbf{c},m]} + S_{m-2-\ell}\big(t_{\mathbf{c}}\big) t_{[\mathbf{c},\ell]}\Big). \tag{II.1}$$

Let $n \in \mathbb{N}$. Using again Proposition II.2 (a) and (c) (with $l = n$), we conclude

$$t_{[\mathbf{c},m,n]} = S_{n+1}\big(t_{[\mathbf{c},m]}\big) t_{[\mathbf{c},m,0]} - S_n\big(t_{[\mathbf{c},m]}\big) t_{[\mathbf{c},m,-1]} = S_{n+1}\big(t_{[\mathbf{c},m]}\big) t_{\mathbf{c}} - S_n\big(t_{[\mathbf{c},m]}\big) t_{[\mathbf{c},m-1]}. \tag{II.2}$$

The case $\ell = 0$ and $m = 1$ need to be treated separately. We treat this case later and first assume that if $\ell = 0$ then $m \geq 2$. Then Proposition II.2 (c) (applied for $n = m-2$ and $l = m-3-\ell \geq -1$) and Equation (II.1) leads to

$$\begin{aligned} t_{[\mathbf{c},m-1]} =& S_{m-2-\ell}\big(t_{\mathbf{c}}\big) t_{[\mathbf{c},1+\ell]} - S_{m-3-\ell}\big(t_{\mathbf{c}}\big) t_{[\mathbf{c},\ell]} \\ =& \frac{S_{m-2-\ell}\big(t_{\mathbf{c}}\big)}{S_{m-1-\ell}\big(t_{\mathbf{c}}\big)} t_{[\mathbf{c},m]} + t_{[\mathbf{c},\ell]} \left(\frac{S_{m-2-\ell}\big(t_{\mathbf{c}}\big)^2}{S_{m-1-\ell}\big(t_{\mathbf{c}}\big)} - S_{m-3-\ell}\big(t_{\mathbf{c}}\big)\right) \\ =& \frac{S_{m-2-\ell}\big(t_{\mathbf{c}}\big)}{S_{m-1-\ell}\big(t_{\mathbf{c}}\big)} t_{[\mathbf{c},m]} + t_{[\mathbf{c},\ell]} \left(\frac{S_{m-2-\ell}\big(t_{\mathbf{c}}\big)^2 - S_{m-3-\ell}\big(t_{\mathbf{c}}\big) S_{m-1-\ell}\big(t_{\mathbf{c}}\big)}{S_{m-1-\ell}\big(t_{\mathbf{c}}\big)}\right). \end{aligned}$$

Since $S_k S_{k-2} - S_{k-1}^2 = -1$ for $k = m-1-\ell$ by Lemma II.1 (a), we conclude

$$t_{[\mathbf{c},m-1)} = \frac{S_{m-2-\ell}\big(t_{\mathbf{c}}\big)}{S_{m-1-\ell}\big(t_{\mathbf{c}}\big)} t_{[\mathbf{c},m]} + t_{[\mathbf{c},\ell]} \frac{1}{S_{m-1-\ell}\big(t_{\mathbf{c}}\big)}.$$

Inserting the latter into Equation (II.2), we get

$$t_{[\mathbf{c},m,n]} = S_{n+1}\big(t_{[\mathbf{c},m]}\big) t_{\mathbf{c}} - S_n\big(t_{[\mathbf{c},m]}\big) \frac{S_{m-2-\ell}\big(t_{\mathbf{c}}\big)}{S_{m-1-\ell}\big(t_{\mathbf{c}}\big)} t_{[\mathbf{c},m]} - t_{[\mathbf{c},\ell]} \frac{S_n\big(t_{[\mathbf{c},m]}\big)}{S_{m-1-\ell}\big(t_{\mathbf{c}}\big)}. \tag{II.3}$$

We claim that the latter identity holds also if $\ell = 0$ and $m = 1$. Indeed, this follows immediately from Equation (II.2), $t_{[\mathbf{c},m-1)} = t_{[\mathbf{c},\ell]}$, $S_{m-1-\ell}\big(t_{\mathbf{c}}\big) = 1$ and $S_{m-2-\ell}\big(t_{\mathbf{c}}\big) = 0$.

Now we can proceed with arbitrary $\ell \in \{-1, 0\}$ and $m \in \mathbb{N}$. Reorganizing Equation (II.3) leads to

$$t_{[\mathbf{c},\ell]} S_n\big(t_{[\mathbf{c},m]}\big) = S_{m-1-\ell}\big(t_{\mathbf{c}}\big)\Big(S_{n+1}\big(t_{[\mathbf{c},m]}\big) t_{\mathbf{c}} - t_{[\mathbf{c},m,n]}\Big) - S_{m-2-\ell}\big(t_{\mathbf{c}}\big) S_n\big(t_{[\mathbf{c},m]}\big) t_{[\mathbf{c},m]}.$$

Since we assumed that $t_{\mathbf{c}} = t_{\mathbf{c}}(E,V) = 2z$, Lemma II.1 (b) implies $S_n(t_{\mathbf{c}}) = z^n(n+1)$ for $n \geq 0$. Thus,

$$\begin{aligned} t_{[\mathbf{c},\ell]} S_n\big(t_{[\mathbf{c},m]}\big) =& z^{m-1-\ell}(m-\ell)\Big(S_{n+1}\big(t_{[\mathbf{c},m]}\big) 2z - t_{[\mathbf{c},m,n]}\Big) \\ & - z^{m-2-\ell}(m-1-\ell) S_n\big(t_{[\mathbf{c},m]}\big) t_{[\mathbf{c},m]} \end{aligned}$$

follows proving the desired identity. □

Lemma II.4. *Let $m, n \in \mathbb{N}$ and $\boldsymbol{c} \in \mathcal{C}$ be such that $[\boldsymbol{c}, m] \in \mathcal{C}$. Let $V \in \mathbb{R}$ and $E \in \mathbb{R}$ be such that*

$$|t_{\boldsymbol{c}}(E,V)| = 2 \qquad \text{and} \qquad |t_{[\mathbf{c},m]}(E,V)| \geq 2.$$

Then for all $n \in \mathbb{N}$ and $\ell \in \{-1, 0\}$,

$$\left|t_{[\mathbf{c},\ell]}(E,V) S_n\big(t_{[\mathbf{c},m]}(E,V)\big)\right| \geq (m-\ell)\big(2 - \left|t_{[\mathbf{c},m,n]}(E,V)\right|\big) + 2\left|S_n\big(t_{[\mathbf{c},m]}(E,V)\big)\right|$$

holds and the estimate is strict if additionally $|t_{[\mathbf{c},m]}(E,V)| > 2$.

Remark. The latter estimate is the general formula that we need to treat backward type A bands ($\ell = 0$) or backward type B bands ($\ell = -1$).

Proof. In order to simplify notation, set $t := t_{[\mathbf{c},m]}(E,V)$, $z_1 := \operatorname{sign}(t)$ and $z_0 := \operatorname{sign}(t_{\mathbf{c}}(E,V))$. Furthermore, we abbreviate the notation and write $t_{\widetilde{\mathbf{c}}} = t_{\widetilde{\mathbf{c}}}(E,V)$ for $\widetilde{\mathbf{c}} \in \mathcal{C}$. Due to Lemma II.3 and $z_1^{2n} = 1$, we have

$$\begin{aligned} &t_{[\mathbf{c},\ell]} S_n\big(t\big) \\ =&z_0^{m-\ell} z_1^{n+1}\Big(2(m-\ell) z_1^{n+1} S_{n+1}\big(t\big) - (m-1-\ell) z_1^{n+1} t S_n\big(t\big) - z_0 z_1^{n+1}(m-\ell) t_{[\mathbf{c},m,n]}\Big). \end{aligned}$$

Hence,

$$\begin{aligned} \left|t_{[\mathbf{c},\ell]} S_n\big(t\big)\right| \geq& \Big|2(m-\ell) z_1^{n+1} S_{n+1}\big(t\big) - (m-1-\ell) z_1^{n+1} t S_n\big(t\big)\Big| - \left|(m-\ell) t_{[\mathbf{c},m,n]}\right| \\ =&2(m-\ell) z_1^{n+1}\Big(S_{n+1}\big(t\big) - \frac{t}{2} S_n\big(t\big)\Big) + z_1^{n+1} t S_n\big(t\big) - (m-\ell)\left|t_{[\mathbf{c},m,n]}\right| \\ \geq& 2(m-\ell) + 2\left|S_n\big(t\big)\right| - (m-\ell)\left|t_{[\mathbf{c},m,n]}\right| \end{aligned}$$

follows by first using the triangle inequality, secondly Lemma II.1 (c) and (d) since $|t| \geq 2$ and finally Lemma II.1 (c) and (d). Note that the last estimate is strict by Lemma II.1 (e) if $|t| > 2$. This leads to the desired estimate. □

Now we can prove Lemma 4.7.

Proof of Lemma 4.7. Recall the assumptions of the proposition. Let $V \in \mathbb{R}$, $m, n \in \mathbb{N}$, $\mathbf{c} \in \mathcal{C}$. Let $I(V)$ be a spectral band of $\sigma_{\mathbf{c}}(V)$ of backward type A or B. Set

$$\ell := \begin{cases} 0, & I(V) \text{ is of backward type } A, \\ -1, & I(V) \text{ is of backward type } B. \end{cases}$$

Let $E \in \{L(I(V)),\ R(I(V))\}$. Then $|t_{[\mathbf{c},\ell]}(E,V)| \leq 2$ follows from Lemma 4.6 and the estimate is strict if $\varphi(\mathbf{c}) \in (0,1)$.

(a) If $|t_{[\mathbf{c},m]}(E,V)| \geq 2$, then Lemma II.4 and $m - \ell \geq 1$ imply

$$2\left|S_n\big(t_{[\mathbf{c},m]}\big)\right| \geq \left|t_{[\mathbf{c},\ell]} S_n\big(t_{[\mathbf{c},m]}\big)\right| \geq (m-\ell)\big(2 - \left|t_{[\mathbf{c},m,n]}\right|\big) + 2\left|S_n\big(t_{[\mathbf{c},m]}\big)\right|$$

and so $|t_{[\mathbf{c},m,n]}| \geq 2$ must hold.

(b) If $|t_{[\mathbf{c},m]}(E,V)| > 2$, then Lemma II.4 and $m - \ell \geq 1$ imply

$$2\left|S_n\big(t_{[\mathbf{c},m]}\big)\right| \geq \left|t_{[\mathbf{c},\ell]} S_n\big(t_{[\mathbf{c},m]}\big)\right| > (m-\ell)\big(2 - \left|t_{[\mathbf{c},m,n]}\right|\big) + 2\left|S_n\big(t_{[\mathbf{c},m]}\big)\right|$$

and so $|t_{[\mathbf{c},m,n]}| > 2$ must hold.

(c) Since $\varphi(\mathbf{c}) \in (0,1)$, Lemma 4.6 asserts $|t_{[\mathbf{c},\ell]}(E,V)| < 2$. If $|t_{[\mathbf{c},m]}(E,V)| \geq 2$, then Lemma II.4 and $m - \ell \geq 1$ imply

$$2\left|S_n\big(t_{[\mathbf{c},m]}\big)\right| > \left|t_{[\mathbf{c},\ell]} S_n\big(t_{[\mathbf{c},m]}\big)\right| \geq (m-\ell)\big(2 - \left|t_{[\mathbf{c},m,n]}\right|\big) + 2\left|S_n\big(t_{[\mathbf{c},m]}\big)\right|.$$

Thus, $|t_{[\mathbf{c},m,n]}| > 2$ must hold. □

Appendix III. A perturbation argument for eigenvalue interlacing

This section is devoted to the proof of Theorem 4.2. Given an $n \times n$ hermitian matrix X, we denote its eigenvalues in non-decreasing order by

$$\lambda_0(X) \leq \lambda_1(X) \cdots \leq \lambda_{n-2}(X) \leq \lambda_{n-1}(X).$$

We first recall a well-known result on interlacing of eigenvalues using classical Weyl inequalities, see e.g. [HJ13, Cor. 4.3.3, Thm. 4.3.6].

Proposition III.1. *Let X and Q be $n \times n$ hermitian matrices, and suppose that Q is a positive semidefinite, rank one matrix.*

(a) For $j = 1, \dots n-1$, we have

$$\lambda_{j-1}(X+Q) \leq \lambda_j(X) \leq \lambda_j(X+Q).$$

(b) For $j = 0, \dots n-2$, we have

$$\lambda_j(X-Q) \leq \lambda_j(X) \leq \lambda_{j+1}(X-Q).$$

From these inequalities we can directly derive the following estimates for traceless rank two perturbations.

Corollary III.2. *Let X and Y be $n \times n$ hermitian matrices, and let $Q = Y - X$. If Q has rank two and trace zero, then*

$$\lambda_{j-1}(Y) \leq \lambda_j(X) \leq \lambda_{j+1}(Y), \qquad j = 1, 2, \dots n-2.$$

Proof. Using matrix diagonalization one may verify that there exist $n \times n$ hermitian, positive semidefinite matrices Q_1, Q_2 of rank one such that $Q = Q_1 - Q_2$. Applying first Proposition III.1 (a) to X and $X+Q_1$, and then Proposition III.1 (b) to $X+Q_1$ and $Y = (X+Q_1)-Q_2$ yields the desired inequalities. $\square$

Let $\mathbf{c} = [0, c_0, c_1, \dots, c_k] \in \mathcal{C}$ be such that $k \in \mathbb{N}_0$ and $c_k \in \mathbb{N}$ if $k \geq 1$. Note that this implies $\varphi(\mathbf{c}) \notin \{-1, \infty\}$. Recall the notations of the matrices $H_{\mathbf{c},V}(\theta)$ and $H_{\mathbf{c},V}^{\times n}(\theta)$ for $\theta \in [0, \pi]$ as introduced in Section 4.1. We aim to apply Corollary III.2 to the matrices

$$H_{[\mathbf{c},m,n],V}\left(\theta_{[\mathbf{c},m,n]}\right) \text{ and } H_{[\mathbf{c},m],V}^{\times n}\left(\theta_{[\mathbf{c},m]}\right) \oplus H_{\mathbf{c},V}(\theta_{\mathbf{c}})$$

with appropriate choice of $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]} \in \{0, \pi\}$. It turns out that if $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]} \in \{0, \pi\}$ are admissible (see Definition 4.3), then these matrices are a rank two perturbation with trace zero of each other. To formalize this statement, we define the matrix

$$H_{[\mathbf{c},m,n]}^{\oplus}(\theta_{[\mathbf{c},m]}, \theta_{\mathbf{c}}) := \begin{cases} \begin{pmatrix} H_{\mathbf{c},V}(\theta_{\mathbf{c}}) & 0 \\ 0 & H_{[\mathbf{c},m],V}^{\times n}\left(\theta_{[\mathbf{c},m]}\right) \end{pmatrix} & \text{if } k \equiv 0 \mod 2, \\ \\ \begin{pmatrix} H_{[\mathbf{c},m],V}^{\times n}\left(\theta_{[\mathbf{c},m]}\right) & 0 \\ 0 & H_{\mathbf{c},V}(\theta_{\mathbf{c}}) \end{pmatrix} & \text{if } k \equiv 1 \mod 2. \end{cases}$$

In the following statements we refer to vectors $x \in \mathbb{R}^q$ as column vectors and use the notation x^t to indicate the transpose of a vector (which is then a row vector). We also use the notation $\langle x, y \rangle$ to denote the (Euclidean) inner product between vectors.

Lemma III.3. *Let $V \in \mathbb{R}$, $m, n \in \mathbb{N}$ and $\boldsymbol{c} = [0, c_0, c_1, \dots, c_k] \in \mathcal{C}$ be such that $k \in \mathbb{N}_0$ and $c_k \in \mathbb{N}$ if $k \geq 1$ be such that $[\boldsymbol{c}, m] \in \mathcal{C}$. Let $\frac{p_1}{q_1} = \varphi(\boldsymbol{c})$, $\frac{p_2}{q_2} = \varphi([\boldsymbol{c}, m])$ and $\frac{p_3}{q_3} = \varphi([\boldsymbol{c}, m, n])$ be such that p_i, q_i are coprime. If $\theta_{\boldsymbol{c}}, \theta_{[\boldsymbol{c},m]}, \theta_{[\boldsymbol{c},m,n]} \in \{0, \pi\}$* are admissible*, then there are $x := x(\theta_{\boldsymbol{c}}, \theta_{[\boldsymbol{c},m]}, \theta_{[\boldsymbol{c},m,n]}), y := y(\theta_{\boldsymbol{c}}, \theta_{[\boldsymbol{c},m]}, \theta_{[\boldsymbol{c},m,n]}) \in \mathbb{R}^{q_3}$ such that*

$$H_{[\boldsymbol{c},m,n],V}\left(\theta_{[\boldsymbol{c},m,n]}\right) - H_{[\boldsymbol{c},m,n]}^{\oplus}(\theta_{[\boldsymbol{c},m]}, \theta_{\boldsymbol{c}}) = xx^t - yy^t$$

is a symmetric rank two perturbation with trace zero. Furthermore, set

$$d_1 := \begin{cases} q_1 & \text{if } k \equiv 0 \mod 2, \\ nq_2 & \text{if } k \equiv 1 \mod 2, \end{cases} \qquad \text{and} \qquad d_2 := \begin{cases} nq_2 & \text{if } k \equiv 0 \mod 2, \\ q_1 & \text{if } k \equiv 1 \mod 2. \end{cases}$$

(a) *If* $w = (w_1, \dots w_{d_1}, 0, \dots, 0)^t \in \mathbb{R}^{q_3}$ *is orthogonal to* x *and to* y *then* $w_1 = w_{d_1} = 0$.
(b) *If* $w = (0, \dots, 0, w_1, \dots w_{d_2})^t \in \mathbb{R}^{q_3}$ *is orthogonal to* x *and to* y *then* $w_1 = w_{d_2} = 0$.

Proof. Let $e_1, \dots, e_{q_3}$ be the standard orthonormal basis of $\mathbb{R}^{q_3}$, namely e_i is the i-th unit vector in $\mathbb{R}^{q_3}$. Recall that for the continued fraction expansion, we have the identity $q_3 = nq_2 + q_1 = d_1 + d_2$, see e.g. Lemma I.1. The statement of the lemma follows by straightforward calculations invoking Lemma I.1, so we just explicitly write here the expressions for the $x, y \in \mathbb{R}^{q_3}$ in the statement of the lemma. Let $k \in \mathbb{N}$ be even. If $(\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}) = (0,0,0)$, then

$$x = \frac{1}{\sqrt{2}}\big(e_1 - e_{nq_2} - e_{nq_2+1} + e_{q_3}\big), \quad y = \frac{1}{\sqrt{2}}\big(-e_1 - e_{nq_2} + e_{nq_2+1} + e_{q_3}\big).$$

If $(\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}) = (\pi,\pi,0)$, then

$$x = \frac{1}{\sqrt{2}}\big(e_1 + e_{nq_2} + e_{nq_2+1} + e_{q_3}\big), \quad y = \frac{1}{\sqrt{2}}\big(-e_1 + e_{nq_2} - e_{nq_2+1} + e_{q_3}\big).$$

If $(\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}) = (\pi,0,\pi)$, then

$$x = \frac{1}{\sqrt{2}}\big(-e_1 + e_{nq_2} + e_{nq_2+1} + e_{q_3}\big), \quad y = \frac{1}{\sqrt{2}}\big(e_1 + e_{nq_2} - e_{nq_2+1} + e_{q_3}\big).$$

If $(\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}) = (0,\pi,\pi)$, then

$$x = \frac{1}{\sqrt{2}}\big(e_1 + e_{nq_2} + e_{nq_2+1} - e_{q_3}\big), \quad y = \frac{1}{\sqrt{2}}\big(-e_1 + e_{nq_2} - e_{nq_2+1} - e_{q_3}\big).$$

The case when $n \in \mathbb{N}$ is odd is treated similarly. □

Remark III.4. Statements (a) and (b) in Lemma III.3 are used to conclude strict inequalities in Theorem 4.2. Towards this, we use that if two consecutive (up to cyclic permutation) entries of a solution (see (a) and (b)) vanish, then the whole solution vanishes, since we have a nearest neighbor interaction.

Now we have all tools at hand to prove Theorem 4.2.

Proof of Theorem 4.2. Recall the statement of the theorem. Let $V > 0$, $m, n \in \mathbb{N}$ and $\mathbf{c} = [0, c_0, \dots, c_k] \in \mathcal{C}$ be such that $\varphi(\mathbf{c}) \notin \{-1, \infty\}$ and $[\mathbf{c}, m] \in \mathcal{C}$. Thus, $c_k \in \mathbb{N}$ if $k \geq 1$. Furthermore, $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]} \in \{0, \pi\}$ are admissible, i.e., they satisfy $\theta_{\mathbf{c}} + \theta_{[\mathbf{c},m]} + \theta_{[\mathbf{c},m,n]} \in \{0, 2\pi\}$. Consider $Y = H_{[\mathbf{c},m,n],V}(\theta_{[\mathbf{c},m,n]})$ and $X = H^{\times n}_{[\mathbf{c},m],V}(\theta_{[\mathbf{c},m]}) \oplus H_{\mathbf{c},V}(\theta_{\mathbf{c}})$. We need to prove that

$$\lambda_{j-1}(Y) \leq \lambda_j(X) \leq \lambda_{j+1}(Y)$$

and that the inequalities are strict whenever $\lambda_j(X)$ is a simple eigenvalue of X. First, observe that by construction X and $Z := H^{\oplus}_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m]}, \theta_{\mathbf{c}})$ share the same eigenvalues (with same multiplicities). Thus, the claimed inequalities follow directly from Corollary III.2 and Lemma III.3. It is left to prove that those inequalities are strict if $\lambda_j(X) = \lambda_j(Z)$ is simple.

Let $n \in \mathbb{N}$ and borrow the notation of Lemma III.3 for $q_1, q_2, q_3 \in \mathbb{N}$ and $d_1, d_2 \in \mathbb{N}$. Following Lemma III.3, there are $x := x(\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]})$, $y := y(\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}) \in \mathbb{R}^{q_3}$ such that $Y - Z = xx^t - yy^t$. Moreover, x and y satisfy the assertions (a) and (b) in Lemma III.3. Set

$$Z(x) := Z + xx^t \qquad \text{and} \qquad Z(y) := Z - yy^t.$$

Then we have

$$Y = Z + xx^t - yy^t = Z(x) - yy^t = Z(y) + xx^t.$$

Recall that $q_3 = nq_2 + q_1 = d_1 + d_2$ (Lemma I.1) and $H^{\times n}_{[\mathbf{c},m],V}(\theta_{[\mathbf{c},m]})$ is an $nq_2 \times nq_2$ matrix while $H_{\mathbf{c},V}(\theta_{\mathbf{c}})$ is an $q_1 \times q_1$.

(a) We prove $\lambda_j(Z) < \lambda_{j+1}(Y)$. Assume by contradiction that $\lambda_j(Z) = \lambda_{j+1}(Y)$ holds and $\lambda_j(Z)$ is a simple eigenvalue of Z. Due to Proposition III.1 (using that xx^t and yy^t are positive semidefinite), the previous identities lead to

$$\lambda_j(Z) \leq \lambda_{j+1}(Z(y)) \leq \lambda_{j+1}(Y) \quad \text{and} \quad \lambda_j(Z) \leq \lambda_j(Z(x)) \leq \lambda_{j+1}(Y).$$

Thus,

$$\lambda := \lambda_j(Z) = \lambda_j(Z(x)) = \lambda_{j+1}(Z(y)) = \lambda_{j+1}(Y).$$

follows by our assumption.

Let $w \in \mathbb{R}^{q_3} \setminus \{0\}$ be an eigenvector of Z corresponding to the eigenvalue λ. Since λ is a simple eigenvalue of Z, then either (1) λ is an eigenvalue of $H^{\times n}_{[\mathbf{c},m],V}(\theta_{[\mathbf{c},m]})$ or (2) λ is an eigenvalue of $H_{\mathbf{c},V}(\theta_{\mathbf{c}})$, but not both. These two cases can be treated similarly using Lemma III.3. We only prove here case (1).

Since λ is an eigenvalue of $H^{\times n}_{[\mathbf{c},m],V}(\theta_{[\mathbf{c},m]})$ but not of $H_{\mathbf{c},V}(\theta_{\mathbf{c}})$, we conclude that the corresponding eigenvector w of Z is of the form $w = (w_1, \ldots, w_{nq_2}, 0, \ldots, 0)^t \in \mathbb{R}^{q_3}$ if $k \in \mathbb{N}$ is odd (where k is determined by the length of the tuple $\mathbf{c}$) and $w = (0, \ldots, 0, w_1, \ldots, w_{nq_2})^t \in \mathbb{R}^{q_3}$ if $k \in \mathbb{N}$ is even. Set $u := (w_1, \ldots, w_{nq_2})^t$.

We claim that $\langle x, w\rangle = 0 = \langle w, y\rangle$ holds. Before proving this identity, let us show how these equalities finish our proof. If $\langle x, w\rangle = 0 = \langle w, y\rangle$, then $w_1 = w_{nq_2} = 0$ follow from Lemma III.3 (a) if k is odd and from Lemma III.3 (b) if k is even. Since $H^{\times n}_{[\mathbf{c},m],V}(\theta_{[\mathbf{c},m]})u = \lambda u$ and each equation in the system involves three consecutive (going cyclically) of the entries of $u \in \mathbb{R}^{nq_2}$, we derive $u = 0$ and so $w = 0$. This is a contradiction as $w \neq 0$ is an eigenvector of Z for the eigenvalue λ.

Now let us prove the claim $\langle x, w\rangle = 0 = \langle w, y\rangle$. Since $\lambda = \lambda_j(Z) = \lambda_j(Z(x))$, there is an eigenvector v of $Z + xx^t$ with eigenvalue λ. Using that $Z + xx^t$ is hermitian and $x^t w = \langle x, w\rangle$, we conclude

$$\lambda\langle v, w\rangle = \langle v, (Z + xx^t)w\rangle = \langle v, Zw\rangle + \langle x, w\rangle\langle v, x\rangle = \lambda\langle v, w\rangle + \langle x, w\rangle\langle v, x\rangle$$

implying $\langle x, w\rangle\langle v, x\rangle = 0$. If $\langle v, x\rangle \neq 0$, we immediately derive $\langle x, w\rangle = 0$ as desired. If $\langle v, x\rangle = 0$, then

$$\lambda v = (Z + xx^t)v = Zv + \langle x, v\rangle x = Zv$$

follows. Thus, $v = Cw$ holds for some $C \in \mathbb{R} \setminus \{0\}$ as λ is a simple eigenvalue of Z with eigenvector w. Hence, $\langle v, x\rangle = 0$ leads to $\langle w, x\rangle = 0$ as claimed.

Similarly, we conclude $\langle w, y\rangle = 0$ using that λ is an eigenvalue of $Z(y)$.

(b) Similarly to case (a), we can prove $\lambda_{j-1}(Y) < \lambda_j(Z)$. Assume by contradiction that $\lambda_{j-1}(Y) = \lambda_j(Z)$ holds and $\lambda_j(Z)$ is a simple eigenvalue of Z. Then Proposition III.1 leads to

$$\lambda_{j-1}(Y) \leq \lambda_{j-1}(Z(x)) \leq \lambda_j(Z) \quad \text{and} \quad \lambda_{j-1}(Y) \leq \lambda_j(Z(y)) \leq \lambda_j(Z).$$

Thus, our assumption yields

$$\lambda := \lambda_j(Z) = \lambda_{j-1}(Z(x)) = \lambda_j(Z(y)) = \lambda_{j-1}(Y).$$

Next, let $w \in \mathbb{R}^{q_3} \setminus \{0\}$ be an eigenvector of Z for the eigenvalue λ. By simplicity of the eigenvalue $\lambda = \lambda_j(Z)$, λ is either an eigenvalue of $H^{\times n}_{[\mathbf{c},m],V}(\theta_{[\mathbf{c},m]})$ or of $H_{\mathbf{c},V}(\theta_{\mathbf{c}})$. Thus, w has either the form $w = (w_1, \ldots, w_{d_1}, 0, \ldots, 0)^t \in \mathbb{R}^{q_3}$ or $w = (0, \ldots, 0, w_1, \ldots, w_{d_2})^t \in \mathbb{R}^{q_3}$. As before one can show that in both cases $\langle x, w\rangle = 0 = \langle w, y\rangle$ holds. Then Lemma III.3 yields $w_1 = w_{d_1} = 0$ (respectively $w_1 = w_{d_2} = 0$) and so $w = 0$ follows, a contradiction. □

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