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The Discourse Mapping Tree as a tool for mapping classroom discussions in linear algebra

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ABSTRACT

Mathematical discussions, along with other learner-centered teaching practices, are becoming more prevalent in tertiary mathematical education, yet the mathematical content of these discussions and the learning involved remain less explored. We developed the Discourse Mapping Tree (DMT), a commognitive based tool for mapping mathematical content during discussions, which allowed us to examine learning of a class as a whole in linear algebra discussions. The DMT displays the objects on which a discussion focuses, the multiple types of realizations of these objects and the narratives authoring links between the realizations. The DMT showed that the division of labor between the instructor and the students in our study was not always symmetrical. This tool provides a succinct overall image of the publicly available mathematics during a classroom discussion and allows for the mapping of complex tertiary level mathematics which include multiple objects and realizations.

1. Introduction

Active student participation in rich mathematical discussions, along with other learner-centered teaching practices, are becoming more prevalent in tertiary mathematical education (Melhuish et al., 2022). In these discussions students are afforded agency to author mathematical narratives; they expound their ideas to the class using verbal and written mediums; they are expected to justify their claims in response to both students' feedback and the instructor's critique; and a collective solution to a given task is authored. During lessons that are built around such discussions, students actively take charge of their own learning, become self-regulated, and develop their own study paths (Pepin et al., 2021). This supports meaningful student participation in all levels of mathematics education (Hershkowitz et al., 2014; Tabach et al., 2014). In the tertiary level mathematical discussions have been used to support students' meaningful mathematical activity of recreating mathematical ideas in a bottom-up manner, for example, in a differential equations course (Stephan & Rasmussen, 2002). Hershkowitz and colleagues (2022) used classroom discussion in small groups and demonstrated that this method supported students' learning about fractals. Yet, most of these studies focused on the social interactions between students and between the instructor and the class. The mathematical content of these discussions and how they afford opportunities for learning remain less explored.

Mapping the mathematical narratives of a classroom discussion allows us to examine how the mathematical content was handled and developed in the discussion and what classroom learning took place (Weingarden et. al., 2019). This can highlight specific

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representations that were missed in the discussion, concepts and skills with which the class felt more comfortable, and areas where more emphasis should have been placed by the instructor to move the class towards less familiar representations and mathematical ideas. This is particularly important in linear algebra, which consists of many abstract mathematical concepts that have multiple representations and intricate connections between them (Talbert, 2014). Students face difficulties linking between these multiple representations, which pose challenges for advancing to more complex mathematical and STEM related disciplines (Chang, 2011; Varsavsky, 2010).

The commognitive framework (Sfard, 2008) is particularly suitable for analyzing classroom discussions since it conceptualizes mathematical learning through the lens of discourse. This enables the treatment of the mathematical content and the inter-personal communication with the same theoretical toolset. Recent studies used commognition to map the mathematical content of discussions using visual representations such as tree diagrams (Knox & Kontorovich, 2023). These have been used to map the mathematical content of textbooks (Haghjoo et al., 2022), to map students' work with technological tools (Weingarden et al., 2024) and to map classroom discussions (Weingarden et al., 2019; Weingarden & Heyd-Metzuyanim, 2023). However, classroom discussions have been mapped only in middle school settings for a few tasks focusing on specific content. The adaptability of these tools to the tertiary setting, with much more complex mathematical discourse, has not yet been examined.

Our goal in this study was to develop a tool for mapping the mathematical content of whole-class discussions in linear algebra, and to use this tool to characterize what mathematical ideas, realizations, and connections become publicly available to the class during these discussions.

2. Theoretical background

2.1. Mathematical discussions and their study

During a whole-class mathematical discussion it is difficult and sometimes outright impossible to track an individual's mathematical progress. Nevertheless, learning processes are understood to occur in this setting. For this reason, some researchers have examined learning in classrooms as a whole, using the class rather than the individual as the unit of analysis (e.g. Nachlieli & Tabach, 2022, Ernest et. al. 2019, Stephan & Rasmussen, 2002).

In such discussions, individual contributions are shaped by what has been previously said and written in the public space, and the mathematical procedures and narratives that emerge are developed collectively over the course of the discussion. Classroom discourse thus develops through individual students' contributions, while not being reducible to an aggregation of individual discourses. Instructors typically address the class as a whole, and students' contributions become part of a shared mathematical space (Nachlieli & Tabach, 2022). Aligned with these studies, we use the term *classroom mathematical discussion* to refer to everything said and done in the public space that relates to mathematics.

In such public, whole-class discussions, much research has shown that students' learning of mathematics is supported, particularly in the K-12 context (e.g. Chen et al., 2020; O'Connor et al., 2017; Stein et al., 2008), as discussions afford opportunities for student struggle and for developing conceptual understanding (Hiebert & Grouws, 2007).

A task appropriate for sparking a discussion is a necessary, but not sufficient, condition for a meaningful discussion. The literature describes discussions around meaningful, engaging tasks with characteristics such as a high level of cognitive demand (Tekkumru-Kisa et al., 2020) and expanding students' mathematical experiences (Koichu & Zazkis, 2021). Multiple studies have established that teaching practices should support meaningful mathematical participation, enable students to struggle in building understanding, emphasize connections between procedures and concepts, and solicit student thinking (Michaels et al., 2008; Schoenfeld, 2014; Stein et al., 2008; Smith & Stein, 2011). Nevertheless, not many studies have engaged in examining what mathematics ensued out of these tasks in the actual discussion.

2.2. Discussions in a tertiary setting

At the practical level, classroom discussions, although less prevalent than in K-12 settings, have received increasing attention in the tertiary setting (e.g. Rasmussen & Kwon, 2007). Studies examining this teaching technique have shown, for example, that classroom discussions about properties of fractals sparked meaningful discussions that supported student learning (Hershkowitz et al., 2022). In another study, Stephen and Rasmussen (2002) showed that during discussions about differential equations there was collective learning through the mathematical practices that emerged. These studies relied on measures of learning outcomes, such as interviews about beliefs and exams measuring knowledge. The relatively rare studies that did focus on classroom discourse (Heyd-Metzuyanim & Cooper, 2023, Saxe & Farid, 2023, Sfard, 2023) examined the classroom interactions in micro-detail, on a turn-by-turn basis, which did not allow an overall view of the mathematical content to which students were exposed throughout the discussion.

In the tertiary context, the productivity of classroom discussion may be even more crucially dependent on the mathematical content that is made available during the discussion than in earlier levels. However, most studies that related to tasks appropriate for classroom discussions did not engage in analyzing which mathematical ideas, representations, or connections actually became publicly articulated in the ensuing discussion (Chang, 2011, Stewart & Thomas, 2009, Zandieh et al., 2017). One line of research that did relate to the gap between what is mathematically possible and what is taken up in practice is didactical engineering research. Artigue (2009) suggested that tasks should be examined by comparing an a priori analysis of a task, independent from its implementation, and an a posteriori analysis of what emerged in classroom activity. While developed in the context of task design, this distinction underscores the importance of differentiating between mathematical potential and what content becomes available during classroom discussions.

3. Theoretical framework–commognition

The commognitive framework allows us to consider the mathematical content of discussions, including the explication of concepts and student agency, with emphasis on the mathematical objects being considered. According to commognition, mathematical objects involved in discussions are discursive constructs; that is, they have no physical manifestation and exist only as part of a discourse (Sfard, 2008). The words or symbols that are used in the discourse are termed *signifiers*. *Realizations of a signifier* are expressions that are all interchangeable, which are all treated in experts' mathematical discourse as denoting "the same" object (Sfard, 2008). For example, the object “two”, which is the number you reach when counting two apples and is a product of counting, can also be signified by the numeral 2, by $4/2$, by $1+1$, and by $\sqrt{4}$.

Objects are part of discourses. A discourse is defined by Sfard (2008) as “set apart by its objects, the kinds of mediators used, and the rules followed by participants” (p. 93). Mathematical discourses are hierarchical and recursive, where their objects (e.g., the rational numbers) are built upon previously established objects (e.g., the whole numbers) (Sfard, 2008).

New mathematical discourses have historically been created either by several existing sub-discourses coalescing into one discourse or by a new discourse subsuming an older one. Sfard (2008) claims that an individual’s adoption of a discourse often proceeds similarly to how discourses developed over centuries. Thus, when learners progress from individualizing one discourse to a subsuming one, the subsuming discourse includes an isomorphic copy of the old discourse, as well as new objects and narratives that can only be realized in the subsuming discourse (Lavie & Sfard, 2019). For example, the discourse of functions is coalesced from its sub-discourses of algebraic formulas, of curves and of physical processes, and includes narratives that can be endorsed in all the sub-discourses and narratives that include pieces of narratives from the sub-discourses (Sfard, 2008).

The commognitive framework is particularly suited to linear algebra, which includes many types of structures that can be represented in different discourses (i.e. manners) and considered as being—or not being—isomorphic (i.e. mathematically equivalent) to other structures (Hillel, 2000).

The discourse commonly referred to as linear algebra includes multiple sub-discourses that, over an extended period of time, were coalesced into a single subsuming discourse. Andrews-Larson (2015) exemplifies this through the history of the notion of systems of linear equations (SLEs). Originally, systems of constraints on everyday problems were described verbally. Next, linear systems and their solutions were described by Chinese mathematicians in 200 BCE and by Gauss (early 19th century) without matrix notation. Significant advances in notation, including matrices and determinants, led to SLEs being described as mathematical objects—not merely a process to a solution. The modern, axiomatic definitions utilize vector spaces and linear transformations to describe SLEs. In accordance with Andrews-Larson’s (2015) explanation, we divide the various narratives regarding SLEs into these five domains: the solution set (constraints), a list of equations, matrix notation, SLEs as objects and vector spaces and transformations. These can be considered sub-discourses, differing in their objects, mediators, key-words and rules.

It is important to note that at the tertiary level narratives can rarely be neatly separated into sub-discourses, as contemporary linear algebra utilizes realizations from assorted sub-discourses interchangably. Still, as previous studies have shown the separate sub-discourses remain perceptible in practice, as students are gradually introduced to realizations from different sub-discourses (Grenier-Boley, 2014). They are also perceptible in evidence describing students’ difficulties in traversing from realizations belonging to different sub-discourses (Duval, 2006). Thus, also at the tertiary level “learning mathematics is a twofold process that involves developing proficiency in the various subsumed discourses … and in the unified discourse”[1]; (Weingarden, 2024, p.264).

An essential step towards unifying different discourses lies in *saming* the different realizations, which is an important part of what Sfard (2008) termed *objectification*. This process should yield, eventually, the objects communicated about in the unified discourse. Through this process, narratives using realizations that belong to different sub-discourses become interchangeable and equivalent. For example, deriving from the fact that the graph of a parabola intersects the x-axis at two points the claim that $ax^2 + bx + c$ has two roots sames between the graphic realization and the algebraic realization. Saming does not just happen between discourses, it can also happen within a single discourse. For example, the claim that $2(3+x) = 2\cdot 3 + 2\cdot x$ sames between the realizations $2(3+x)$ and $2\cdot 3 + 2\cdot x$, both belonging to the algebraic discourse.

Weingarden (2024) described communicative actions that included saming in the classroom, using the term saming narratives. In this study, we build on this to define saming narratives as narratives that translate one realization to the other, or narratives that use as their protagonists objects that manifest through at least two realizations. Further, we distinguish between intra-discursive saming narratives, where the two realizations are of the same discourse and inter-discursive saming narratives, in which the two realizations belong to different discourses. For example, “the parabola meets the x axis at three, therefore p(3) = 0” is an inter-discursive saming narrative since it contains as its protagonists two realizations of the same object (“the parabola” and “p(3) = 0”) belonging to two disparate discourses – the graphs discourse and the algebraic discourse.

3.1. Visual mapping using realization trees

Relying on the idea that visualization tools can convey complex ideas more efficiently than prose (Gropper, 1963), in recent years advancements in the domain of commognitive studies have led to the development of visual tools for mapping classroom discourse.

[1] Weingarden (2024) refers to this twofold process as object-level learning and meta-level learning, a term we also used in our previous writings (Wallach, 2022). However, in this study the meta-level aspects of the discourse are less apparent, thus we revert to simply observing the two processes (intra-discursive and inter-discursive) as ones worthy of delineation.

The commognitive visual tools use the notion of an object being a "signifier together with its realization tree" (Sfard, 2008). A realization tree is a visual representation of realizations of a mathematical object and the connections between them.

Realization trees have been used to examine the processes that are involved in derivatives (Park, 2016) and used to examine what type of objects are introduced to students in textbooks (Haghjoo et al., 2022). These gave a visual image of what can be included in a classroom discussion. Realization trees have also been used to map oral discussions, allowing the participants' narratives to be examined. Weingarden and colleagues (2019) mapped the opportunities and the agency afforded to middle school students to construct narratives about the perimeter of trains of hexagons; Miragliotta and Lisarelli (2022) mapped available opportunities and the authored realizations for the height of a triangle; Knox and Kontorovich (2023) mapped discursive gaps between group members and individual learners' development of objects. Realization trees can be used both for a priori mapping of what is possible to include in a discussion (e.g., what is available in a textbook), and for a posteriori mapping of what was included in a discussion (e.g., what a student said).

Despite the usefulness of realization trees presented above, they are limited by their focus on a single mathematical object. For example, Weingarden and colleagues (2019) mapped the perimeter of a train of hexagons, referring to this perimeter object and its various realizations as linear algebraic expressions, visual mediators of the sides of the "trains", and verbal descriptions of how the hexagon sides should be counted. This characteristic of realization trees is particularly limiting for tertiary discussions which often pertain to families of objects. Such discussions can include, for example, the properties of continuous functions or of row-reduced matrices, while giving multiple examples of functions or matrices for which these properties hold or do not hold. Mapping all the mathematics that these discussions include necessitated developing a new tool to capture discourses and sub-discourses, rather than a single object with its realizations.

Another limitation of the realization trees is the lack of differentiation between a priori analysis and a posteriori analysis. This is particularly important when one wishes to map the expected mathematical content, not just the discourse observed in the discussion. Researchers in the past dealt with this limitation in various ways. Weingarden and colleagues (2019) constructed the expected mathematics by empirically mapping more than 30 classroom discussions and by determining whether to expect a certain realization in a specific classroom discussion based on the assumption that the full set of what was to be expected could be found in the 30 discussions. Miragliotta and Lisarelli (2022) used similar assumptions to construct the tree used in mapping a discussion a posteriori. Such construction is limited for two reasons. First, it necessitates compiling a large database of classroom discussions as the "potential". Second, even with a large database, not all the potential realizations and narratives might occur. Thus, there is a need to develop the realization tree tool to include the potential mathematical content of a task, independent of any implementation.

In this study, we examined the mathematical content of whole-class discussions through a visual mapping of the sub-discourses involved. Given our view of mathematical discussions, their mapping should include the objects being discussed, the types of realizations of these objects (i.e. the sub-discourses) and the types of connections the saming-narratives describe (intra-discursive or inter-discursive).

Our research question is thus: What can a visual mapping of the discourses tell us about the mathematical content of a whole-class discussion?

4. Method

4.1. Context

This study is part of a larger project intended to develop teaching practices and materials for discussion-based learning in linear algebra (Wallach, 2022). In parallel to the regular lectures and recitations, workshops led by the first author were planned and offered to students. The basis for each workshop was a task with which the students engaged and which formed the basis for the whole-class discussion. The tasks were designed by the first and third author (instructors of linear algebra with many years of experience) to prompt and foster meaningful mathematical discussions. Over the course of two semesters 11 workshops were held, using the seven tasks that were designed for them.

The project was conducted at a science and engineering university where students take a linear algebra course during their first semester. The workshops were one academic hour and participation was voluntary. The number of students participating in each workshop varied from 7 to 60. Some students were in all or most of the sessions and some students were only in a single session. The attendance fluctuated based on the amount of advertising for the sessions, due to the students' workload (homework, midterms, etc.) and dependent on the point of the semester the session was held.

The workshops were an adaptation of the launch, explore and discuss (LED) structure and Smith and Stein's (2011) suggested practices for orchestrating productive mathematics discussions described above. Each workshop started with a short (around 5 min) introduction, that included a summary of definitions and theorems presented in the lectures; these were written on the board and available to the students throughout the workshop. Next, the students were given a printed worksheet with a task to work on in groups of 2 or 3 students for 15–20 min. Finally, students presented their solutions to the class and a whole-class discussion was moderated by the instructor discussing the proffered solutions, connecting the various solutions suggested by the students, and discussing other related topics brought up by the students' questions and examples. The discussion was 15–20 min long. Data was collected with several video cameras focused on the board and at student groups.

4.2. Analysis

The tasks and the video recordings of the whole-class discussions in the workshops were analyzed with a visual mapping tool developed for this purpose—a Discourse Mapping Tree (DMT). Constructing a DMT for a discussion includes two phases. First, an a priori analysis of the task, as it is presented to the students, directs the construction. The analysis is carried out by experts (mathematicians) who represent the canonical discourse, that is the discourse deemed mathematically correct by the community of mathematicians, and is supported by textbooks and curricula. The analysis includes determining the root node—the object at the center of the task, considering possible narratives that would be involved in a discussion around the task to determine the objects the task deals with, listing these objects' realizations, and finally grouping together realizations of a similar type. Each type of realization belongs to a specific discourse as it is associated with its own keywords, mediators, narratives and routines of manipulation.[2] For example, a realization utilizing row reduction to determine a given matrix's rank includes keywords (e.g., rows, reduction), mediators (e.g., $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$), narratives (e.g., row reduction preserves the row space) and routines (e.g., switching rows of a matrix $R_1 \leftrightarrow R_2$) which have no meaning within the functions discourse. Each realization is placed according to its type of realization on a separate branch of the DMT. For example, a realization utilizing row reduction would be placed on the matrices branch of the DMT. This phase, which considers hypothetical mathematical narratives, was used to prepare the analysis of the classroom discussion in the second phase.

The second phase is the a posteriori construction of a DMT based on a video recording of a discussion. In this phase, for each mathematical narrative authored in the discussion the objects realized by the narrative are determined and the type of realizations of the objects are classified by the types of realizations identified a priori. The narratives are written in boxes and placed in the branches of the tree, which were determined in the first phase. The root of the tree is the object towards which the discussion is aimed. Connections between narratives authored or implied during the discussion are indicated by lines linking between boxes. See Fig. 2 for an example DMT. Light boxes and solid lines signal narratives that were authored by students and dark boxes and broken lines signal narratives that were authored by the instructor, similar to other realization tree mappings (e.g. Weingarden et al., 2019). The construction process is exemplified in detail in the findings section.

The trustworthiness of this analysis is ensured through prolonged engagement with the data and expert debriefings. The first author functioned as a participant-observer by moderating the discussion sessions, teaching regular tutorial sessions in the course and analyzing the data. The analysis was from recordings to promote greater objectivity.

5. Findings

In this section we exemplify the process of building a DMT for a task that was posed to a class during a workshop and the discussion around it. We start with the a priori analysis of the tasks, then we exemplify the a posteriori analysis of the classroom discussion that took place around this task and finally we describe what a DMT exposes about a discussion. We then briefly discuss the DMTs of other tasks and classroom discussions.

5.1. A Priori analysis – mathematics available in the task

We exemplify the a priori mapping for the following task.

Task: T: $Z_5^4 \rightarrow Z_5^4$ is a linear transformation such that T(1,2,3,4) = (0,0,0,0).

For which values n ϵ N does there exist such a T where dim Ker T = n? For those values give an example of such a T and find a basis for Ker T.

The first step of constructing a DMT (see Fig. 2 for an example) is determining the object most central to the solution to use as the root node of the tree. We considered what mathematical objects the task examines and what is described in the solution process. Solving the above task includes defining a linear transformation with certain properties, thus we determined the main object to be the "linear transformation" object. Additionally, the task asks to find a basis for the kernel of this transformation, thus objects from the vector space discourse (e.g., basis, spanning set, linear independent set) are invoked by solving this task as well as objects from the function discourse (e.g., image, domain, kernel).

The next step is considering narratives that might be included in a discussion around the task and classifying the objects that form the protagonists of these narratives according to their possible realizations. These realizations were classified according to the type of realization based on the following analysis:

Function realization: A linear transformation is, firstly, a transformation. That is, it is a function or a mapping between sets. Thus, the object $T : U \rightarrow V$ has realizations using sets and functions irrespective of the other properties of T. The narrative that T preserves linearity realizes T solely as a function, as this describes function type properties and does not pertain to other types of realizations. Other narratives that realize T as a function include *T(1,2,3) = (3,3,3), the image of a vector (x,y,z) is (x + y,x + y,x + y)* and *the transformation is injective*.

Vector space realization: A linear transformation is a transformation between vector spaces. The object $T : U \rightarrow V$ includes realizations of the vector spaces object, as the range and domain of the transformation are vector spaces. Defining the transformation also

[2] In practice the distinction between the types is rarely as clear as in theory, yet it is sufficiently perceptible for analysis.

defines other vector spaces, such as *Im T* and *Ker T*. These can be realized exclusively within the vector subspace discourse, albeit using the label from functions realizations. Narratives within the discourse of vector spaces include *the space being considered is* $sp\{v_1, v_2, v_3\}$ or *the basis of the kernel is {(1,0,0,0), (0,0,1,0)}*.

Matrix realization: A linear transformation can be realized by a matrix representation. This describes a linear transformation as a product of a $m \times n$ matrix and a $n \times 1$ matrix. This includes narratives such as *a given matrix is invertible* and *the rank of the representative matrix characterizes the linear transformation*.

There are other types of realizations that we did not include as branches in the DMT. Linear transformations can be realized as elements of Hom(V,W), so there are narratives with objects using this type of realization (e.g., R^4 *is isomorphic to* R_3[x]). However, this notion is not included in the curriculum of the linear algebra course examined in this study and these narratives and objects do not appear in the recorded discussions. Thus, these types of objects and realizations are not included in the DMTs constructed.

Additionally, there are multiple objects that are not considered in this analysis. This includes objects, realizations and routines assumed to be individualized for university students (e.g., fraction notation, basic arithmetic routines) and narratives considered as previous knowledge at the stage of the semester when the task is presented (e.g., properties of fields, properties of sets). Thus, these types of realizations are also not discussed in this paper.

As detailed above, our a priori analysis revealed that a discussion around this task could include narratives about objects including functions, vector spaces, and matrices as realizations of linear transformations. These were thus chosen as the branches of the tree. The a priori DMT below (Fig. 1) displays the types of realizations available for the objects on which the task focuses via the branches shown and a possible narrative for each type of realization.

5.2. A posteriori analysis – mapping the mathematical narratives of a discussion

We now turn to mapping a discussion around this task in a classroom. The following discussion took place during a workshop held in the eleventh week of a thirteen-week semester and 7 students attended. The task[3] asked for which values of n does there exist a linear transformation T such that T(1,2,3,4) = (0,0,0,0) and dim Ker T = n.

During the discussion the students realized the linear transformation in the vector space discourse, while the instructor included realizations from other discourses and attempted to connect the students' realizations in the vector space discourse to realizations in the functions discourse.

We see this in the DMT in Fig. 2, which maps part of the discussion. The numbers (e.g., (10)) signify authored narratives and the order in which they first appeared in the discussion; Roman numerals (e.g., (II)) signify authored connections between different types of realizations; And letters (e.g., (D)) signify connections between narratives utilizing realizations of the same type. The smaller rectangles are not discussed in the analysis in this section and are thus smaller.

During the small group discussions (not shown on the DMT nor analyzed here), prior to the whole-class discussion, students constructed examples of linear transformations for some values of n for which dim Ker T = n. Building on that, the instructor initiated the whole-class discussion by focusing on specific values of n and asked, "Is there a transformation for which the dimension of the kernel is 0?" After some discussion the class agreed that if T(1,2,3,4) = (0,0,0,0) the dimension of the kernel must be at least one. The instructor then asked whether a linear transformation T would satisfy the task's conditions if dim(ker T)= 1.

In answer to this question, Dan went to the board to show such a linear transformation. "We can define the linear transformation by its behavior on the basis", he said. He then wrote four column vectors, which he later labelled a_1, a_2, a_3, and a_4. He justified that *{a_1, a_2, a_3, a_4}* was a basis by stating that the set was linearly independent and maximal. Then, to meet the task's condition, he set T(1,2,3,4) = (0,0,0,0) and claimed that to ensure dim kerT = 1 it was sufficient that the image of the remaining three vectors be anything other than 0. He then wrote images of these four vectors on the board[4] and concluded that his construction fulfilled all the conditions (see Fig. 3 for a reconstruction of his work).

In the DMT (Fig. 2) Dan's communication sits entirely within the vector space discourse. His first narrative – that the set of four vectors {a_1, a_2, a_3, a_4} form a basis appears as Box [**3**]; the next, that defining the images of that basis determines the transformation, as Box [**4**]. The link between them (Link [**A**]) binds the two realizations: both realize the transformation through the properties of vectors. Yet, realizing the transformation as four specific mappings does not make use of the function discourse, in which a linear transformation acts on every vector of Z_5^4. From this partial framing came his misstep: using the description on the board (Box [**4**]), he asserted that dim Ker T_1= 1 (Box [**5**]). The kernel, treated as a vector space, was taken for granted rather than derived from its functional definition as ($Ker\ T = \{v \in Z_5^4 \mid T(v) = \overrightarrow{0}\}$). The bond between Boxes [**4**] and [**5**] (Link [**B**]) rests on Dan's mistaken claim that any non-zero images for the other three vectors guarantee that result. In short, Boxes [**3**], [**4**], and [**5**] and Links [**A**] and [**B**] trace a coherent but self-contained thread within the vector-space discourse.

Dan's argument remained coherent within the vector space discourse, yet it relied on a justification – the behavior of T as a function on every vector – that belongs to the functions discourse and was never invoked. In fact, assigning images to a basis $\{b_1, \ldots, b_n\}$ does uniquely determine a linear transformation T such that $T(b_i) = v_i$; but that relies on *linearity* – a property that only becomes visible when the transformation is realized as a function acting on every vector in Z_5^4. Within the vector space discourse, it is common to treat

[3] This task is worded using the vector space Z_5^4, that is the scalars are from the finite field Z_5. However, the scalars used (i.e. the field) were not mentioned during the discussion and are in any case mathematically irrelevant.

[4] Both transformations presented in the discussion were defined as T, for the purpose of this analysis they are labeled as T_1 and T_2.

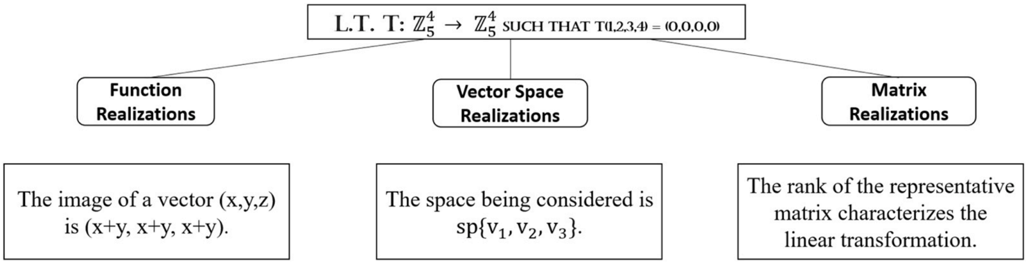

Fig. 1. Branches of tree mapping for linear transformation task.

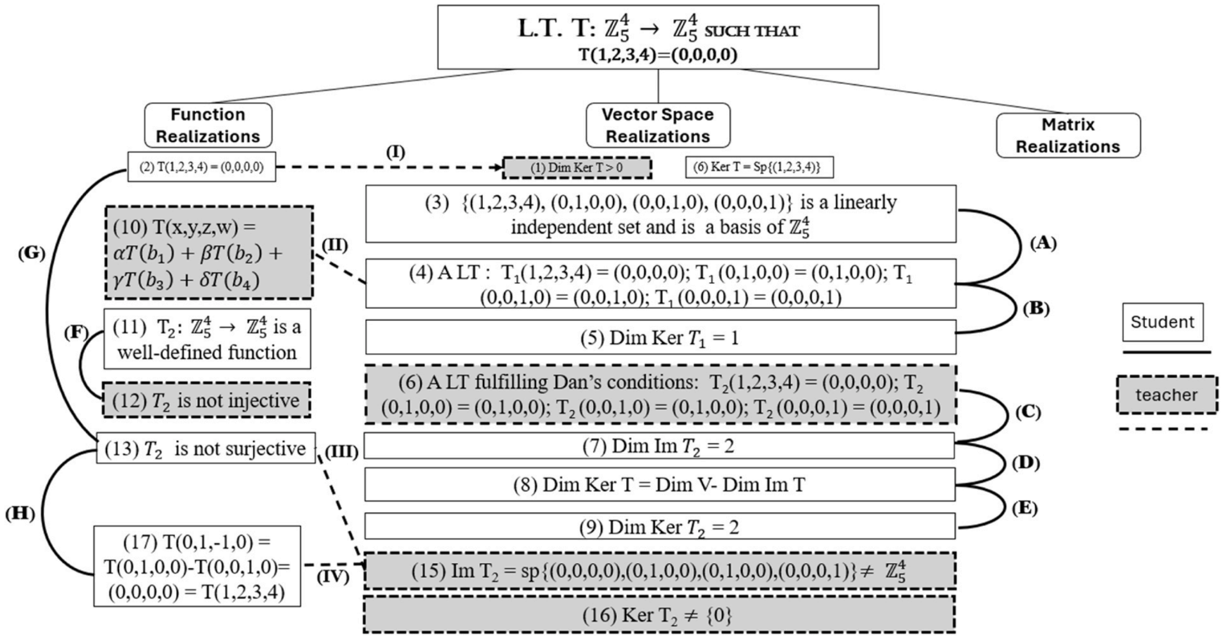

Fig. 2. DMT of a part of the whole-class discussion.

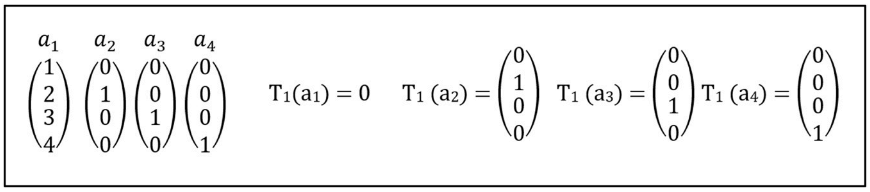

Fig. 3. Reconstruction of whiteboard.

properties of a finite set of vectors (particularly a basis) as representative of properties of the entire vector space. However, within the functions discourse, that leap demands proof through the structure of the mapping itself. Dan's statement that "only the image of (1,2,3,4) is zero" stretched a local property – true to those four vectors – into a global one, without invoking linearity. The DMT captures this gap: the sequence of Boxes [**3**]-[**5**] coheres inside one discourse but does not include the cross-branch link between discourses that would have anchored his claim in the functions discourse and exposed the mathematical error.

Noticing that the class aligned with Dan and no one voiced objections, the instructor recognized that the class was realizing the kernel entirely within the vector space discourse. Ker T is a label for a set of vectors ($Ker\ T = \{v \in Z_5^4 \mid T(v) = \overrightarrow{0}\}$) which the students realized as a vector space, relating only to its properties as vector space and disregarding its construction from the function realization. The discourse of vector spaces was the one with which the students felt comfortable, thus to challenge the students' complacency without leaving the familiar discourse, the instructor proposed a counterexample. Using Dan's construction T_1 from the board, she defined a new linear transformation T_2, based on her implicit realization of transformations in the function discourse, for which $T_2(a_2)$

= $T_2(a_3)$ (Box [**6**]). Using the linearity property of the transformation, realized as a function, it holds that $T_2(a_2 - a_3) = 0$, and the kernel includes other vectors. However, the presentation to the class of T_2, similar to the presentation of T_1, relied on a vector space realization: specifying images of basis vectors. The instructor did not expose her realization from the function discourse but stayed within the discourse comfortable to the students. T_2 met all the students' stated conditions and still produced a different outcome in the vector space discourse. The image of T_2 is Sp{(0,1,0,0), (0,0,0,1)}, so that dim Ker $T_2 = 2$, and not 1 as suggested by the students.

With some prompting, the instructor led the class to author the narratives supporting and justifying the claims about T_2: dim Im T_2 = 2 (Box [**7**]), the dimension theorem for linear transformations (dim Ker T + dim Im T = dim V) (Box [**8**]), and dim Ker T_2 = 2 (Box [**9**]). The links among these [Links [**C**], [**D**] and [**E**]) represent justifications still framed inside the vector space discourse. On the DMT, this episode ([**7**], [**8**], [**9**,], [**C**], [**D**], [**E**]), including the counterexample, appears as a compact cluster of boxes on the same branch as Dan's original claim.

Sensing that the counterexample still had not shifted the class's discourse, the instructor turned the students attention toward the functions discourse. "The transformation needs to be given with the general term", she reminded them, "and we know how to do that". This remark introduced a *function type realization* of a linear transformation – "the general term" (Box [**10**]), and hinted at the connection (Link [**II**]) between what was written on the board – a *vector space realization*, and the broader realization of T_2 as a mapping on all of Z_5^4. The instructor's reference to "We know how" pointed back to the workshop's opening, when she had demonstrated this very procedure through an example (T(*x,y,z,w*) = (*x-y,x-y,x-y*)). That procedure depends on two theorems stated earlier in the course: any vector can be written uniquely as a linear combination of any basis and a linear transformation preserves linear combinations (T*(au+v)*=*a*T*(u)*+T*(v)*). In the workshop's introduction, the emphasis had been on the technical side of linking these two realizations, expressing T both through its action on basis vectors and through a description of the transformation on a general element. However, the bridge between the discourses was not discussed explicitly. Now, assuming the students shared the implicit narratives regarding bases and preservation of linear combination, the instructor invoked the connection without making it explicit. On the DMT (Fig. 2), the instructor's move appears as a link (Link [**II**]) between Boxes [**4**] and [**10**]: a bridge authored by the instructor rather than by the students. It marks her attempt to bring the function discourse into play, even if the narratives connecting these two discourses remained implicit.

The turning point came when the discussion finally entered the functions discourse. A student, puzzled, asked, "is what we defined even a linear transformation?" – a question that invited the class to check T_2 against the formal definition of a function. The instructor followed this opening, asking, "what is the definition of a linear transformation?" This prompted a *function type of realization*: to describe T_2 as a mapping on all of Z_5^4. One student then said that defining T_2 such that T_2(0,1,0,0) = T_2(0,0,1,0) = (0,1,0,0) contradicts the definition of T_2 as a function (Box [**11**]). That statement allowed the class to describe the difference (Link [**F**]) between a well-defined function and an injective function (Box [**12**]), terms that describe properties of transformations on the whole space and use a *function type of realization*. From here, the class also articulated a narrative describing the surjective property of T_2 (Box [**13**]) connecting it (Link [**G**]) to the given property of T(1,2,3,4) = (0,0,0,0) (Box [**2**]). A comment which was a vague doubt thus became a hinge in the discussion, drawing the class into the functions discourse and grounding their earlier, vector space narratives in the behavior of T_2 as a function on the whole space.

Building on this opening, the instructor began to weave the two discourses together. She used the students' new narratives from within the functions discourse to author explicit links back to the vector space discourse. The function based statement about T_2 being surjective (Box [**13**]) was connected (Link [**III**]) to determining if the vector space ImT_2 equals Z_5^4 – a question expressed through a *vector space type realization* of finding the linear span of images of the basis vectors (Box [**15**]).

Next, the class examined injectivity. The instructor invited them to look for another vector, besides (1,2,3,4), whose image under T_2 is (0,0,0,0) (Box [**17**]), a move grounded in the *functions discourse*. She asked the students to calculate T(0,1,-1,0) using what was given about T(0,1,0,0) and T(0,0,1,0). The students concluded that T_2(1,2,3,4)=T_2(0,1,-1,0) and therefore T_2 is not injective (Link [**H**]). The instructor then connected this realization (Link [**IV**]) to a vector-space narrative – that the kernel of T_2 contains more than just the zero vector (Box [**16**]).

On the DMT, these paired connections – [**III**] and [**IV**] – mark the instructor's deliberate effort to bridge the two branches. Although it is not clear whether any student ultimately objectified the linear transformation within the coalesced discourse of linear algebra, the discussion shows movement in that direction. The students, who began the episode communicating solely within the vector space discourse, eventually authored several narratives using *function type of realizations* (Boxes [**11**], [**13**] and [**17**]). The DMT in Fig. 2 makes this visible: most student authored boxes remain on the vector space branch, connected through links of the same type, while the instructor's interventions introduce crossings between branches. Once those crossings appeared, students began to use function type realizations themselves, extending their earlier narratives about vectors to statements about mappings on the whole space.

By the end of the class discussion, the linear transformation was no longer realized only as the image of four specific vectors but as a function defined on all of Z_5^4 – an object whose properties drew on both discourses and on the connections between them.

5.3. Zooming out – what we learn from additional DMTs

We next zoomed out to examine the DMT of a discussion as a whole. The analysis in the previous section showed us that the connections between the types of realizations visualize aspects of the discussion that we found crucial in terms of the narrative development, more so than the connections between the same types of realizations. Thus, in this section we focus on the branches and the connections between them. The images (Figs. 4, 5 and 6) were thus simplified by omitting the connections between narratives on the same branch.

The DMT of the entire linear transformation discussion (Fig. 4) consolidates the mathematical activity of the class discussion into a single view. From this DMT, we can see the broader structure of the discourse in this discussion: nearly all student contributions remained within the vector-space branch, with the function and matrix realizations appearing only through the instructor's interventions. This overview highlights the difference between the a priori map (Fig. 1), namely what types of realizations were possible, with what was realized in practice.

Fig. 5 describes a discussion in which the task was to determine the conditions on parameters $a_1, a_2, \ldots a_n$ for which a given $n \times n$ matrix would not be diagonalizable. As in the DMT of linear transformations, the a priori map included three potential branches – matrix realizations, vector space realizations, and algebraic realizations. Yet the a posteriori map again shows an uneven distribution of activity within the available three branches.

In this final workshop of the semester 25 students participated. The discussion revolved around the matrix's diagonalizability and, as Fig. 5 shows, included *matrix type realizations* and *algebraic type realizations*, but no *vector space type realizations*. Students initially reasoned within the algebraic branch, working with the characteristic polynomial and algebraic manipulations of its coefficients and roots. The sole explicit link between the algebraic and matrix branches was authored by the instructor, who attempted to broaden the discourse and connect the symbolic work on polynomials to properties of a matrix such as similarity or invertibility.

In the a priori map (visible as the branches of the a posteriori map in Fig. 5), each branch offered a distinct realization of a diagonalizable matrix: as a matrix, it could be *similar to another matrix*; in the vector space discourse, it could be realized as *there exists a basis of eigenvectors for the vector space F^n*; and in the algebraic discourse, it could be described through its characteristic polynomial and eigenvalues. The a posteriori DMT (Fig. 5), however, makes visible that the classroom discussion actualized only part of this landscape. Students realized the diagonalizable matrix algebraically, while the instructor, who had access to all three discourses, drew the cross-branch connection but did not open the vector space realization to public discussion.

In contrast to the linear transformation discussion (Fig. 4) and the diagonalizable matrix discussion (Fig. 5), the DMT for another lesson shows a different pattern of student participation and mathematical diversity of the narratives. In the eighth week of the semester, 24 students took part in a discussion about linear independence. The task asked them to either prove claims about given sets of vectors or provide counterexamples. The discussion was mapped onto a DMT (Fig. 6). The a priori analysis revealed four types of realizations, seen in Fig. 6 as four branches.

The DMT in Fig. 6 displays four distinct types of realizations and the links connecting them. The narrative, "0 is an element of the set" realizes a linearly independent set via a *set type realization*. The narrative using a linear combination of them, "$\alpha \cdot v_1 + \beta \cdot v_2 + \gamma \cdot v_3 + \delta \cdot v_4 = 0$" uses a *vector type of realization*. "Row reduction to echelon form results in no zero rows" realizes the object in matrix discourse. Finally, "none of the vectors is in the span of the others" exemplifies a *vector space type realization*.

Unlike the previous discussions, the DMT in Fig. 6 shows that the students themselves authored narratives across all four branches and linked them without instructor mediation. The instructor's contributions are not mapped at all in this figure, as she only moderated the discussion in this lesson. This pattern suggests the cross-branch integration occurred within the students' own discourse. Linear independence was realized simultaneously as a property of a set, as a vector relation, as a matrix structure, and as a subspace configuration. This notable flexibility may have been a result of the lesson occurring at a relatively late stage of the students' learning about linear independence in the course.

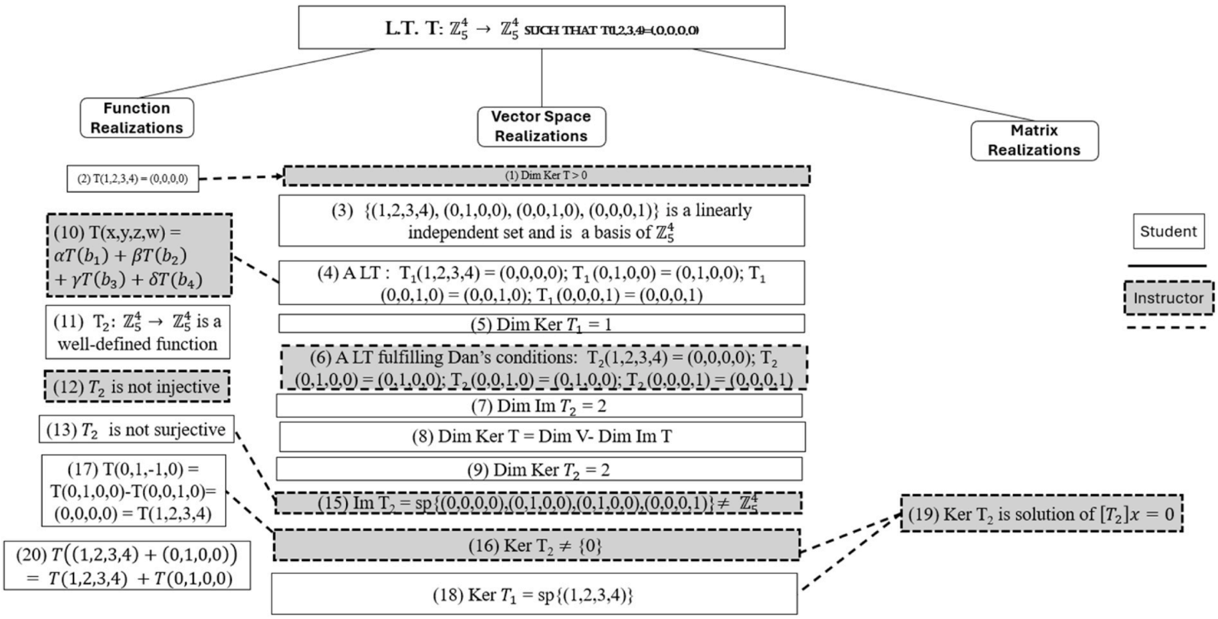

Fig. 4. DMT Linear Transformation Discussion.

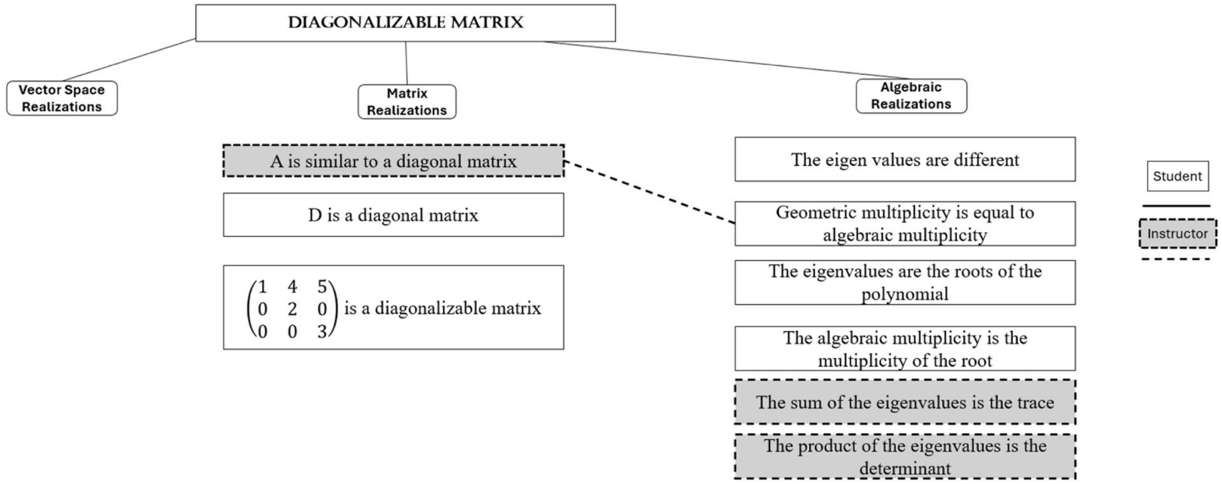

Fig. 5. DMT for Diagonalizable Matrix Discussion.

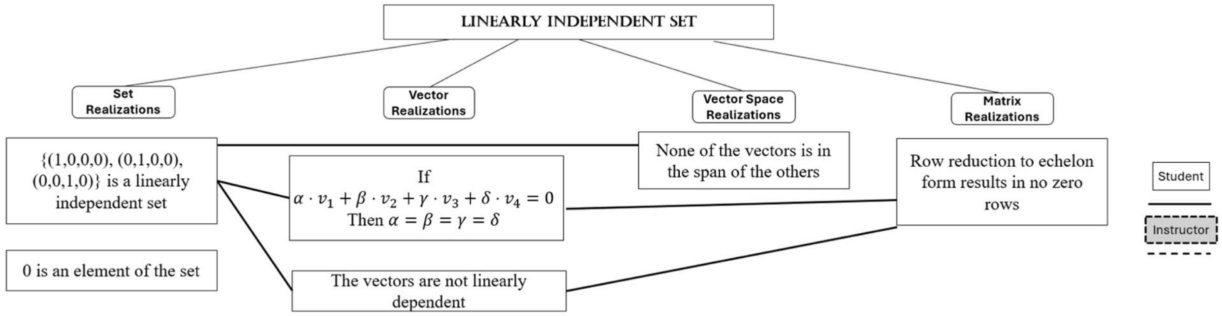

Fig. 6. DMT Linearly Independent Set Discussion.

6. Discussion

In this study we developed a tool for mapping the mathematical content of whole-class discussions in linear algebra and applied it to several such discussions. The discussions occurred during workshops designed to encourage student participation and involved specially designed tasks that could support rich mathematical discussions. Our aim was to characterize what mathematical narratives, realizations, and connections become publicly available to the class during these discussions. To this end, the Discourse Mapping Tree (DMT) represents the public mathematical discourse by mapping the narratives that are articulated, the objects construed by them, the types of realizations through which these objects are expressed, and the links authored between different realizations. By mapping these aspects together in a single representation, the DMT enables an overview of the mathematical content that is articulated across an entire discussion, making it possible to examine which realizations are taken up, by whom (teacher or students), how they are connected, and which potential connections remain unarticulated.

In our application of the DMT to several whole-class discussions in a linear algebra workshop, one pattern that became visible concerns the division of labor between students and the instructor in authoring different types of mathematical connections. In two of the discussions analyzed in detail, the DMT revealed a clear asymmetry: students predominantly authored intra-discursive links, connecting realizations within the same discourse, whereas inter-discursive links (connecting realizations belonging to different discourses) were primarily authored by the instructor. The synoptic representation provided by the DMT made this pattern visible across the discussion as a whole, rather than as a sequence of isolated turns. Although we also observed a discussion that did not conform to this pattern, the analysis of additional lessons from the same course (Wallach, 2022) suggested that this division of labor was prevalent in this instructional context.

The important role of the instructor in authoring inter-discursive links is consistent with previous commognitive studies suggesting that, for certain types of mathematical content, experts (teachers or instructors) may play an indispensable role. For example, Sfard (2007) argued that learners often struggle to independently establish new meta-rules, norms concerning how mathematical narratives are endorsed, such as those involved in arithmetic with negative numbers. Similarly, Weingarden and colleagues (2019) showed that teachers' interventions were crucial in encouraging connections between different realizations of objects.

The present analysis does not allow us to determine why inter-discursive links were primarily instructor-authored in the discussions examined. However, the DMT makes visible that linking between realizations belonging to different discourses poses a recurring challenge in whole-class discussions, even in contexts where students have previously encountered relevant discourses. In this sense, the DMT mapping highlights the *location* of potential learning demands, namely, the transition between discourses, without claiming that such demands were resolved or that new meta-rules were adopted. Future research would be needed to examine whether and how inter-discursive linking involves meta-level learning processes.

The analysis presented here highlights three main contributions of the Discourse Mapping Tree. First, the DMT extends existing commognitive visualization tools by enabling the mapping of multiple mathematical objects and their realizations within a single whole-class discussions. Whereas prior realization trees have typically focused on a single object (e.g. Weingarden et al., 2019; Knox & Kontorovich, 2023), the DMT is suited to tertiary mathematics contexts in which several objects and realizations may be invoked and intertwined within one task or discussion. This affordance is particularly important in linear algebra, where the coordination of multiple objects and representations is central.

A second contribution of the DMT concerns its differentiation between types of links authored during a discussion, specifically, intra-discursive and inter-discursive links. This differentiation is particularly useful for making their authorship (instructor or students) visible. As demonstrated above, this differentiation enabled the identification of recurring patterns in the division of labor between students and the instructor, which would have been difficult to discern from turn-by-turn analyses alone. Differentiating between intra-discursive links and inter-discursive links might be used in future studies to explore the tension that exists between the necessity of instructors presenting accurate mathematical content through delivery methods versus the aspiration for student responsibility for their learning via active engagement in the learning process. This tension can be placed within the context of the ongoing debate in other settings between those that claim students can independently invent mathematics and those that insist mathematical accuracy necessitates being explicitly stated by experts. These two disparate paradigms are exemplified in two guides for teachers: One states, "without formal instruction on specific algorithms or procedures, children **can** construct viable solutions to a variety of problems" (Carpenter et al., 1996, p.6). While the opposing view is exemplified by the statement, "information required by the citizens of an intellectually advanced society […] requires direct, explicit instruction" (Sweller et al., 2007, p. 121). The DMT mapping of classroom discussions hints that this tension might be resolved by utilizing different pedagogical approaches for the different types of links. That is, independent, student invention of intra-discursive links and explicit, accurate introduction by an expert of inter-discursive links.

A third contribution of the DMT lies in its capacity to render visible saming demands in whole-class discussions through the mapping of realizations and the links authored between them. By foregrounding when and how realizations belonging to different discourses are brought into relation, the DMT supports a characterization of objectification as it arises in classroom discussions. For example, in the discussion of linear transformations, the DMT made visible that students' narratives consistently realized the linear transformation within the vector space discourse, treating the kernel as a vector space characterized by its basis and dimension. The saming between this realization and a function realization of the transformation, necessary for endorsing claims about the kernel as the preimage of zero, was introduced only through instructor-authored narratives. In this sense, the DMT contributes to existing work that examines mathematics learning at the level of the classroom (Rasmussen & Kwon, 2007; Ernest et. al., 2019). Future studies, however, will be needed to develop this tool to record actual objectification processes as they occur during classroom discussions. Such developments may necessitate combining the relatively coarse-grained analysis of the DMT, with more micro-analytic methods of turn-by-turn discourse analysis.

The DMT builds on recent commognitive visualization tools that have been used to map realizations and saming narratives in classroom discussions (Weingarden et al., 2019; Weingarden, 2024). While these tools have primarily focused on single mathematical objects or narrowly defined content, the DMT extends this line of work by supporting the mapping of multiple objects, realizations, and links within a single whole-class discussion. This extension is particularly relevant in tertiary mathematics contexts, such as linear algebra, where the mathematical activity often involves coordinating multiple objects and realizations, a well-documented source of difficulty (Grenier-Boley, 2014; Harel, 2002).

By providing a concise visual representation of the mathematical content articulated in a whole-class discussion, the DMT may support analytic and reflective work on classroom discourse. Teachers and observers often recall whether a discussion was lively or whether multiple solutions were proposed, yet reconstructing which mathematical ideas, realizations, and connections were made publicly available is considerably more difficult. The DMT offers a way to make such aspects of a discussion explicit, thereby supporting reflection on the mathematical content that was made available in the public space. Importantly, this does not mean that the DMT should be used as an evaluative tool; rather, it is best understood as supporting analytic and reflective engagement with the mathematical content of classroom discussions.

Building on this analytic affordance, recent work has shown that realization-tree-based tools can function as productive teaching representations for supporting teacher learning, by directing teachers' attention to the mathematical objects, realizations, and connections that structure classroom discourse (Weingarden, 2024). In this sense, the DMT may serve not only as a research tool, but also as a resource for teachers' professional learning, supporting reflection on the mathematical content of discussions without presupposing particular instructional moves or learning outcomes.

Constructing DMTs from video recordings, while less time consuming than micro-analytical discourse analysis, still requires substantial analytic work. This raises the questions regarding the feasibility of employing the tool for purposes such as lesson preparation, lesson reflection and other practical uses. Prior work suggests that realization-tree-based tools can be adapted as teaching representations when appropriately simplified and adapted for reflective purposes (Weingarden, 2024); exploring such adaptations of the DMT remains a direction for future research. Regardless, extending the use of the DMT to additional contexts should be approached with care, considering the differences in content, level, and classroom norms.

Another limitation of the DMT lies in its deliberate focus on the mathematical content, which entails backgrounding other important aspects of classroom discussion. The DMT operates at the level of the class and does not trace individual student's contributions or temporal unfolding of narratives, nor does it account for social or affective dimensions of interaction. Nevertheless, given the availability of analytic tools that foreground such aspects of classroom discourse (e.g. Chen et al., 2015; Bouton & Asterhan, 2023), the DMT's focus on the mathematical dimension can be viewed as complementary rather than deficient. In the future, studies combining tools that analyze social aspects of classroom discussion with the DMT's analysis of the mathematical aspects may yield additional insights.

To conclude, this study contributes to ongoing efforts to better understand whole-class mathematical discussions by offering a tool for mapping the mathematical content that becomes publicly available during such discussions. Through this mapping, the DMT provides a means for characterizing the mathematical landscape of classroom discourse without reducing it to individual learning trajectories or social interactional patterns. In doing so, it opens new possibilities for comparative analyses of tasks, discussions, and instructional contexts, and for more precise theorizing about the demands placed on students and instructors in coordinating mathematical discussions at the tertiary level.

Author statement

All authors contributed to the study conception and design. All authors read and approved the final manuscript.

On behalf of all authors, the corresponding author states that there is no conflict of interest.

This project was approved by the Behavioral Sciences Research Ethics Committee of the Technion - Israel Institute of Technology.

CRediT authorship contribution statement

Ram Band: Validation, Data curation, Conceptualization. **Einat Heyd-Metzuyanim:** Writing – review & editing, Supervision, Methodology, Conceptualization. **Miriam N. Wallach:** Writing – review & editing, Writing – original draft, Visualization, Validation, Project administration, Methodology, Investigation, Formal analysis, Data curation, Conceptualization.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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