JOHNSON–SCHWARTZMAN GAP LABELLING FOR METRIC AND DISCRETE DECORATED GRAPHS

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Abstract. We study Schrödinger operators on metric and discrete decorated graphs. The values taken by the integrated density of states (IDS) on spectral gaps are called gap labels. A natural question is which gap labels can occur. We answer this for graphs arising from uniquely ergodic one-dimensional dynamical systems by proving Johnson–Schwartzman gap-labelling theorems in both the metric and discrete settings.

Our results extend Johnson–Schwartzman gap labelling beyond the standard one-dimensional setting. Unlike in one dimension, these graphs may contain cycles, which prevent the use of Sturm oscillation theory and require different spectral methods.

We also analyze discontinuities of the IDS for certain graph families and show that not every admissible label corresponds to an open spectral gap. This reveals a mechanism of gap closing driven by graph geometry rather than by the underlying dynamics.

1. Introduction and main results

This paper studies the integrated density of states (IDS) of Schrödinger operators on discrete and metric graphs constructed through ergodic one-dimensional dynamical systems. The IDS, which roughly counts the number of eigenstates per unit volume below a given energy level, is a widely studied object in spectral theory, quantum mechanics, and solid state physics. It is an important tool for characterizing the spectral gaps – the connected components of the spectrum's complement.

Since the IDS is monotone increasing and is constant at spectral gaps, each gap can be assigned a specific label based on the value of the IDS within the gap. The *gap labels* of an operator are of significant physical importance, for instance for characterizing the Hall conductance in the Integer Quantum Hall Effect [AOS03].

Traditionally, the first step in deriving the gap labels is by determining the set of their allowed values, in the form of a gap labelling theorem (GLT). Historically, proving gap labelling theorems often involved using K-theory, as originated in [BLT83, BBG92] (see also [Kel24] for a modern review). Nevertheless, for one-dimensional ergodic systems, there is an alternative approach to gap labelling. This was first done by Johnson [Joh86] (see [JM82, Sch57] for additional background), who showed that for certain Schrödinger operators on $\mathbb{R}$, the IDS takes values in a countable group that can be explicitly computed via a homomorphism introduced by Schwartzman. Since then, and especially in recent years, this approach (known as Johnson–Schwartzman gap labelling) has been successfully extended to ergodic Schrödinger operators on $\mathbb{Z}$, Jacobi matrices, and CMV matrices (see [DF22, DF23, DFZ23, DL25] and references therein). It is known that for one-dimensional systems and whenever both approaches (K-theory and Johnson–Schwartzman) provide a well-defined label set, these label sets agree [Kel].

While Johnson–Schwartzman gap labelling has been developed for various one-dimensional systems, many physical systems are modeled by more complicated network-like structures. This naturally leads to the study of Schrödinger operators on discrete and metric (quantum) graphs. These serve as models for various physical systems, and often exhibit interesting spectral properties which are not found in standard one-dimensional systems. With this in mind, the goal of this paper is to extend the Johnson–Schwartzman gap labelling to ergodic Schrödinger operators on metric and discrete graphs. In contrast to K-theoretic methods, the Johnson–Schwartzman gap-labelling is more directly tied to oscillatory properties of eigenfunctions, making it particularly suitable for Schrödinger operators on graphs. The graphs studied here, called decorated $\mathbb{Z}$-graphs, are inspired by one-dimensional tilings, see Figure 1.1. Here, the ergodic dynamical system determines the geometry of the graph itself, rather than just the potential. Developing Johnson–Schwartzman gap labelling for these graphs requires going beyond classical arguments based on Sturm's oscillation theorem, since these graphs contain cycles. We do so via analysis of these graphs' non-trivial nodal count, and with further tools from the spectral analysis on metric graphs. Finally, we show that for certain non-generic metric graphs, not all predicted gap labels actually appear as IDS gap labels, due to jump discontinuities in the IDS. We explicitly express all the energies of these discontinuities and the corresponding jump values for the class of Sturmian comb graphs.

The remainder of the paper is structured as follows: the following subsections provide the necessary background for stating our main results, which are then presented in Subsection 1.6. Section 2 presents additional necessary tools, followed by a proof of the GLT for metric graphs (Theorem 1.7). Section 3 then presents the proof of the GLT for discrete graphs (Theorem 1.9). Section 4 studies the existence of discontinuities in the IDS for Sturmian comb graphs. Appendix A presents results about the existence of the IDS for metric decorated graphs (Proposition 1.4), and Appendix B contains the proof of Lemma 2.1, which is needed for proving the metric GLT.

1.1. **Discrete and metric graphs.** The discrete graphs in this work are denoted by $G = (\mathcal{V}, \mathcal{E})$ (with the vertex and edge sets sometimes denoted $\mathcal{V}_G$, $\mathcal{E}_G$ for emphasis). For a vertex $v \in \mathcal{V}_G$, let $\mathcal{E}_v$ denote the set of edges incident to v. The degree of a vertex v, denoted $\deg(v)$, is the number of edges in $\mathcal{E}_v$. The discrete graphs here are equipped with the *normalized discrete Laplacian* Δ, acting on $\ell^2(G)$ as

$$\Delta\psi(v) = \psi(v) - \sum_{u\in\mathcal{E}_v} \frac{1}{\sqrt{\deg(v)\deg(u)}}\psi(u). \tag{1.1}$$

We consider infinite, connected graphs, with vertex degrees $\deg(v)$ uniformly bounded from above. With these assumptions, Δ is bounded and self-adjoint, and its spectrum $\mathrm{Spec}(\Delta)$ is contained in $[0, 2]$.

A *metric graph* $\Gamma = (G, \ell)$ consists of a discrete graph G, together with a length function $\ell : \mathcal{E} \to \mathbb{R}_+$ which assigns a positive length ℓ_e to each edge $e \in \mathcal{E}$. This equips Γ with the natural structure of a metric space, by identifying each edge $e \in \mathcal{E}$ with the interval $[0, \ell_e]$.

A *quantum graph* is a metric graph Γ equipped with a self-adjoint differential operator H acting on the Sobolev space $H^2(\Gamma) := \oplus_{e\in\mathcal{E}} H^2(0,\ell_e)$. The most common example is the Schrödinger operator $H = -\frac{d^2}{dx^2} + q(x)$, where $q \in L^\infty(\Gamma)$ is real-valued, along with vertex conditions which render H self-adjoint. The most common choice for the vertex conditions is known as the *Neumann-Kirchhoff* conditions (or standard conditions), which require:

1. The function f is continuous at each $v \in \mathcal{V}$, i.e.,

$$f|_e(v) = f|_{e'}(v), \forall e, e' \in \mathcal{E}_v. \tag{1.2}$$

2. The sum of the derivatives of f at v, taken in the outward direction along each edge, is zero:

$$\sum_{e\in\mathcal{E}_v} f'|_e(v) = 0. \tag{1.3}$$

In this work, we assume that the edge lengths ℓ_e are uniformly bounded from above and below. Under this condition (together with the assumptions above regarding the combinatorial structure of G), the associated Neumann-Kirchhoff Laplacian is self-adjoint and non-negative (see [Ber17, BK13, GS06, Kur24] for an extensive introduction to quantum graphs).

1.2. **Dynamics.** The graphs considered here are defined through one-dimensional dynamical systems, which govern their geometric structure. We now introduce the relevant definitions, and refer to [BG13, DF, DF22, Lot02] for additional background.

Let $\mathcal{A}$ be a finite set, called an *alphabet*, and consider the space of bi-infinite sequences $\mathcal{A}^{\mathbb{Z}}$. We equip this space with the product topology, as induced by the following metric:

$$d(\omega,\omega') = \sum_{n\in\mathbb{Z}} \frac{1-\delta_{\omega(n),\omega'(n)}}{2^{|n|}}, \tag{1.4}$$

where $\delta_{i,j}$ is the Kronecker delta. The space $\mathcal{A}^{\mathbb{Z}}$ is naturally equipped with the *shift* map (or left shift):

$$T : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}, \tag{1.5}$$

$$T\omega(n) = \omega(n+1). \tag{1.6}$$

Definition 1.1. A *subshift* is a closed, T-invariant subset $\Omega \subset \mathcal{A}^{\mathbb{Z}}$. We say that Ω is uniquely ergodic if there exists a unique T-invariant probability measure μ on Ω.

We define the letter counting function for $a \in \mathcal{A}$ on $\omega \in \Omega$, by

$$\#_a^N(\omega) := \#\{n \in \{0,...,N-1\} \ : \ \omega(n) = a\}. \tag{1.7}$$

For a uniquely ergodic subshift Ω, the letter frequencies

$$\nu_a = \lim_{N\to\infty} \frac{\#_a^N(\omega)}{N}, \tag{1.8}$$

are well-defined, independent of $\omega \in \Omega$, and satisfy $\sum_{a\in\mathcal{A}} \nu_a = 1$ ([BG13, prop. 4.4], [Oxt52]).

Example 1.2. Let $\alpha \in (0,1) \setminus \mathbb{Q}$ and $\theta \in [0,1)$. A Sturmian sequence $\omega_{\alpha,\theta}$ over the alphabet $\mathcal{A} = \{0,1\}$ is defined by

$$\omega_{\alpha,\theta}(n) = \chi_{[1-\alpha,1)}(n\alpha + \theta \bmod 1). \tag{1.9}$$

The *Sturmian subshift* is then given by

$$\Omega_\alpha := \overline{\{\omega_{\alpha,\theta} \ : \ \theta \in [0,1)\}} \subset \mathcal{A}^{\mathbb{Z}},$$

and is a uniquely ergodic subshift, with letter frequencies α and $1-\alpha$ for 1 and 0, respectively, [Que10].

1.3. **Decorated $\mathbb{Z}$-graphs.** We now introduce the class of *decorated $\mathbb{Z}$-graphs*, which are the graphs whose IDS is analyzed in this paper. To define those graphs, we fix a uniquely ergodic subshift (Ω, T) over a finite alphabet $\mathcal{A}$. We use these dynamics to define both metric and discrete families of graphs.

1.3.1. *Metric decorated $\mathbb{Z}$-graphs.* To each $a \in \mathcal{A}$, we associate a compact metric graph Γ_a, which we call a *decoration*; it may consist of just a single vertex. We also select a distinguished *base vertex* $v_a \in \mathcal{V}_{\Gamma_a}$ in each decoration. Given $L > 0$, we construct a family of infinite metric graphs $\Gamma_\Omega := (\Gamma_\omega)_{\omega\in\Omega}$ as follows: for each $\omega \in \Omega$, we begin with the bi-infinite chain graph whose vertices are $L\mathbb{Z}$. To each vertex $Ln \in L\mathbb{Z}$, attach the graph $\Gamma_{\omega(n)}$, by identifying the base vertex $v_a \in \mathcal{V}_{\Gamma_{\omega(n)}}$ with the vertex Ln (see Figure 1.1). This produces an infinite metric graph Γ_ω, obtained by decorating the chain graph $\mathbb{Z}$ with the graphs $\{\Gamma_a\}_{a\in\mathcal{A}}$ according to $\omega \in \Omega$.

We define the normalized length which is assigned to the graph family Γ_Ω by

$$\overline{L}(\Gamma_\Omega) := L + \sum_{a\in\mathcal{A}} \nu_a \ell_a, \tag{1.10}$$

where L is the horizontal distance between consecutive decorations, ν_a is the frequency of $a \in \mathcal{A}$ (1.8), and ℓ_a is the total length of the decoration Γ_a. Since the frequencies v_a are independent of $\omega \in \Omega$, the normalized length (1.10) may also be expressed through the average growth rate of geodesic balls (which is independent of the choice of $\omega \in \Omega$):

$$\overline{L}(\Gamma_\Omega) = \lim_{r\to\infty} \frac{\left| \Gamma_\omega|_{B(x,r)} \right|}{2\frac{r}{L}}, \qquad \forall \omega \in \Omega, \quad x \in \Gamma_\omega, \tag{1.11}$$

where $B_\omega(x,r)$ is the geodesic ball of radius r around $x \in \Gamma_\omega$, and $|\cdot|$ is the standard Lebesgue measure.

Example 1.3 (Sturmian comb)**.** Taking the Sturmian subshift Ω_α from Example 1.2, one may construct for each $\omega \in \Omega_\alpha$ a decorated $\mathbb{Z}$-graph Γ_ω by taking the bi-infinite chain with vertices $L\mathbb{Z}$, and attaching a dangling edge of length $\ell > 0$ at all vertices Ln such that $\omega(n) = 1$ (see Figure 1.1). In the notations above, this means that Γ_1 is a single edge graph of length ℓ and Γ_0 is the single vertex graph. In this case,

$$\overline{L}(\Gamma_\Omega) = L + \alpha\ell. \tag{1.12}$$

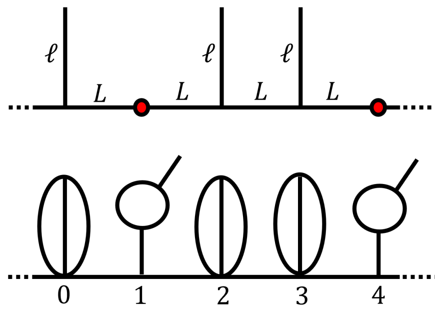

FIGURE 1.1. A Sturmian comb (top), along with a decorated $\mathbb{Z}$-graph with more complex decorations.

Any family of graphs Γ_Ω is equipped with a naturally induced shift,

$$T : \Gamma_\Omega \to \Gamma_\Omega, \tag{1.13}$$

$$T\Gamma_\omega = \Gamma_{T\omega}, \tag{1.14}$$

where we abuse the notation T. One can further define

$$T : C(\Gamma_\omega) \to C(\Gamma_{T\omega}), \tag{1.15}$$

$$(Tf)(x) = f\left(T^{-1}x\right), \tag{1.16}$$

where we once again abuse the notation T. Equipping each decorated graph Γ_ω with the Kirchhoff Laplacian H_ω, one can consider the family $H_\Omega := (H_\omega)_{\omega\in\Omega}$ as a dynamical system of operators, and get that the family H_Ω is covariant, i.e.,

$$H_{T\omega} = TH_\omega T^{-1},\ \forall \omega \in \Omega, \tag{1.17}$$

which means that the operators H_ω and $H_{T\omega}$ are unitarily equivalent. Unique ergodicity implies that $\mathrm{Spec}\,(H_\omega)$ is in fact almost-surely independent of $\omega \in \Omega$ (see also [BSon]), and so we can simply denote it by $\mathrm{Spec}\,(H_\Omega)$.

Remark. Most results in this work should also hold true when the decorated graphs are equipped with Schrödinger operators whose potentials and vertex conditions are naturally compatible with the subshift (Ω, T). For simplicity, we focus here on the Kirchhoff Laplacian.

1.3.2. *Discrete decorated $\mathbb{Z}$-graphs.* We consider a discrete version of decorated $\mathbb{Z}$-graphs, constructed in the same manner. Let (Ω, T) be a uniquely ergodic subshift over an alphabet $\mathcal{A}$. Let $(G_a)_{a\in\mathcal{A}}$ be a set of discrete graphs (the possible decorations), each assigned a base vertex $v_a \in \mathcal{V}_{G_a}$. Form a family of *discrete decorated* $\mathbb{Z}$-graph*s* $G_\Omega := (G_\omega)_{\omega\in\Omega}$ as follows: the graph G_ω is constructed from the chain graph $\mathbb{Z}$ by attaching to each vertex $n \in \mathbb{Z}$ the decoration $G_{\omega(n)}$, via the identification of the vertex $n \in \mathbb{Z}$ with the base vertex of $G_{\omega(n)}$. Each graph G_ω is equipped with the normalized discrete Laplacian, Δ_ω. We get the operator family $\Delta_\Omega := (\Delta_\omega)_{\omega\in\Omega}$, and as above, $\mathrm{Spec}\,(\Delta_\omega)$ is almost-surely independent of $\omega \in \Omega$, and is simply denoted by $\mathrm{Spec}\,(\Delta_\Omega)$.

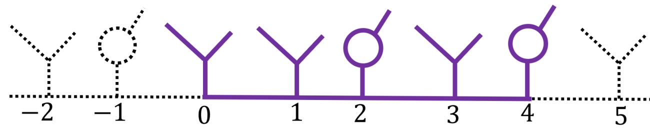

FIGURE 1.2. The compact graph $\Gamma_\omega|_{[0,4]}$, constructed by truncating the infinite graph Γ_ω and keeping five decorations.

In this setting, the analogue of the normalized length will be the average number of vertices:

$$\overline{V}\left(G_\Omega\right) := \sum_{a\in\mathcal{A}} \nu_a \left|\mathcal{V}_{G_a}\right| . \tag{1.18}$$

1.4. **Integrated density of states (IDS).**

1.4.1. *IDS for metric graphs.* Let (Ω, T) be a uniquely ergodic subshift, with an associated family of metric decorated $\mathbb{Z}$-graphs Γ_Ω. For $\omega \in \Omega$ and $n \in \mathbb{N}$, we restrict Γ_ω to a compact graph by removing the edges $(-1,0)\cdot L$ and $(n,n+1)\cdot L$ from Γ_ω, and denote by $\Gamma_\omega|_{[0,n]}$ the resulting compact connected component (see Figure 1.2). At the cut vertices, 0 and nL, impose Neumann-Kirchhoff vertex conditions (though the results below do not depend on the vertex conditions as long as they render the operator self-adjoint). The corresponding Kirchhoff Laplacian $H_\omega|_{[0,n]}$ is bounded from below and has a compact resolvent, and thus has purely discrete spectrum accumulating at infinity. Denote the associated normalized spectral counting function by

$$N_\omega^{(n)}\left(E\right) := \frac{\#\left\{\lambda \in \mathrm{Spec}\left(H_\omega|_{[0,n]}\right) : \lambda \le E\right\}}{\left|\Gamma_\omega|_{[0,n]}\right|}. \tag{1.19}$$

Proposition 1.4. *For almost all $\omega \in \Omega$, the sequence of functions $N_\omega^{(n)}\left(E\right)$ converges uniformly as $n \to \infty$ to a function $N_\Omega\left(E\right) : \mathbb{R} \to \mathbb{R}$. We call the function $N_\Omega\left(E\right)$ the integrated density of states (IDS) of the family H_Ω.*

The proof of the proposition above relies on an adaptation of a method from [GLV07] and appears in Appendix A.

The IDS is a nondecreasing function which is constant at each connected component of the complement of the spectrum (called *spectral gaps*). We are interested in the gap labels of N_Ω:

$$\mathcal{GL}\left(N_\Omega\right) := \left\{N_\Omega\left(E\right) : E \in \mathbb{R}\backslash \mathrm{Spec}\left(H_\Omega\right)\right\} \subset \mathbb{R}. \tag{1.20}$$

1.4.2. *IDS for discrete graphs.* The IDS $N_\Omega^\Delta\left(E\right)$ for the discrete Laplacian is defined similarly to the metric case discussed above. Remove from G_ω the two edges $(-1,0)$ and $(n,n+1)$ and denote by $G_\omega|_{[0,n]}$ the resulting compact connected component. The resulting operator $\Delta_\omega|_{[0,n]}$ is a self-adjoint matrix. We define the IDS as the limit of associated normalized spectral counting functions,

$$N_\Omega^\Delta(E) := \lim_{n\to\infty} \frac{\#\left\{\lambda \in \mathrm{Spec}\left(\Delta_\omega|_{[0,n]}\right) : \lambda \le E\right\}}{\left|\mathcal{V}_{G_\omega|_{[0,n]}}\right|}, \tag{1.21}$$

where the limit exists for almost all $\omega \in \Omega$ and its value is independent of ω (the proof is similar to that of Proposition 1.4, see Appendix A).

1.5. **The Schwartzman group.** We now define the Schwartzman group, which plays a central role in Theorems 1.7 and 1.9. For more details, see [DEF23, DF22, DF23, DFZ23] and references therein.

Let (Ω, T) be a uniquely ergodic subshift, equipped with a (unique) invariant probability measure μ. We associate with this dynamical system a suspension space:

$$X_\Omega := \Omega \times [0,1] \,/\, \{(\omega,1) \sim (T\omega, 0)\}. \tag{1.22}$$

The space X_Ω is naturally endowed with the translation flow in the second factor:

$$\tau^t : X_\Omega \to X_\Omega \quad (t \in \mathbb{R}), \tag{1.23}$$

$$\tau^t(\omega, s) = (\omega, t + s \bmod 1), \tag{1.24}$$

and with a probability measure η:

$$\int_{X_\Omega} f\, d\eta = \int_0^1 \int_\Omega f([\omega, t])\, d\mu(\omega)\, dt. \tag{1.25}$$

Let $C^\sharp(X_\Omega)$ be the space of homotopy classes of functions from X_Ω to the one-dimensional torus $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. For a given function $\phi : X_\Omega \to \mathbb{T}$, let ϕ_x denote the restriction of ϕ to the orbit of a point $x = (\omega, s) \in X_\Omega$ under the flow τ^t:

$$\phi_x : \mathbb{R} \to \mathbb{T}, \tag{1.26}$$

$$\phi_x(t) = \phi\left(\tau^t x\right). \tag{1.27}$$

Since $\mathbb{R}$ is the universal cover of $\mathbb{T}$, the function ϕ_x is naturally lifted to a map $\widetilde{\phi}_x(t) : \mathbb{R} \to \mathbb{R}$. With this in mind, define the Schwartzman homomorphism by

$$S_\Omega : C^\sharp(X_\Omega) \to \mathbb{R}, \tag{1.28}$$

$$S_\Omega([\phi]) = \lim_{t\to\infty} \frac{\widetilde{\phi}_x(t)}{t}, \tag{1.29}$$

where the limit above is μ almost-surely independent of ω (where $x = (\omega, s)$) [DF22, thm. 3.9.13]. In other words, the Schwartzman homomorphism is the average rate of rotation of ϕ_x along the flow.

Definition 1.5. The *Schwartzman group* $\mathfrak{S}_\Omega$ is the image of S_Ω.

The Schwartzman groupis a countable subgroup of $\mathbb{R}$, which depends on the full dynamical system (Ω, T, μ), but for brevity we denote it by $\mathfrak{S}_\Omega$.

Example 1.6. Let $\alpha \in (0,1) \setminus \mathbb{Q}$ and let (Ω_α, T) be the Sturmian subshift from Example 1.2. Its Schwartzman group is given by

$$\mathfrak{S}_{\Omega_\alpha} = \{\alpha n + m \ : \ m, n \in \mathbb{Z}\}, \tag{1.30}$$

see [DF, thm. 10.9.3].

1.6. **Main results.** Our first main result is a gap labelling theorem (GLT) for the metric graph operator family H_Ω.

Theorem 1.7. *Let (Ω, T) be a uniquely ergodic subshift, with an associated family of metric decorated $\mathbb{Z}$-graphs Γ_Ω, equipped with the Kirchhoff Laplacian. Then,*

$$\mathcal{GL}(N_\Omega) \subset \frac{1}{\overline{L}(\Gamma_\Omega)} \mathfrak{S}_\Omega \cap [0, \infty), \tag{1.31}$$

where $\mathfrak{S}_\Omega$ is the Schwartzman group.

The following is an immediate application of the theorem above to the Sturmian subshift Ω_α:

Corollary 1.8. *For a Sturmian decorated $\mathbb{Z}$-graph, the possible gap labels are given by*

$$\mathcal{GL}(N_{\Omega_\alpha}) \subset \left\{ \frac{\alpha n + m}{L + \alpha \ell_1 + (1-\alpha)\ell_2} \ : \ m, n \in \mathbb{Z} \right\} \cap [0, \infty), \tag{1.32}$$

where ℓ_1, ℓ_2 are the total lengths of the decorations, and L is the horizontal distance between the decorations.

Our next main result is a GLT for discrete decorated $\mathbb{Z}$-graphs:

Theorem 1.9. *Let (Ω, T) be a uniquely ergodic subshift, with an associated family of discrete decorated $\mathbb{Z}$-graphs G_Ω, equipped with the normalized discrete Laplacian. Then*

$$\mathcal{GL}\left(N_\Omega^\Delta\right) \subset \frac{1}{\overline{V}(G_\Omega)} \mathfrak{S}_\Omega \cap [0, 1]. \tag{1.33}$$

As we shall see, for some non-generic choices of the edge lengths, $\mathrm{Spec}(H_\Omega)$ may contain isolated eigenvalues, corresponding to jump discontinuities in the IDS. For Sturmian comb graphs (defined in Example 1.3), these eigenvalues and IDS jumps are fully characterized:

Theorem 1.10. *Let Γ_{Ω_α} be a metric Sturmian comb graph.*

If $E \in \mathrm{Spec}(H_{\Omega_\alpha})$ is an eigenvalue then E is of infinite multiplicity, with compactly supported eigenfunctions.

The resulting jump in the IDS takes one of the following values

$$\Delta N_{\Omega_\alpha}(E) \in \frac{1}{\ell\alpha + L} \{(c_1+1)\alpha - 1, \ -c_1\alpha + 1, \ \alpha\}, \tag{1.34}$$

where ℓ is the length of the decoration, L is the decoration spacing, and c_1 is the first digit in the continued fraction expansion of α:

$$\alpha = \frac{1}{c_1 + \frac{1}{c_2 + \ldots}}. \tag{1.35}$$

A more detailed version of Theorem 1.10, which also contains the explicit expressions for the eigenvalues, is presented as Theorem 4.1.

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2. Gap labelling for metric graphs - Proof of theorem 1.7

We begin by introducing the nodal count of metric graphs, which is later used for the proof of Theorem 1.7.

2.1. **The nodal surplus of a tile.** Let Γ_a be a decoration of type $a \in \mathcal{A}$ with base vertex v_a. For an energy $E \in \mathbb{R}$, consider the differential equation on Γ_a,

$$-\frac{d^2}{dx^2} f = Ef, \tag{2.1}$$

subject to the Kirchhoff conditions at all vertices of Γ_a, except for v_a, where we impose only a continuity condition (1.2), but no condition on the derivatives. Then for all but a discrete subset of $E \in \mathbb{R}$, this equation has a unique solution (up to a scalar multiple), denoted by f_E. This discrete set is exactly the spectrum of the Kirchhoff Laplacian on Γ_a with a Dirichlet condition imposed at v_a (see e.g., [BBS12, thm. 2.1 and cor. 2.4]). For E values outside of this discrete set we denote

$$m_a(E) := \sum_{e \in \mathcal{E}_{v_a}} \frac{f'_E|_e(v_a)}{f_E(v_a)}. \tag{2.2}$$

We use this to define an (energy dependent) Robin vertex condition at v_a:

$$g|_e(v_a) = g|_{e'}(v_a), \qquad \forall e, e' \in \mathcal{E}_{v_a}, \tag{2.3}$$

$$\sum_{e \in \mathcal{E}_{v_a}} g'|_e(v_a) = m_a(E)\, g(v_a). \tag{2.4}$$

By construction, (E, f_E) is an eigenpair of $-\frac{d^2}{dx^2}$ on Γ_a, with Kirchhoff condition imposed at all vertices except for v_a, where the Robin condition (2.3), (2.4) is imposed. We denote the resulting operator by $H|_{\Gamma_a}$, keeping in mind that this operator depends on E (but do not indicate this in the notation for brevity).

Denoting the (non-normalized) spectral counting function of $H|_{\Gamma_a}$ by

$$n^{(a)}(E) := \#\left\{\lambda \in \operatorname{Spec}\left(H|_{\Gamma_a}\right) : \lambda \le E\right\}, \tag{2.5}$$

we define the nodal surplus of E in Γ_a by

$$\sigma^{(a)}(E) := \#\left\{\text{zeros of } f_E \text{ in } \Gamma_a\right\} - \left(n^{(a)}(E) - 1\right). \tag{2.6}$$

Outside a discrete set of E values, the eigenfunction f_E does not vanish at any vertex of Γ_a [BBS12, cor. 2.4] (This discrete set contains the discrete set mentioned above,

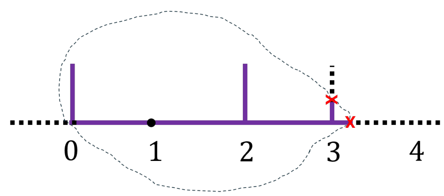

FIGURE 2.1. The compact graph $\Gamma_\omega(t)$ (2.13). Here, $s_\omega(t)$ (2.7) consists of two points, marked by x signs.

and in general might be larger). Restricting E to be outside the mentioned set, and using the unique continuation of f_E at every edge of Γ_a, we conclude that the zero set of f_E is discrete. Hence the surplus $\sigma^{(a)}(E)$ is well-defined for all such E values.

The nodal surplus has been extensively studied for quantum graphs starting from [GSW04]. For additional background see [ABB18, ABB22, Alo20, BBS12, Ber08].

2.2. **Right propagation along Γ_ω.** For the proof in the next subsection we need to establish a notion of propagation along Γ_ω. Fix $\omega\in\Omega$, and set the origin $o(\Gamma_\omega)$ to be the vertex with 0 coordinate of the $\mathbb{Z}$-graph (which is identified with the base vertex of $\Gamma_{\omega(0)}$).

Assume first that for all $a\in\mathcal{A}$ the total metric length of Γ_a is smaller than L (the horizontal distance between adjacent decorations). Under this assumption we set for $t\geq 0$,

$$s_\omega(t) := \left\{x\in\Gamma_\omega^+ \;:\; d(o(\Gamma_\omega),x)=tL\right\}, \tag{2.7}$$

see Figure 2.1, where Γ_ω^+ is the right part of Γ_ω, i.e., the half positive ray, $[0,\infty)$, together with all decorations attached to it. In particular, we note that $s_\omega(0)=\{o(\Gamma_\omega)\}$ and for $k\in\mathbb{N}$, $s_\omega(k)=\{kL\}$. We wish to maintain the property $s_\omega(k)=\{kL\}$ at integer radii even when some decorations Γ_a have total length exceeding L. To do so, if needed, we can rescale the metric inside each decoration Γ_a used in (2.7) by a factor of $\frac{L}{2|\Gamma_a|}$, while leaving the metric unchanged at the horizontal $\mathbb{Z}$-graph.

2.3. **Proof of Theorem 1.7.** The proof proceeds in three main steps: first, for each gap of H_ω we define an appropriate function on the suspension space X_Ω for which the Schwartzman homomorphism will be evaluated. Second, we relate this function to the nodal count of a generalized eigenfunction and finally, we combine these results to express the IDS value at the gap in terms of the Schwartzman group.

Step one: Defining an appropriate function on the suspension. Fix $E\in\mathbb{R}\backslash\mathrm{Spec}(H_\Omega)$. For $\omega\in\Omega$, consider the differential equation on Γ_ω^+,

$$-\frac{d^2}{dx^2}u(x)=Eu(x), \tag{2.8}$$

with the Kirchhoff condition imposed at all vertices, except the origin where no boundary condition is imposed. Since $E\notin\mathrm{Spec}(H_\omega)$, this equation has a unique solution (up to a scalar multiple) which is in $L^2(\Gamma_\omega^+)$, denoted $f_{\omega,E}$ (the proof is similar to that of [Tes14, lem. 9.7]). In addition, the uniqueness of the solution guarantees that, up to normalization, $f_{T\omega,E}|_{T\Gamma_\omega^+} = Tf_{\omega,E}|_{T\Gamma_\omega^+}$, where T acts on $f_{\omega,E}$ as in (1.15). Each solution $f_{\omega,E}$ may be also extended to the left (i.e., to $\Gamma_\omega \backslash \Gamma_\omega^+$) by solving the ODE (2.8), starting from the initial conditions given by the value and derivative of $f_{\omega,E}$. We hence may adopt the notation $f_{\omega,E}$ for a function on the whole Γ_ω and we get that $f_{T\omega,E}|_{\Gamma_\omega^+} \in L^2(\Gamma_\omega^+)$ and $f_{T\omega,E} = Tf_{\omega,E}$.

We use the function $f_{\omega,E}$ to define a function from the suspension space to the one-dimensional torus. Consider the following form of the Cayley transform,

$$C(t) = \frac{t+\mathrm{i}}{t-\mathrm{i}}, \tag{2.9}$$

which maps the left to right oriented real line $\overline{\mathbb{R}}$ (augmented with $\pm\infty$) onto the clockwise oriented unit circle. Using this we define the following function on the suspension space:

$$\phi : X_\Omega \to \mathbb{T}, \tag{2.10}$$

$$\phi(\omega, t) = \frac{1}{2\pi} Arg\left[C\left(\sum_{x\in s_\omega(t)} \frac{f'_{\omega,E}(x)}{f_{\omega,E}(x)}\right)\right], \tag{2.11}$$

where $s_\omega(t)$ is given in (2.7), and Arg is the argument function mapping complex numbers onto the one dimensional torus $[0, 2\pi)$. In the sum over $x \in s_\omega(t)$ above, a special emphasis should be given to the case when x is a vertex. The derivative $f'_{\omega,E}(x)$ at a vertex x is defined by parameterizing the elements of $s_\omega(t)$ as $x(t)$ and setting $f'_{\omega,E}(x(t)) := \lim_{\tilde{t}\to t^-} f'_{\omega,E}\left(x\left(\tilde{t}\right)\right)$. For $t = 0$ and $x = o(\Gamma)$, we similarly set $f'_{\omega,E}(x) := \lim_{\tilde{x}\to x^-} f'_{\omega,E}(\tilde{x})$. The function ϕ may be considered as a generalized Prüfer angle. Straightforward computation shows that ϕ is well-defined on X_Ω, as

$$\begin{aligned}\phi(\omega, 1) &= \frac{1}{2\pi} Arg\left[C\left(\sum_{x\in s_\omega(1)} \frac{f'_{\omega,E}(x)}{f_{\omega,E}(x)}\right)\right] \\ &= \frac{1}{2\pi} Arg\left[C\left(\sum_{x\in s_{T\omega}(0)} \frac{f'_{T\omega,E}(x)}{f_{T\omega,E}(x)}\right)\right] = \phi(T\omega, 0),\end{aligned} \tag{2.12}$$

where we have used that $f_{T\omega,E} = Tf_{\omega,E}$ and also the equivalence between $s_\omega(1)$ in Γ_ω and $s_{T\omega}(0)$ in $\Gamma_{T\omega}$ (both consist of a single point, which is the same up to the isomorphism between Γ_ω and $\Gamma_{T\omega}$). We further argue that ϕ is continuous on X_Ω. First, we show that $\phi(\omega, t)$ is continuous in $\omega \in \Omega$ using the Titchmarsh-Weyl m-function of the half infinite graph Γ_ω^+, denoted $m_\omega^+(z)$. . The values $f_{\omega,E}(x)$ and $f'_{\omega,E}(x)$ depend continuously on the Robin boundary condition $\frac{f'_{\omega,E}\left(o(\Gamma_w^+)\right)}{f_{\omega,E}\left(o(\Gamma_w^+)\right)}$ at the origin, as solutions of the ODE (2.8). Therefore the RHS of (2.11) depends continuously on $m_\omega^+(E) = \frac{f'_{\omega,E}\left(o(\Gamma_w^+)\right)}{f_{\omega,E}\left(o(\Gamma_w^+)\right)}$, and since the m-function $m_\omega^+(E)$ is continuous in ω (see e.g., [BSon]), we conclude that $\phi(\omega, t)$ is continuous in ω. Next, we show that $\phi(\omega, t)$ is also continuous in t. Clearly the expression $\sum_{x\in s_\omega(t)} \frac{f'_{\omega,E}(x)}{f_{\omega,E}(x)}$ is continuous in t, when $s_\omega(t)$ does not contain any vertex of Γ_ω^+. In addition, the Kirchhoff vertex conditions (1.2),(1.3) ensure the continuity of $\sum_{x\in s_\omega(t)} \frac{f'_{\omega,E}(x)}{f_{\omega,E}(x)}$ in t also when $s_\omega(t)$ contains a vertex. Overall, we conclude that the function ϕ is well-defined and continuous on X_Ω.

Step 2: Expressing $\phi(\omega,t)$ using the nodal count of $f_{\omega,E}$. Having defined $\phi : X_\Omega \to \mathbb{T}$ in (2.11) we wish to apply the Schwartzman homomorphism to it via (1.29). At this point, fix $\omega \in \Omega$ to be in the full measure set for which (1.29) holds. Define

$$\Gamma_\omega(t) := \left\{ x \in \Gamma_\omega^+ \ : \ d\left(o\left(\Gamma_\omega\right), x\right) \le tL \right\}, \tag{2.13}$$

see Figure 2.1, and note that $s_\omega(t)$ forms part of the boundary of $\Gamma_\omega(t)$. By (2.11) the function $\phi(\omega,t)$ equals $0 \in \mathbb{T}$ precisely when $f_{\omega,E}(x) = 0$ for some $x \in s_\omega(t)$. With this observation we use the values of $\phi(\omega,t)$ (or more precisely its lift) to count the zeros of $f_{\omega,E}$. To do so, recall the notation $\phi_{(\omega,0)}(t) := \phi(\tau^t(\omega,0))$ and $\widetilde{\phi}_{(\omega,0)}(t)$ for its lift (see Section 1.5). With this notation, the number of zeros of $f_{\omega,E}$ in $\Gamma_\omega(t)$ is equal to the number of times that the function $\widetilde{\phi}_{(\omega,0)}$ intersects 0 mod 1 in the interval $[0,t]$. We use this observation to connect between the (average) zero count of $f_{\omega,E}$ and the value of the Schwarzman homomorphism $S_\Omega([\phi])$. Explicitly, using the notation

$$Z_{\omega,t} := \#\left\{ x \in \Gamma_\omega(t) \ : \ f_{\omega,E}(x) = 0 \right\}, \tag{2.14}$$

we have

$$Z_{\omega,t} = \left\lfloor \widetilde{\phi}_{(\omega,0)}(t) \right\rfloor, \tag{2.15}$$

where $\lfloor \ \rfloor$ denotes the floor function. Therefore, by (1.29),

$$S_\Omega([\phi]) = \lim_{t\to\infty} \frac{\tilde{\phi}_{(\omega,0)}(t)}{t} = \lim_{t\to\infty} \frac{1}{t} \left\lfloor \widetilde{\phi}_{(\omega,0)}(t) \right\rfloor = \lim_{t\to\infty} \frac{1}{t} Z_{\omega,t}, \tag{2.16}$$

where in the first equality we used that $\omega \in \Omega$ is in the full measure set for which (1.29) holds.

Having this connection between the Schwartzman homomorphism and the nodal count, we analyze $Z_{\omega,t}$. In what follows we decompose the total nodal count on $\Gamma_\omega(t)$ via the nodal count of its subgraphs: the decorations, and the horizontal path. Outside a discrete set of energies E, the solution to the ODE (2.1) on each decoration Γ_a is unique up to scalar multiple (as discussed in Subsection 2.1). Hence, the nodal count on each decoration of a given type does not depend on the location of this decoration within $\Gamma_\omega(t)$. Denoting this nodal count function by $Z^{(a)}(E)$, we write

$$Z_{\omega,t} = Z_{\omega,t}^{\text{horiz}} + \sum_{a\in\mathcal{A}} \#_a^t(\omega)\, Z^{(a)}, \tag{2.17}$$

where $Z_{\omega,t}^{\text{horiz}}$ is the nodal count function of $f_{\omega,E}$ on the path graph $[0,tL]$ (which is a subgraph of $\Gamma_\omega(t)$), and we extend the definition of the letter counting function (1.7) to non-integer t values by setting $\#_a^t(\omega) := \#_a^{\lfloor t\rfloor}(\omega)$ to be the number of decorations of type a in $\Gamma_\omega(t)$. Note that (2.17) is an equality between functions in E, but for brevity we omit the E-dependence. As already mentioned, these functions are well-defined up to a discrete set of E values. We further use the spectral counting functions (2.5) and nodal surplus functions (2.6) to write

$$Z_{\omega,t} = Z_{\omega,t}^{\text{horiz}} + \sum_{a\in\mathcal{A}} \#_a^t(\omega)\left(n^{(a)} + \sigma^{(a)} - 1\right), \tag{2.18}$$

We next express the nodal counting functions $Z_{\omega,t}$ and $Z_{\omega,t}^{\text{horiz}}$ through spectral counting functions of suitable operators. Towards this, we define the corresponding operators. First, consider the restriction of H_ω to the finite graph $\Gamma_\omega(t)$. At the vertices $u \in s_\omega(t) \cup o(\Gamma_\omega)$ we impose the Robin condition

$$\frac{f'(u)}{f(u)} = \frac{f'_{\omega,E}(u)}{f_{\omega,E}(u)}, \tag{2.19}$$

and at all other vertices of $\Gamma_\omega(t)$ we impose the Neumann-Kirchhoff vertex conditions as in H_ω. We naturally denote the resulting operator $H_\omega|_{\Gamma_\omega(t)}$. We describe now an operator associated with the horizontal subgraph $[0, tL]$. Let v_m be an interior vertex of $[0, tL]$, which is positioned at mL, where $m \in \mathbb{Z} \cap (0, t)$. Denote its two neighboring edges by $e_m^\pm$. We impose at the vertex v_m the Robin-type conditions

$$f|_{e_v^+}(v_m) = f|_{e_v^-}(v_m) =: f(v_m), \tag{2.20}$$

$$f'|_{e_v^+}(v_m) + f'|_{e_v^-}(v_m) = -m_{\omega(m)}(E)\, f(v_m), \tag{2.21}$$

where the Robin parameter $m_{\omega(m)}(E)$ is as in (2.2), and takes into account that in $\Gamma_\omega(t)$ the decoration $\Gamma_{\omega(m)}$ is glued to v_m. At the boundary vertices $u \in \{o(\Gamma_\omega), tL\}$ we impose the same Robin conditions (2.19) as were imposed for $H_\omega|_{\Gamma_\omega(t)}$. Overall these vertex conditions render the one-dimensional Laplacian on $[0, tL]$ a self-adjoint operator, which we denote by $H_\omega|_{[0,tL]}$. These particular choices of vertex conditions guarantee that $\left(E, f_{\omega,E}|_{[0,tL]}\right)$ is an eigenpair of $H_\omega|_{[0,tL]}$ and $\left(E, f_{\omega,E}|_{\Gamma_\omega(t)}\right)$ is an eigenpair of $H_\omega|_{\Gamma_\omega(t)}$.

Denoting the spectral counting function of $H_\omega|_{[0,tL]}$ by

$$n_{\omega,t}^{\text{horiz}} := \#\left\{\lambda \in \operatorname{Spec}\left(H_\omega|_{[0,tL]}\right) \;:\; \lambda \le E\right\}, \tag{2.22}$$

Sturm's oscillation theorem (see [Ber08] and [Sch06]) yields

$$n_{\omega,t}^{\text{horiz}}(E) = Z_{\omega,t}^{\text{horiz}}(E) + 1. \tag{2.23}$$

Substituting this in (2.18) gives

$$Z_{\omega,t} = n_{\omega,t}^{\text{horiz}} - 1 + \sum_{a\in\mathcal{A}} \#_a^t(\omega)\left(n^{(a)} + \sigma^{(a)} - 1\right). \tag{2.24}$$

We next relate the spectral counting functions of the three operators $H_\omega|_{\Gamma_\omega(t)}$, $H_\omega|_{[0,tL]}$, $H_\omega|_{\Gamma_a}$ discussed above (the operator $H_\omega|_{\Gamma_a}$ was presented in Section 2.1, where it was denoted by $H|_{\Gamma_a}$).

Lemma 2.1. *Let $E \notin \operatorname{Spec}(H_\omega)$. Assume that for all $a \in \mathcal{A}$, the spectrum of the Kirchhoff Laplacian on Γ_a with Dirichlet condition imposed at v_a does not contain E.*

Denote the spectral counting functions of $H_\omega|_{\Gamma_\omega(t)}$*,* $H_\omega|_{[0,tL]}$ *and* $H_\omega|_{\Gamma_a}$ *by* $n_{\omega,t}$ *,* $n_{\omega,t}^{horiz}$ *and* $n^{(a)}$ *respectively. Then,*

$$n_{\omega,t}(E) = n_{\omega,t}^{horiz}(E) + \sum_{a\in\mathcal{A}} \#_a^t(\omega)\left(n^{(a)}(E) - 1\right). \tag{2.25}$$

The proof of the lemma involves a continuous interpolation between the relevant operators. While this is an interesting method, the proof is somewhat technical and is postponed to Appendix B.

Step 3: Computing the Schwartzman homomorphism of ϕ***.*** Using Lemma 2.1, Equation (2.24) gives

$$Z_{\omega,t} = n_{\omega,t} - 1 + \sum_{a\in\mathcal{A}} \#_a^t(\omega)\,\sigma^{(a)}. \tag{2.26}$$

Now, computing the Schwartzman homomorphism as in (2.16) gives

$$\begin{aligned}
S_\Omega([\phi]) &= \lim_{t\to\infty}\frac{1}{t}Z_{\omega,t}(E)\\
&= \lim_{t\to\infty}\frac{1}{t}\left(n_{\omega,t}(E) - 1 + \sum_{a\in\mathcal{A}}\#_a^t(\omega)\,\sigma^{(a)}(E)\right)\\
&= \lim_{t\to\infty}\frac{n_{\omega,t}(E)-1}{t} + \sum_{a\in\mathcal{A}}\lim_{t\to\infty}\frac{\#_a^t(\omega)}{t}\sigma^{(a)}(E)\\
&= \lim_{t\to\infty}\left[\left(\frac{n_{\omega,t}(E)}{|\Gamma_{\omega,t}|} - \frac{1}{|\Gamma_{\omega,t}|}\right)\cdot\frac{|\Gamma_{\omega,t}|}{t}\right] + \sum_{a\in\mathcal{A}}\nu_a\sigma^{(a)}(E)\\
&= N_\Omega(E)\cdot\overline{L}(\Gamma_\Omega) + \sum_{a\in\mathcal{A}}\nu_a\sigma^{(a)}(E). \qquad (2.27)
\end{aligned}$$

We thus finally obtain

$$N_\Omega(E) = \frac{S_\Omega([\phi]) - \sum_{a\in\mathcal{A}}\nu_a\sigma^{(a)}(E)}{\overline{L}(\Gamma_\Omega)}. \tag{2.28}$$

To complete the proof we need to show that the numerator of (2.28) belongs to the Schwartzman group, $\mathfrak{S}_\Omega$. By definition this group is the image of the Schwartzman homomorphism so that $S_\Omega([\phi]) \in \mathfrak{S}_\Omega$. Since $\mathfrak{S}_\Omega$ is an additive group, it is left to prove that $\sum_{a\in\mathcal{A}}\nu_a\sigma^{(a)}(E) \in \mathfrak{S}_\Omega$. From [DF23, thm. 7.1], we know that $\mathfrak{S}_\Omega$ is the $\mathbb{Z}$-module generated by

$$\{\mu(\Xi) : \Xi \text{ is a cylinder set in } \Omega\}, \tag{2.29}$$

where a cylinder set is a subset of Ω, for which a finite subword is fixed to be a given value. In particular, we consider cylinder sets with a single letter being fixed, which are of the form

$$\Xi_a := \{\omega \in \Omega : \omega(0) = a\}, a \in \mathcal{A}. \tag{2.30}$$

Since $\mu(\Xi_a) = \nu_a$ for a uniquely ergodic subshift, and $\sigma^{(a)}(E)$ is an integer for all $a \in \mathcal{A}$, we get $\sum_{a\in\mathcal{A}}\nu_a\sigma^{(a)}(E) \in \mathfrak{S}_\Omega$, as required. □

3. Gap labelling for discrete graphs - Proof of Theorem 1.9

In this section we prove the gap labelling theorem for discrete decorated graphs (Theorem 1.9). The main tool is the well-known spectral relation between the discrete Laplacian and the Kirchhoff Laplacian on the corresponding equilateral metric graph, summarized below.

Theorem 3.1. *Let Γ be an equilateral metric graph with all edge lengths equal to 1, equipped with the Kirchhoff Laplacian H. Let G be the associated discrete graph, equipped with the normalized discrete Laplacian Δ.*

(1) *For all $k \notin \{\pi m : m \in \mathbb{N}\}$,*

$$k^2 \in \mathrm{Spec}\,(H) \iff 1 - \cos(k) \in \mathrm{Spec}\,(\Delta)\,. \tag{3.1}$$

Furthermore, if the corresponding points in the spectrum (k^2 and $1-\cos(k)$) are eigenvalues, then they have the same multiplicities.

(2) *If, in addition, Γ is compact and connected, then its spectral counting function at $k^2 = \pi^2 m^2$ equals*

$$\#\left\{\lambda \in \mathrm{Spec}\,(H)\ :\ \lambda \le \pi^2 m^2\right\} = |\mathcal{E}_\Gamma|\, m + M, \tag{3.2}$$

where $M \in \{0,1\}$ is the multiplicity of $1 - \cos(\pi m) \in \{0,2\}$ in $\mathrm{Spec}\,(\Delta)$.

The first part of the theorem is standard (see e.g., [Cat97, LP08, Pan06, vB85]). The second part follows from the case-by-case eigenvalue count in [LP08, prop. 6.2], together with some basic properties of the normalized discrete Laplacian.

Using Theorem 3.1, we relate the IDS of the discrete and metric decorated graphs. The "conversion factor" which connects between the discrete and metric IDS is given by

$$C\,(G_\Omega) := \frac{\overline{\mathcal{V}}\,(G_\Omega)}{\overline{\mathcal{E}}\,(G_\Omega)} = \frac{\sum_{a\in\mathcal{A}} \nu_a\, |\mathcal{V}_{G_a}|}{1 + \sum_{a\in\mathcal{A}} \nu_a\, |\mathcal{E}_{G_a}|}, \tag{3.3}$$

which is the ratio between the average number of vertices and the average number of edges.

Proposition 3.2. *Let (Ω, T) be a uniquely ergodic subshift. Let $\{\Gamma_\omega\}_{\omega\in\Omega}$ be a family of decorated $\mathbb{Z}$-graphs, such that each Γ_ω is an equilateral graph with all edge lengths equal to 1. Let $\{G_\omega\}_{\omega\in\Omega}$ be the associated discrete graphs. Denote the corresponding IDS functions by $N_\Omega^H(E), N_\Omega^\Delta(E)$. Then at every point E, where N_Ω^H is continuous we have*

$$N_\Omega^H(E) = \left\lfloor \frac{\sqrt{E}}{\pi} \right\rfloor + C\,(G_\Omega)\cdot \begin{cases} N_\Omega^\Delta\left(1 - \cos\left(\sqrt{E}\right)\right), & \left\lfloor \frac{\sqrt{E}}{\pi} \right\rfloor \text{ is even,} \\ 1 - N_\Omega^\Delta\left(1 - \cos\left(\sqrt{E}\right)\right), & \left\lfloor \frac{\sqrt{E}}{\pi} \right\rfloor \text{ is odd.} \end{cases} \tag{3.4}$$

Proof. We first relate the spectral counting functions of compact discrete and metric graphs. We then take the limit as in Proposition 1.4 and (1.21) in order to compare the corresponding IDS.

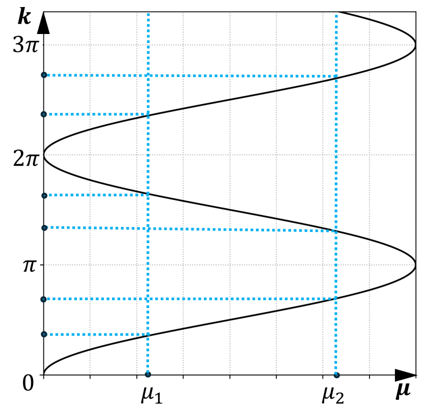

FIGURE 3.1. The dispersion relation $k(\mu) = \arccos(1-\mu)$ from Theorem 3.1 relating $\mathrm{Spec}(\Delta)$ (horizontal) and $\mathrm{Spec}(H)$ (vertical). Each $\mu \in \mathrm{Spec}(\Delta)$ corresponds to a point $k(\mu)^2 \in \mathrm{Spec}(H)$. The dispersion relation $k(\mu)$ is either monotone increasing or monotone decreasing, depending on the parity of $\left\lfloor \frac{k(\mu)}{\pi} \right\rfloor$.

Let Γ be an equilateral compact metric graph with all edge lengths equal to 1, and equipped with the Kirchhoff Laplacian H. Let G be the associated discrete graph equipped with Δ.

Let $E \geq 0$. Write $E = k^2$, and count the total number of square roots of eigenvalues of H in $[0,k]$ (with multiplicity). We write $k = \pi m + r$, with $m \in \mathbb{N}$ and $r \in [0,\pi)$. We use Theorem 3.1 to present the number of (square roots of) eigenvalues in $[0,k]$ as the sum of eigenvalue counts in $[0,\pi m]$ and in $[\pi m, k]$. Noting that $m = \left\lfloor \frac{k}{\pi} \right\rfloor$ and applying Theorem 3.1 we get

$$\#\left\{\lambda \in \mathrm{Spec}(H) \,:\, \lambda \leq k^2\right\} - \left\lfloor \frac{k}{\pi} \right\rfloor |\mathcal{E}_G| =$$

$$= \begin{cases} \#\left\{\mu \in \mathrm{Spec}(\Delta) \;:\; \mu \leq 1 - \cos(k)\right\}, & \left\lfloor \frac{k}{\pi} \right\rfloor \text{ is even,} \\ \#\left\{\mu \in \mathrm{Spec}(\Delta) \;:\; \mu \geq 1 - \cos(k)\right\}, & \left\lfloor \frac{k}{\pi} \right\rfloor \text{ is odd,} \end{cases}$$

$$= \begin{cases} \#\left\{\mu \in \mathrm{Spec}(\Delta) \;:\; \mu \leq 1 - \cos(k)\right\}, & \left\lfloor \frac{k}{\pi} \right\rfloor \text{ is even,} \\ |\mathcal{V}_G| - \#\left\{\mu \in \mathrm{Spec}(\Delta) \;:\; \mu < 1 - \cos(k)\right\}, & \left\lfloor \frac{k}{\pi} \right\rfloor \text{ is odd,} \end{cases} \tag{3.5}$$

where all eigenvalue counts above are with multiplicities (i.e., $\#\{\ \}$ is considered as element counting of a multi-set). In the first equality above we need to separate cases according to the parity of $m = \left\lfloor \frac{k}{\pi} \right\rfloor$, since the dispersion relation $1-\cos(k)$ in (3.1) is monotone increasing when $\left\lfloor \frac{k}{\pi} \right\rfloor$ is even and decreasing when $\left\lfloor \frac{k}{\pi} \right\rfloor$ is odd. See Figure 3.1, where the inverse dispersion relation $\lambda(\mu) = \arccos(1-\mu)$ is depicted.

Next, let $\omega \in \Omega$ be in the full measure set for which Proposition 1.4 holds and choose the sequences of compact graphs $\Gamma^{(n)} := \Gamma_\omega|_{[0,n]}$ and $G^{(n)} := G_\omega|_{[0,n]}$ as in (1.19),(1.21) and take the limit $n \to \infty$ to get the IDS. We perform the computation only for the case of odd $\left\lfloor \frac{k}{\pi} \right\rfloor$. The complementary case involves a similar (and slightly simpler) computation. Using the convergence stated in Proposition 1.4 and applying (3.5) we compute:

$$\begin{aligned}
N_{\Omega}^{H}\left(k^{2}\right) &= \lim_{n\to\infty}\frac{\#\left\{\lambda\in\operatorname{Spec}\left(H_{\omega}|_{\Gamma^{(n)}}\right)\ :\ \lambda\leq k^{2}\right\}}{\left|\Gamma^{(n)}\right|}\\
&= \lim_{n\to\infty}\frac{\#\left\{\lambda\in\operatorname{Spec}\left(H_{\omega}|_{\Gamma^{(n)}}\right)\ :\ \lambda\leq k^{2}\right\}}{\left|\mathcal{E}_{G^{(n)}}\right|}\\
&= \lim_{n\to\infty}\frac{1}{\left|\mathcal{E}_{G^{(n)}}\right|}\cdot\left\lfloor\frac{k}{\pi}\right\rfloor\left|\mathcal{E}_{G^{(n)}}\right|\\
&\quad+\lim_{n\to\infty}\frac{\left|\mathcal{V}_{G^{(n)}}\right|-\#\left\{\mu\in\operatorname{Spec}\left(\Delta|_{G^{(n)}}\right)\ :\ \mu<1-\cos\left(k\right)\right\}}{\left|\mathcal{E}_{G^{(n)}}\right|}
\end{aligned}$$

$$= \left\lfloor\frac{k}{\pi}\right\rfloor+\lim_{n\to\infty}\frac{\left|\mathcal{V}_{G^{(n)}}\right|}{\left|\mathcal{E}_{G^{(n)}}\right|}\frac{\left|\mathcal{V}_{G^{(n)}}\right|-\left\{\mu\in\operatorname{Spec}\left(\Delta|_{G^{(n)}}\right)\ :\ \mu<1-\cos\left(k\right)\right\}}{\left|\mathcal{V}_{G^{(n)}}\right|}, \tag{3.6}$$

where in the first equality we used that $\Gamma^{(n)}$ has all edge lengths equal to 1 and so $\left|\Gamma^{(n)}\right|=\left|\mathcal{E}_{G^{(n)}}\right|$. The prefactor inside the limit above is

$$\lim_{n\to\infty}\frac{\left|\mathcal{V}_{G^{(n)}}\right|}{\left|\mathcal{E}_{G^{(n)}}\right|}=\lim_{n\to\infty}\frac{\sum_{a\in\mathcal{A}}\#_{a}^{n}\left(\omega\right)\left|\mathcal{V}_{G_{a}}\right|}{n+\sum_{a\in\mathcal{A}}\#_{a}^{n}\left(\omega\right)\left|\mathcal{E}_{G_{a}}\right|}$$

$$=\lim_{n\to\infty}\frac{\sum_{a\in\mathcal{A}}\frac{\#_{a}^{n}(\omega)}{n}\left|\mathcal{V}_{G_{a}}\right|}{1+\sum_{a\in\mathcal{A}}\frac{\#_{a}^{n}(\omega)}{n}\left|\mathcal{E}_{G_{a}}\right|}=\frac{\sum_{a\in\mathcal{A}}\nu_{a}\left|\mathcal{V}_{G_{a}}\right|}{1+\sum_{a\in\mathcal{A}}\nu_{a}\left|\mathcal{E}_{G_{a}}\right|}=C\left(G_{\Omega}\right). \tag{3.7}$$

Substituting this above and recalling that $k=\sqrt{E}$ gives

$$N_{\Omega}^{H}\left(E\right)=\left\lfloor\frac{\sqrt{E}}{\pi}\right\rfloor+C\left(G_{\Omega}\right)\lim_{n\to\infty}\frac{\left|\mathcal{V}_{G^{(n)}}\right|-\left\{\mu\in\operatorname{Spec}\left(\Delta|_{G^{(n)}}\right)\ :\ \mu<1-\cos\left(\sqrt{E}\right)\right\}}{\left|\mathcal{V}_{G^{(n)}}\right|}$$

$$=\left\lfloor\frac{\sqrt{E}}{\pi}\right\rfloor+C\left(G_{\Omega}\right)\left[1-N_{\Omega}^{\Delta}\left(1-\cos\left(\sqrt{E}\right)\right)\right]. \tag{3.8}$$

Note that by definition,

$$N_{\Omega}^{\Delta}\left(1-\cos\left(\sqrt{E}\right)\right)=\lim_{n\to\infty}\frac{1}{\left|\mathcal{V}_{G^{(n)}}\right|}\left\{\mu\in\operatorname{Spec}\left(\Delta|_{G^{(n)}}\right)\ :\ \mu\leq1-\cos\left(\sqrt{E}\right)\right\}. \tag{3.9}$$

Nevertheless, the last equality above is justified (even though the strict inequality $\mu<1-\cos\left(k\right)$ appears), since we assume that $N_{\Omega}^{H}\left(E\right)$ is continuous at E. The distinction between strict and non-strict inequality in the spectral counting functions matters only when there is a discontuity in the IDS (see more on jump discontinuities of the IDS in Section 4).

For the case when $\left\lfloor\frac{\sqrt{E}}{\pi}\right\rfloor$ is even a similar computation gives $N_{\Omega}^{H}\left(E\right)=\left\lfloor\frac{\sqrt{E}}{\pi}\right\rfloor+C\left(G_{\Omega}\right)\cdot N_{\Omega}^{\Delta}\left(1-\cos\left(\sqrt{E}\right)\right)$. ☐

Proof of Theorem 1.9. By Proposition 3.2,

$$N_{\Omega}^{\Delta}\left(1-\cos\left(\sqrt{E}\right)\right)=\begin{cases}\frac{1}{C(G_{\Omega})}\left(N_{\Omega}^{H}(E)-\left\lfloor\frac{\sqrt{E}}{\pi}\right\rfloor\right), & \left\lfloor\frac{\sqrt{E}}{\pi}\right\rfloor \text{ is even,}\\ 1-\frac{1}{C(G_{\Omega})}\left(N_{\Omega}^{H}(E)-\left\lfloor\frac{\sqrt{E}}{\pi}\right\rfloor\right), & \left\lfloor\frac{\sqrt{E}}{\pi}\right\rfloor \text{ is odd.}\end{cases} \tag{3.10}$$

By Theorem 1.7, if $E\notin\operatorname{Spec}(H_{\Omega})$ then

$$N_{\Omega}^{H}(E)\in\frac{\mathfrak{S}_{\Omega}}{\overline{\mathcal{E}}(G_{\Omega})}, \tag{3.11}$$

where we used $\overline{L}(\Gamma_{\Omega})=\overline{\mathcal{E}}(G_{\Omega})$ which holds since the metric graphs are equilateral with each edge length equal to 1. We fix $\omega\in\Omega$ to be in the full measure set for which $\operatorname{Spec}(H_{\omega})=\operatorname{Spec}(H_{\Omega})$ and $\operatorname{Spec}(\Delta_{\omega})=\operatorname{Spec}(\Delta_{\Omega})$ (see Section 1.3) and the spectral counting functions converge to the IDS as in Proposition 1.4 and Equation (1.21). From Theorem 3.1 we conclude that E is inside a spectral gap of H_{ω} if and only if $1-\cos(\sqrt{E})$ is inside a spectral gap of Δ_{ω}. Therefore, the possible gap labels of Δ_{ω} (and hence of $\operatorname{Spec}(\Delta_{\Omega})$) may be obtained by substituting (3.11) in (3.10). For this, recall that $\overline{\mathcal{E}}(G_{\Omega})=1+\sum_{a\in\mathcal{A}}\nu_a\mathcal{E}(G_a)$ and that $\nu_a\in\mathfrak{S}_{\Omega}$ for all $a\in\mathcal{A}$ and $\mathbb{Z}\subset\mathfrak{S}_{\Omega}$ (as is explained in the end of the proof of Theorem 1.7). Therefore $\overline{\mathcal{E}}(G_{\Omega})\in\mathfrak{S}_{\Omega}$ and so for $E\notin\operatorname{Spec}(H_{\Omega})$,

$$N_{\Omega}^{H}(E)+\mathbb{Z}\in\frac{\mathfrak{S}_{\Omega}+\mathbb{Z}\,\overline{\mathcal{E}}(G_{\Omega})}{\overline{\mathcal{E}}(G_{\Omega})}\subset\frac{\mathfrak{S}_{\Omega}}{\overline{\mathcal{E}}(G_{\Omega})}, \tag{3.12}$$

and using $C(G_{\Omega})=\frac{\overline{\mathcal{V}}(G_{\Omega})}{\overline{\mathcal{E}}(G_{\Omega})}$ we get

$$\frac{1}{C(G_{\Omega})}\left(N_{\Omega}^{H}(E)+\mathbb{Z}\right)\subset\frac{\mathfrak{S}_{\Omega}}{\overline{\mathcal{V}}(G_{\Omega})}. \tag{3.13}$$

Using again that $\nu_a\in\mathfrak{S}_{\Omega}$ for all $a\in\mathcal{A}$, we get $\overline{\mathcal{V}}(G_{\Omega})\subset\mathfrak{S}_{\Omega}$, which yields that

$$1-\frac{1}{C(G_{\Omega})}\left(N_{\Omega}^{H}(E)+\mathbb{Z}\right)\subset\frac{\mathfrak{S}_{\Omega}}{\overline{\mathcal{V}}(G_{\Omega})}. \tag{3.14}$$

From (3.13) and (3.14), both cases in (3.10) yield the same gap labels,

$$\mathcal{GL}\left(N_{\Omega}^{\Delta}\right)\subset\frac{\mathfrak{S}_{\Omega}}{\overline{\mathcal{V}}(G_{\Omega})}\cap[0,1]. \tag{3.15}$$

□

4. Discontinuities of the IDS - Proof of Theorem 1.10

Theorems 1.7 and 1.9 provide the set of all possible gap labels for operators on metric and discrete decorated $\mathbb{Z}$-graphs. A well-known problem is to find whether all gap labels predicted by such gap labelling theorems actually occur. This is called the dry ten Martini problem, originating in a question by Mark Kac about the almost Mathieu operator [Sim82]. In this section we discuss a specific form of obstructions for the appearance of the predicted gaps. Since the predicted gap labels form a dense set, any discontinuity of the IDS implies the existence of labels that are not realized (also known as closed gaps). We illustrate this by completely analyzing the IDS jumps for metric Sturmian combs (see Example 1.3). The necessary and sufficient conditions for IDS discontinuities of these graphs are given in Theorem 4.1. In addition, the theorem explicitly states all the energies at which such discontinuities occur and the size of the IDS jump at those energies. Theorem 1.10 is an immediate corollary of Theorem 4.1.

Theorem 4.1. *Let $\alpha \in (0,1)\setminus\mathbb{Q}$, written as the following infinite continued fraction:*

$$\alpha = \frac{1}{c_1 + \frac{1}{c_2 + \ldots}}. \tag{4.1}$$

Let Ω_α be the corresponding Sturmian subshift. Then the IDS for the associated family of Sturmian combs $(\Gamma_\omega)_{\omega\in\Omega_\alpha}$ has discontinuities if and only if one of the following holds:

(1) $\frac{\ell}{L} = \frac{2m+1}{2n}(c_1+1)$ *for some $m, n \in \mathbb{N}$.*
In this case the IDS is discontinuous at $E = \left(\frac{\pi n}{L(c_1+1)}\right)^2$, and the associated jump in the IDS value is

$$\Delta N_{\Omega_\alpha}(E) = \frac{1 - c_1\alpha}{L + \alpha\ell}, \tag{4.2}$$

or

(2) $\frac{\ell}{L} = \frac{2m+1}{2n}c_1$ *for some $m, n \in \mathbb{N}$.*
In this case, the IDS is discontinuous at $E = \left(\frac{\pi n}{Lc_1}\right)^2$, and the associated jump in the IDS value is

$$\Delta N_{\Omega_\alpha}(E) = \frac{(c_1+1)\alpha - 1}{L + \alpha\ell}. \tag{4.3}$$

If both conditions on ℓ/L above hold simultaneously, i.e., $\frac{\ell}{L} = \frac{2m_1+1}{2n_1}(c_1+1) = \frac{2m_2+1}{2n_2}c_1$ for $m_1, n_1, m_2, n_2 \in \mathbb{N}$, then the IDS is discontinuous at $E = \left(\frac{\pi n_1}{L(c_1+1)}\right)^2 = \left(\frac{\pi n_2}{Lc_1}\right)^2$, and the associated jump in the IDS value is the sum of (4.3) and (4.2), i.e.,

$$\Delta N_{\Omega_\alpha}(E) = \frac{\alpha}{L + \alpha\ell}. \tag{4.4}$$

Remark. Note that if either case in the theorem occurs, it does so for infinitely many pairs (m, n), hence the IDS has jumps at infinitely many energies.

We show that the IDS discontinuities are caused by compactly supported eigenfunctions. A detailed resolution to the dry ten Martini problem for Sturmian metric graphs is given in [BS]. Two intriguing recent works [DEFM23, SFFG24] explore IDS discontinuities in aperiodic discrete graphs, which are also due to compactly supported eigenfunctions. Some fundamental results on this phenomenon for random operators on aperiodic discrete graphs appeared already in [KLS03]. Similar phenomena is observed also in periodic graphs models, as was analyzed for discrete graphs [PT21] and metric graphs [LPPV09] (see also [PTV17] where continuous models are confronted with metric and discrete graphs). The most recent work on the IDS of quantum graphs and their discontinuities appear in [BL26, Lev26] where periodic metric trees are analyzed.

To prove Theorem 4.1 we need two lemmas. Lemma 4.2 shows that all the compactly supported eigenfunctions of Γ_ω are supported on specific subgraphs. These subgraphs are associated with particular subwords of $\omega \in \Omega_\alpha$ and Lemma 4.3 expresses the frequencies of these subwords.

Lemma 4.2. *Let $\alpha \in (0,1)\setminus\mathbb{Q}$, $\omega \in \Omega_\alpha$ and $E \in \mathbb{R}$. A compactly supported E-eigenfunction of the Sturmian comb Γ_ω exists if and only if there exists an E-eigenfunction which is supported between two adjacent teeth in Γ_ω.*

Proof. One direction is trivial. For the converse, let f be a compactly supported solution to $-\frac{d^2 f}{dx^2} = Ef$ on Γ_ω, satisfying Neumann-Kirchhoff vertex conditions. Since f is compactly supported, choose the two farthest teeth on which it is supported, and get that f must vanish at the base of each of these two teeth (i.e., the vertex which connects them to the $\mathbb{Z}$-graph). Since f has a vanishing derivative at the other vertex of each of these teeth (i.e., the boundary vertex), we get that $k\ell = \frac{\pi}{2} + \pi m$ for some $m \in \mathbb{N}$, where $k := \sqrt{E}$ and ℓ is the tooth length. This implies that f in fact vanishes at the base of all teeth of the comb. Now, choose a (horizontal) path $\tilde{e}$ between two adjacent teeth e_1, e_2, such that f does not identically vanish on this path. At the bases of these teeth v_1, v_2 we have that $f(v_1) = f(v_2) = 0$, by the argument given above. Construct a new eigenfunction $\tilde{f}$ as follows:

1. At the horizontal path set $\tilde{f} = f$.
2. At the two mentioned teeth e_1, e_2, set $\tilde{f}\Big|_{e_i}(x) = A_i \sin\left(\left(\frac{\frac{\pi}{2}+\pi m}{\ell}\right)x\right)$ for $i \in \{1,2\}$.

Choose A_1, A_2 so that $\tilde{f}'\Big|_{e_i}(v_i) + \tilde{f}'\Big|_{\tilde{e}}(v_i) = 0$.

3. Extend $\tilde{f}$ to be identically 0 everywhere else.

The resulting function is an E-eigenfunction supported between two adjacent teeth of Γ_ω. □

Towards the next lemma, we define the frequency of a subword. Let Ω_α be a Sturmian subshift, and $W = W_0 \ldots W_k$ a finite subword over the alphabet $\mathcal{A} = \{0,1\}$. Let $\omega \in \Omega_\alpha$. We denote

$$\nu_W := \lim_{N\to\infty} \frac{\#\left\{n \in \{0,\ldots,N-1\} \ : \ \omega|_{[n,n+k]} = W\right\}}{N}, \tag{4.5}$$

which is the frequency with which the subword W occurs in ω, and is actually invariant with respect to $\omega \in \Omega$ due to unique ergodicity (see Subsection 1.2 and [BG13, prop. 4.4]). We therefore refer to ν_W as the frequency with which W occurs in the subshift Ω_α.

Lemma 4.3. *Let $\alpha \in (0,1)\setminus\mathbb{Q}$ with the continued fraction expansion (4.1). Then there exist only two subwords of the form*

$$W = 1\underbrace{0\ldots0}_{k}1 \tag{4.6}$$

which occur in the subshift Ω_α:

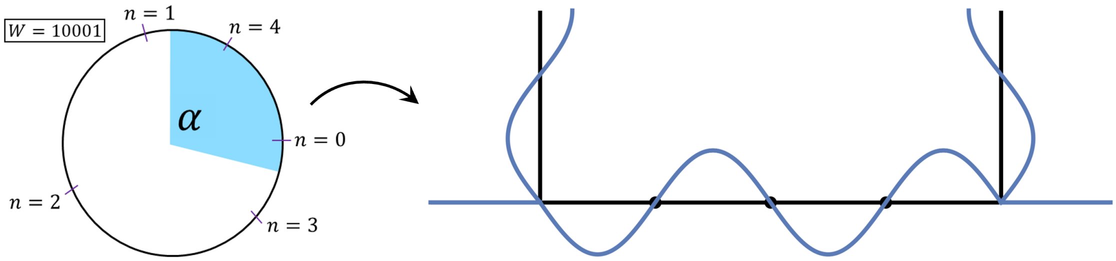

FIGURE 4.1. Illustration of how subwords of the form $W = 10...01$ give rise to a compactly supported eigenfunction. Here, taking $\alpha \approx 0.29$ and the initial angle to be $\theta = 1 - \alpha + \varepsilon$, the first subword of length 5 of the associated Sturmian sequence (1.9) gives rise to the compactly supported eigenfunction on the right.

(1) A subword W with $k = c_1$ zeros, which appear with frequency $1 - c_1\alpha$ in Ω_α.
(2) A subword W with $k = c_1 - 1$ zeros, which appears with frequency $(c_1+1)\alpha - 1$.

Proof. Given a finite word W we consider the following subset of S^1:

$$I_W := \left\{ \theta \in S^1 \ : \ \omega_{\alpha,\theta}|_{[0,\dots,|W|-1]} = W \right\}, \tag{4.7}$$

where $\omega_{\alpha,\theta}(n) := \chi_{(1-\alpha,1]}(n\alpha + \theta \mod 1)$ is a Sturmian (infinite) word such that $\omega_{\alpha,\theta} \in \Omega_\alpha$. By [Lot02, sec. 2.2.3] (see also [BGM24, sec. 5]), the frequency of the subword W in Ω_α is equal to the Lebesgue measure of I_W. We therefore compute the Lebesgue measure I_W for all admissible subwords of the form $W = 10....01$. We accompany the proof with Figure 4.1.

First, note that

$$\frac{1}{c_1+1} < \alpha < \frac{1}{c_1}. \tag{4.8}$$

By the definition of the sequence $\omega_{\alpha,\theta}$ we have that $\omega_{\alpha,\theta}(0) = 1$ iff $\theta \in (1-\alpha, 1]$. By (4.8) we get that $n\alpha + \theta \in \left[\frac{n-1}{c_1+1} + 1, \frac{n}{c_1} + 1\right)$ for all $\theta \in [1-\alpha, 1)$. In particular we get that $n\alpha + \theta \mod 1 \in [0, 1-\alpha)$, for all $1 \le n \le c_1 - 1$. We conclude the argument above by

$$\omega_{\alpha,\theta}(0) = 1 \quad \Leftrightarrow \quad \theta \in [1-\alpha, 1) \quad \Leftrightarrow \quad \omega_{\alpha,\theta}|_{[0,\dots,c_1-1]} = 1\underbrace{0....0}_{c_1-1}. \tag{4.9}$$

As we wish that $\omega_{\alpha,\theta}(0) = W(0) = 1$, we may assume the above equivalent conditions and split into two cases:

(1) Assume $\omega_{\alpha,\theta}(c_1) = 0$, which is equivalent to $c_1\alpha + \theta \mod 1 \in [0, 1-\alpha)$. In addition, from $\theta \in [1-\alpha, 1)$ we have $c_1\alpha + \theta \in [1 + (c_1-1)\alpha, 1 + c_1\alpha)$ and so $c_1\alpha + \theta \mod 1 \in [(c_1-1)\alpha, c_1\alpha)$. Intersecting both intervals gives $c_1\alpha + \theta \mod 1 \in [(c_1-1)\alpha, 1-\alpha)$. From here we get $(c_1+1)\alpha + \theta \mod 1 \in [c_1\alpha, 1)$,

and this implies $\omega_{\alpha,\theta}(c_1+1)=1$. Concluding we get that in this case

$$\omega_{\alpha,\theta}|_{[0,\ldots,c_1+1]} = W = 1\underbrace{0....0}_{c_1}1. \tag{4.10}$$

We need also to know the range of θ for this case, namely what is I_W for the subword W above. In the current case, we got $c_1\alpha+\theta \mod 1 \in [(c_1-1)\alpha, 1-\alpha)$. This means that $\theta \in [-\alpha, 1-(c_1+1)\alpha)$. The Lebesgue measure of this interval is $1-c_1\alpha$, which is the frequency of the word W above.

(2) Assume $\omega_{\alpha,\theta}(c_1)=1$, which is equivalent to $c_1\alpha+\theta \mod 1 \in [1-\alpha,1)$. Repeating the arguments as in the case above we get that $I_W = [2-(c_1+1)\alpha, 1)$ for $W = 1\underbrace{0....0}_{c_1-1}1$. The Lebesgue measure of this interval is $(c_1+1)\alpha-1$, which is the frequency of that word.

The two cases above exhaust all subwords of the form $W = 10....01$ occurring in Ω_α. □

Proof of Theorem 4.1. We start the proof by referring to Corollary A.7, whose hypothesis holds because all finite subwords of a Sturmian subshift have positive frequency (see beginning of proof of Lemma 4.3, or similar arguments in [Lot02, sec. 2.2.3] and [BGM24, sec. 5]). We conclude from Corollary A.7 that N_{Ω_α} has a jump discontinuity at energy E if and only if E admits a compactly supported eigenfunction. By Lemma 4.2 compactly supported E-eigenfunctions exist precisely when there is an E-eigenfunction supported between two adjacent teeth of the graph, see Figure 4.1.

Let f be an E-eigenfunction which is supported between adjacent teeth and denote $k := \sqrt{E}$. The following holds:

(1) By the proof of Lemma 4.2, f vanishes at the base of each tooth, and $k\ell = \frac{\pi}{2}+\pi m$ for some $m \in \mathbb{N}$.

(2) By Lemma 4.3, the (horizontal) distance between adjacent teeth in Γ_ω is either $c_1 L$ or $(c_1+1)L$. By (1) above, this implies that $k\,(c_1+1)\,L = \pi n$ or $kc_1L = \pi n$ for $n \in \mathbb{N}$. It may be that both equalities hold for the same value of k, but with different n values.

We now examine the two cases in (2). First consider $k\,(c_1+1)\,L = \pi n$. Combining this with the $k\ell = \frac{\pi}{2}+\pi m$, translates into the following condition:

$$\frac{\ell}{L} = \frac{(2m+1)\,(c_1+1)}{2n},\ m,n \in \mathbb{N}, \tag{4.11}$$

and the corresponding eigenvalue is $E = k^2 = \left(\frac{\pi n}{L(c_1+1)}\right)^2$. This is precisely the case demonstrated in Figure 4.1 with $n=4$, $m=1$, and $c_1=3$.

Similarly, the case $kc_1L = \pi n$ translates into the following condition:

$$\frac{\ell}{L} = \frac{(2m+1)\,c_1}{2n},\ m,n \in \mathbb{N}, \tag{4.12}$$

and the corresponding eigenvalue is $E = k^2 = \left(\frac{\pi n}{Lc_1}\right)^2$.

These are exactly the two possible conditions on ℓ/L and the corresponding energies in the theorem. It remains to compute the jump size. Using (1.19), we count the number of compactly supported eigenfunctions for the finite truncations $H_\omega|_{[0,N]}$. For the energy $E = \left(\frac{\pi n}{L(c_1+1)}\right)^2$, we need to consider the subword $W = 1\underbrace{0....0}_{c_1}1$ and the eigenfunctions supported on the corresponding subgraphs. These eigenfunctions are linearly independent and so

$$\dim\ker\left(H_\omega|_{[0,N]} - E\right) = \#\left\{j \in \{0, ..., N - c_1 - 1\} \ : \ \omega|_{[j,j+c_1+1]} = W\right\}. \tag{4.13}$$

The jump in the IDS at E is given by

$$\begin{aligned}
&\Delta N_{\Omega_\alpha}(E) \\
&= \lim_{N\to\infty} \frac{\#\left\{\lambda \in \operatorname{Spec}\left(H_\alpha|_{[0,N]}\right) \ : \ \lambda \le E\right\} - \#\left\{\lambda \in \operatorname{Spec}\left(H_\alpha|_{[0,N]}\right) \ : \ \lambda < E\right\}}{\left|\Gamma_\alpha|_{[0,N]}\right|} \\
&= \lim_{N\to\infty} \frac{\dim\ker\left(H_\alpha|_{[0,N]} - E\right)}{\left|\Gamma_\alpha|_{[0,N]}\right|} \\
&= \lim_{N\to\infty} \frac{\#\left\{j \in \{0, ..., N - c_1 - 1\} \ : \ \omega_\alpha|_{[j,j+c_1+1]} = W\right\}}{\left|\Gamma_\alpha|_{[0,N]}\right|} \\
&= \lim_{N\to\infty} \frac{(N - c_1)\,\nu_W}{NL + \alpha N\ell} = \frac{1 - c_1\alpha}{L + \alpha\ell},
\end{aligned} \tag{4.14}$$

where the last line is obtained by Lemma 4.3 according to which $\nu_W = 1 - c_1\alpha$ (see also the definition of word frequency, (4.5)).

Repeating the same computation for the energy $E = \left(\frac{\pi n}{L c_1}\right)^2$ whose eigenfunctions correspond to the subword $W = 1\underbrace{0....0}_{c_1-1}1$. The only change which is required in the computation is in using the word frequency which is now $\nu_W = (c_1 + 1)\alpha - 1$, and we get

$$\Delta N_{\Omega_\alpha}(E) = \frac{(c_1 + 1)\alpha - 1}{L + \alpha \cdot \ell}. \tag{4.15}$$

It may happen that both (4.11) and (4.12) hold (but for different n, m values). Namely,

$$\frac{\ell}{L} = \frac{(2m_1 + 1)(c_1 + 1)}{2n_1} = \frac{(2m_2 + 1)\, c_1}{2n_2} \tag{4.16}$$

for some $m_1, n_1, m_2, n_2 \in \mathbb{N}$. The corresponding energy is then $E = \left(\frac{\pi n_1}{L(c_1+1)}\right)^2 = \left(\frac{\pi n_2}{L c_1}\right)^2$ and the associated eigenfunctions are supported on subgraphs corresponding to both subwords $1\underbrace{0 \cdot .... \cdot 0}_{c_1}1$ and $1\underbrace{0 \cdot .... \cdot 0}_{c_1-1}1$. These eigenfunctions are linearly independent and so the dimensions of the corresponding eigenspaces sum up (and the same holds for the frequencies). Therefore, the IDS jump at such energies is the sum of (4.14) and (4.15),

$$\Delta N_{\Omega_\alpha}(E) = \frac{1-c_1\alpha}{L+\alpha\cdot\ell} + \frac{(c_1+1)\alpha-1}{L+\alpha\cdot\ell} = \frac{\alpha}{L+\alpha\cdot\ell}. \tag{4.17}$$

□

Appendix A. Proof of Proposition 1.4

In this appendix we prove Proposition 1.4, namely that the IDS for metric decorated $\mathbb{Z}$-graphs is well-defined and given by the limit of the spectral counting functions. The discrete case is analogous and omitted.

In the following, we denote by $\mathcal{F}$ the set of finite subsets of $\mathbb{Z}$. For any subset $Q \in \mathcal{F}$, let $H_\omega|_Q$ represent the restriction of the operator H_ω to the compact subgraph $\Gamma_\omega|_Q$ of the decorated $\mathbb{Z}$-graph Γ_ω induced by Q (as in Subsection 1.4). We impose the Dirichlet condition at the boundary vertices of $\Gamma_\omega|_Q$ where the decorated $\mathbb{Z}$-graph Γ_ω is truncated, although other self-adjoint boundary conditions would yield the same results. We prove the following Pastur–Shubin-type trace formula, which gives Proposition 1.4 as an immediate corollary:

Proposition A.1. *Let (Ω, T) be a uniquely ergodic subshift. Denote by μ the unique shift-invariant probability measure on Ω. For almost every $\omega \in \Omega$, the sequence of normalized counting functions $N_\omega^{(n)}(E)$ in (1.19) converges uniformly to a limiting function $N_\Omega(E)$. For an arbitrary finite $Q \in \mathcal{F}$, the function N_Ω can be expressed as*

$$N_\Omega(E) = \frac{1}{|Q|\,\overline{L}(\Gamma_\Omega)} \int_\Omega tr\left[\chi_{\Gamma_\omega|_Q}\chi_{(-\infty,E]}(H_\omega)\right] d\mu(\omega), \tag{A.1}$$

where $\overline{L}(\Gamma_\Omega)$ denotes the average metric length, as defined in (1.10).

Our proof relies on an adaptation of the method presented in [GLV07], which utilizes an ergodic theorem proven in [LMV08] (see also [GLV08]). Notably, the proof can be generalized to many other graph families, including graphs with random potentials and vertex conditions, higher dimensional decorated graphs (i.e. $\mathbb{Z}^d$ with $d > 1$), and tiling graphs, as studied in [BSon].

A.1. Background and definitions.

We start by introducing essential definitions, and refer to [LMV08] for more details.

We denote the spectral counting function (and normalized spectral counting function) for H_ω^Q by

$$n_\omega^Q(E) := \#\left\{\lambda \in \mathrm{Spec}\left(H_\omega|_Q\right) : \lambda \le E\right\}, \tag{A.2}$$

$$N_\omega^Q(E) := \frac{1}{\left|\Gamma_\omega|_Q\right|} n_\omega^Q(E). \tag{A.3}$$

To decouple the graph into its decorations, we further introduce the operator $H_{\omega,D}|_Q$, obtained by imposing Dirichlet conditions at the centers of all edges of the horizontal path. The corresponding spectral counting function is

$$n^Q_{\omega,D}(E) := \#\left\{\lambda \in \operatorname{Spec}\left(H_{\omega,D}|_Q\right) : \lambda \le E\right\}. \tag{A.4}$$

To simplify notation, let $n^a_D(E)$ represent the counting function for the operator $n^Q_{\omega,D}$ when $Q = \{a\}$ for $a \in \mathcal{A}$. The overall counting function for $H^Q_{\omega,D}$ can then be written as:

$$n^Q_{\omega,D}(E) = \sum_{a\in\mathcal{A}} \#^Q_a(\omega)\, n^a_D(E), \tag{A.5}$$

where $\#^Q_a(\omega)$ is the number of occurrences of the letter a in the subword $\omega|_Q$ (extending the definition of the letter counting function from (1.7)). With the above, we define the *spectral shift* function

$$\xi^Q_\omega(E) := n^Q_\omega(E) - n^Q_{\omega,D}(E) = n^Q_\omega(E) - \sum_{a\in\mathcal{A}} \#^Q_a(\omega)\, n^a_D(E). \tag{A.6}$$

Lastly, we provide a few definitions which are required for the proofs.

Definition A.2. A *van Hove* sequence is a sequence $(Q_j)_{j\in\mathbb{N}} \subset \mathcal{F}$ such that

$$\lim_{j\to\infty} \frac{|\partial Q_j|}{|Q_j|} = 0, \tag{A.7}$$

where the boundary ∂Q is defined as

$$\partial Q := \{n \in Q \ : \ n+1 \notin Q \text{ or } n-1 \notin Q\}. \tag{A.8}$$

Definition A.3. A function $b : \mathcal{F} \to [0,\infty)$ is called a *boundary term* if

(1) $b(Q) = b(m+Q)$ for all $m \in \mathbb{Z}$ and $Q \in \mathcal{F}$,
(2) there exists $D > 0$ such that $b(Q) \le D|Q|$ for all $Q \in \mathcal{F}$,
(3) for any van Hove sequence $(Q_j)_{j\in\mathbb{N}}$, the following holds:

$$\lim_{j\to\infty} \frac{b(Q_j)}{|Q_j|} = 0. \tag{A.9}$$

Definition A.4.

(1) Let X be a Banach space. $F : \mathcal{F} \to X$ is called almost-additive if there exists a boundary term b such that

$$\left\| F\left(\cup^l_{k=1} Q_k\right) - \sum_{k=1}^{l} F(Q_k) \right\| \le \sum_{k=1}^{l} b(Q_k) \tag{A.10}$$

for all $l \in \mathbb{N}$ and pairwise disjoint sets Q_k.

(2) For a subshift element $\omega \in \Omega$, F is said to be ω-equivariant if $F(Q)$ depends only on the local pattern of ω at Q, i.e.,

$$F(Q) = F(m+Q), \tag{A.11}$$

whenever $m \in \mathbb{Z}$ and Q obeys $\omega|_{m+Q} = \omega|_Q$.

(3) F is said to be bounded if there exists $C>0$ such that

$$\left\Vert F\left(Q\right)\right\Vert \leq C\left|Q\right|. \tag{A.12}$$

A.2. **Proving the main result.** The following paraphrase on the ergodic theorem [LMV08, thm. 1] is a main key to the proof of Proposition A.1:

Theorem A.5. *Let (Ω,T) be a uniquely ergodic subshift over $\mathcal{A}$, and let $\omega\in\Omega$. Let $(X,\left\Vert \cdot\right\Vert)$ be a Banach space, and let $\left(Q_{j}\right)_{j\in\mathbb{N}}$ be a van Hove sequence. Suppose that $F:\mathcal{F}\to X$ is an ω-equivariant, almost-additive bounded function. Then the following limit exists:*

$$\overline{F}:=\lim_{j\to\infty}\frac{F\left(Q_{j}\right)}{\left|Q_{j}\right|}. \tag{A.13}$$

Remark. [LMV08, thm. 1] also assumes existence of all subword frequencies, which here follows from unique ergodicity (see [BG13, prop. 4.4], [Oxt52]).

The following lemma provides the function F on which Theorem A.5 is applied.

Lemma A.6. *On the Banach space $(X,\left\Vert \cdot\right\Vert _{\infty})$ of right-continuous bounded functions, define the function*

$$F:\mathcal{F}\to X, \tag{A.14}$$

$$\left(F\left(Q\right)\right)\left(E\right)=\frac{\xi_{\omega}^{Q}\left(E\right)}{\overline{L}\left(\Gamma_{\Omega}\right)}, \tag{A.15}$$

where ξ_{ω}^{Q} is the spectral shift function (A.6). Then F is ω-equivariant, bounded, and almost-additive.

The proof is similar to [GLV07, lem. 22]. Boundedness follows since H_{ω}^{Q} and $H_{\omega,D}^{Q}$ differ by a finite rank perturbation. Similarly, almost-additivity holds since the disjoint decomposition $Q=\sqcup_{k=1}^{l}Q_{k}$ yields finite rank perturbations between the associated operators.

The proof of Proposition A.1 now follows, using conceptually the same arguments as in [GLV07, thm. 3].

Proof of Proposition A.1. By Lemma A.6, the function $\left(F\left(Q\right)\right)\left(E\right)=\xi_{\omega}^{Q}\left(E\right)/\overline{L}\left(\Gamma_{\Omega}\right)$ is ω-equivariant, almost-additive and bounded. Applying (A.6) along a van Hove sequence $\left(Q_{j}\right)_{j\in\mathbb{N}}$, we get for all $j\in\mathbb{N}$

$$n_{\omega}^{Q_{j}}\left(E\right)=\xi_{\omega}^{Q_{j}}\left(E\right)+\sum_{a\in\mathcal{A}}\#_{a}^{Q_{j}}\left(\omega\right)n_{D}^{a}\left(E\right). \tag{A.16}$$

Dividing both sides by $\left|\Gamma_\omega^{Q_j}\right|$ and taking the limit $j \to \infty$, we obtain using (1.19):

$$
\begin{aligned}
N_\omega(E) &= \lim_{j\to\infty} \frac{n_\omega^{Q_j}(E)}{\left|\Gamma_\omega^{Q_j}\right|} =_{(1.11)} \frac{1}{\overline{L}(\Gamma_\Omega)} \lim_{j\to\infty} \frac{n_\omega^{Q_j}(E)}{|Q_j|} \\
&= \frac{1}{\overline{L}(\Gamma_\Omega)} \lim_{j\to\infty} \left(\frac{\xi_\omega^{Q_j}(E)}{|Q_j|} + \frac{1}{|Q_j|} \sum_{a\in\mathcal{A}} \#_a^{Q_j}(\omega)\, n_D^a(E) \right) \\
&= \frac{1}{\overline{L}(\Gamma_\Omega)} \lim_{j\to\infty} \frac{\xi_\omega^{Q_j}(E)}{|Q_j|} + \frac{1}{\overline{L}(\Gamma_\Omega)} \sum_{a\in\mathcal{A}} \nu_a n_D^a(E), \qquad \text{(A.17)}
\end{aligned}
$$

and the limit exists by Theorem A.5. The convergence is uniform, since the Banach-space norm in Lemma A.6 is $\|\cdot\|_\infty$.

We now prove formula (A.1), beginning with the Q-independence of the right-hand side. This is clearly true if $Q = \{m\}$, i.e., Q is a singleton. This follows from the translation invariance of the ergodic measure μ, since for all $m \in \mathbb{Z}$:

$$
\begin{aligned}
&\int_\Omega tr\left[\chi_{\Gamma_\omega^{\{m\}}} \chi_{(-\infty,E]}(H_\omega)\right] d\mu(\omega) \\
&= \int_\Omega tr\left[\chi_{\Gamma_{T\omega}^{\{m\}}} \chi_{(-\infty,E]}(H_{T\omega})\right] d\mu(\omega) \\
&=_{(1.17)} \int_\Omega tr\left[\chi_{\Gamma_{T\omega}^{\{m\}}} \chi_{(-\infty,E]}\left(T^{-1} H_\omega T\right)\right] d\mu(\omega) \\
&= \int_\Omega tr\left[T^{-1} \chi_{\Gamma_{T\omega}^{\{m\}}} \chi_{(-\infty,E]}(H_\omega)\, T\right] d\mu(\omega) \\
&=_{(*)} \int_\Omega tr\left[\chi_{\Gamma_{T\omega}^{\{m\}}} \chi_{(-\infty,E]}(H_\omega)\right] d\mu(\omega) \\
&= \int_\Omega tr\left[\chi_{\Gamma_\omega^{\{m+1\}}} \chi_{(-\infty,E]}(H_\omega)\right] d\mu(\omega), \qquad \text{(A.18)}
\end{aligned}
$$

where $(*)$ follows from the cyclic property of the trace. For arbitrary Q, the claim follows by writing $\frac{1}{|Q|}\chi_{\Gamma_\omega^Q} = \frac{1}{|Q|}\sum_{m\in Q} \chi_{\Gamma_\omega^{\{m\}}}$.

To prove (A.1), we first show that

$$
\lim_{j\to\infty} \left[\frac{1}{|Q_j|\,\overline{L}(\Gamma_\Omega)} tr\left\{\chi_{Q_j} f_z(H_\omega)\right\} - \frac{1}{\left|\Gamma_\omega|_{Q_j}\right|} tr\left\{ f_z\left(H_\omega|_{Q_j} \right) \right\} \right] = 0, \qquad \text{(A.19)}
$$

for all f_z of the form $f_z(t) = (t-z)^{-1}$ for $z \in \mathbb{C}\backslash\mathbb{R}$, where we use χ_{Q_j} as an abbreviated notation for $\chi_{\Gamma_\omega|_{Q_j}}$.

For a given box $Q = Q_j$, the graph Γ_ω naturally splits into two components, $\Gamma_\omega|_Q$ and $\Gamma_\omega|_{\mathbb{Z}\backslash Q}$. In this case, the operators H_ω and $H_\omega|_Q \oplus H_\omega|_{\mathbb{Z}\backslash Q}$ only differ by the boundary conditions imposed at the set ∂Q. We consider the operator

$$
D := f_z(H_\omega) - f_z\left(H_\omega|_Q \oplus H_\omega|_{\mathbb{Z}\backslash Q} \right). \qquad \text{(A.20)}
$$

As it is a difference of two self-adjoint operator and $z \notin \mathbb{R}$, we get $\|D\| \leq 2\,|Im\,(z)|^{-1}$. In addition, by the second resolvent identity we get $rank\,D \leq |\partial Q|$. Both bounds imply $|tr\,D| \leq 2\,|\partial Q|\,|Im\,(z)|^{-1}$. We use this bound to get

$$\lim_{j\to\infty}\left|\frac{1}{|Q_j|\overline{L}(\Gamma_\Omega)}tr\{\chi_{Q_j}f_z(H_\omega)\} - \frac{1}{|\Gamma_\omega|_{Q_j}|}tr\{f_z(H_\omega|_{Q_j})\}\right|$$

$$= \lim_{j\to\infty}\frac{1}{|Q_j|\overline{L}(\Gamma_\Omega)}\left|tr\{\chi_{Q_j}f_z(H_\omega) - f_z(H_\omega|_{Q_j})\}\right|$$

$$= \lim_{j\to\infty}\frac{1}{|Q_j|\overline{L}(\Gamma_\Omega)}\left|tr\{\chi_{Q_j}\big(f_z(H_\omega) - f_z\big(H_\omega|_{Q_j}\oplus H_\omega|_{\mathbb{Z}\setminus Q_j}\big)\big)\}\right|$$

$$\leq \frac{2}{\overline{L}(\Gamma_\Omega)\,|Im\,(z)\,|}\lim_{j\to\infty}\frac{|\partial Q_j|}{|Q_j|} = 0. \tag{A.21}$$

where in the last equality we used that Q_j is van Hove.

A Stone-Weierstrass argument then upgrades $(A.19)$ to indicator functions $\chi_{(-\infty,E]}$. Applying this to $Q_j = [0,j]$ and recalling that the normalized spectral functions were defined as $N_\omega^{(j)}(E) = \frac{1}{\left|\Gamma_\omega|_{Q_j}\right|}\,tr\left\{\chi_{(-\infty,E]}\left(H_\omega|_{Q_j}\right)\right\}$ we get

$$\int_\Omega \lim_{j\to\infty} N_\omega^{(j)}(E)\ \mathrm{d}\mu(\omega) = \int_\Omega \lim_{j\to\infty}\frac{1}{\left|\Gamma_\omega|_{Q_j}\right|}\,tr\left\{\chi_{(-\infty,E]}\left(H_\omega|_{Q_j}\right)\right\}\ \mathrm{d}\mu(\omega)$$

$$= \int_{\omega\in\Omega}\lim_{j\to\infty}\frac{1}{|Q_j|\,\overline{L}\,(\Gamma_\Omega)}\,tr\left\{\chi_{\Gamma_\omega|_{Q_j}}\chi_{(-\infty,E]}\,(H_\omega)\right\}\ \mathrm{d}\mu(\omega)$$

$$= \frac{1}{|Q|\,\overline{L}\,(\Gamma_\Omega)}\int_{\omega\in\Omega} tr\left\{\chi_{\Gamma_\omega|_Q}\chi_{(-\infty,E]}\,(H_\omega)\right\}\ \mathrm{d}\mu(\omega)\,, \tag{A.22}$$

where the second equality follows due to $(A.19)$, and the third equality follows from the Q-independence of the trace. This completes the proof. □

Corollary A.7. *Assume that the frequencies of all finite subwords in Ω are positive. The IDS N_Ω has a jump discontinuity at $E \in \mathbb{R}$ if and only if E admits a compactly supported eigenfunction.*

The proof follows the same arguments as in [KLS03, thm. 2] and [GLV07, cor. 7]: a jump discontinuity of the IDS is equivalent to the dimension of the eigenspace of $E \in \mathrm{Spec}\left(H_\omega|_{[0,n]}\right)$ growing "sufficiently quickly" in n, which then allows one to construct compactly supported eigenfunctions for H_ω.

Appendix B. Proof of Lemma 2.1

This appendix proves Lemma 2.1, which compares the spectral counting functions for the Kirchhoff Laplacian on the graph $\Gamma_\omega(t)$ with those of some of its subgraphs.

The proof relies on continuously interpolating between the full graph $\Gamma_\omega(t)$ and the disjoint union of the corresponding subgraphs of $\Gamma_\omega(t)$. This is done using a one-parameter family of operators $(H_\tau)_{\tau\in[0,\pi]}$. This operator family transitions between the

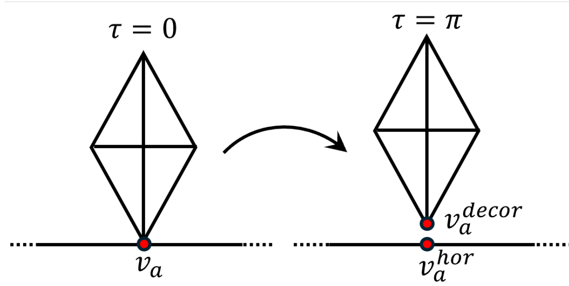

FIGURE B.1. The family $(H_\tau)_{\tau\in[0,\pi]}$ which continuously disconnects the graph Γ_ω^t into two subgraphs as $\tau \to \pi$.

full graph and the split graph while keeping $f_{\omega,E}$ an eigenfunction for all the graphs in the transition process, allowing to relate the spectral counting functions of the graphs.

To describe the construction, we start by fixing some $E \notin \mathrm{Spec}\,(H_\omega)$ such that for each $a \in \mathcal{A}$, it holds that E does not belong to the spectrum of Γ_a with a Dirichlet condition at the base vertex v_a of Γ_a (as in the statement of Lemma 2.1). Take on Γ_ω the unique solution $f_{\omega,E}$, as is described in step one of the proof of Theorem 1.7. Consider a compact truncation $\Gamma_\omega\,(t)$ of the infinite graph Γ_ω (see (2.13)). The vertices at which the cut is made are $s_\omega(t) \cup o(\Gamma_\omega)$ (see (2.7), (2.13), and Figure 2.1). For each vertex $u \in s_\omega(t) \cup o(\Gamma_\omega)$ we denote

$$\alpha(u) := \frac{\sum_{e\in\mathcal{E}_u\cap\Gamma_\omega(t)} f'_{\omega,E}\big|_e(u)}{f_{\omega,E}(u)}, \tag{B.1}$$

which is considered to be the Robin parameter of $f_{\omega,E}$ at u when restricted to $\Gamma_\omega(t)$. We consider an arbitrary decoration Γ_a which is attached to $\Gamma_\omega(t)$. We can split the graph $\Gamma_\omega\,(t)$ into two subgraphs: Γ_a and $\Gamma_\omega\,(t)\setminus\Gamma_a$, each containing a respective copy of the base vertex v_a of the decoration. We label these copies $v_a^{\mathrm{dec}}, v_a^{\mathrm{hor}}$, as in Figure B.1.

We describe a family of operators $(H_\tau)_{\tau\in[0,\pi]}$ on $\Gamma_\omega(t)$ which is now considered as a disjoint union of Γ_a and $\Gamma_\omega(t)\setminus\Gamma_a$. Each operator H_τ acts as the Laplacian on each edge; at the vertices $s_\omega(t)\cup o(\Gamma_\omega)$ and $v_a^{\mathrm{dec}}, v_a^{\mathrm{hor}}$ it satisfies continuity conditions (for this purpose v_a^{dec} and v_a^{hor} are considered as separate vertices) and also the following vertex conditions:

$$\sum_{e\in\mathcal{E}_u} f'|_e\,(u) = \alpha(u)f(u), \quad \forall u \in s_\omega(t)\cup o(\Gamma_\omega), \tag{B.2}$$

$$\sum_{e\sim v_a^{\mathrm{dec}}} f'_e\left(v_a^{\mathrm{dec}}\right) = m_a(E)f\left(v_a^{\mathrm{dec}}\right) + \cot(\tau/2)\left(f\left(v_a^{\mathrm{hor}}\right) - f\left(v_a^{\mathrm{dec}}\right)\right), \tag{B.3}$$

$$\sum_{e\sim v_a^{\mathrm{hor}}} f'_e\left(v_a^{\mathrm{hor}}\right) = -m_a(E)f\left(v_a^{\mathrm{hor}}\right) - \cot(\tau/2)\left(f\left(v_a^{\mathrm{hor}}\right) - f\left(v_a^{\mathrm{dec}}\right)\right), \tag{B.4}$$

where the Robin parameter $m_a\,(E)$ is a fixed number determined from $f_{\omega,E}$ restricted to Γ_a as in (2.2), and the Kirchhoff conditions is imposed at all other vertices of $\Gamma_\omega(t)$. At $\tau = 0$ we also add the requirement $f(v_a^{\mathrm{dec}}) = f(v_a^{\mathrm{hor}})$. At $\tau = \pi$, the graph $\Gamma_\omega\,(t)$ is effectively split at the vertex v_a, with the Robin conditions imposed at both v_a^{dec} and v_a^{hor} (but with opposite signs of the coupling coefficient).

One can verify that $f_{\omega,E}|_{\Gamma_\omega(t)}$ satisfies the vertex conditions (B.2),(B.3),(B.4) for all $\tau \neq 0$ and so it is an eigenfunction of H_τ for all $\tau \neq 0$. At $\tau = 0$, the additional condition $f(v_a^{\mathrm{dec}}) = f(v_a^{\mathrm{hor}})$, together with (B.3),(B.4) simplifies to Kirchhoff, and therefore $f_{\omega,E}|_{\Gamma_\omega(t)}$ is an eigenfunction of H_0 as well. As a matter of fact, this is the part of the rationale behind the particular choice of the operator family $(H_\tau)_{\tau\in[0,\pi]}$. We may also describe this operator family via its quadratic form (the connection between vertex conditions and quadratic forms for quantum graphs is standard, see e.g. [BK13]):

$$\begin{aligned} Q_\tau\left[f\right] = &\int_{\Gamma_\omega(t)} \left|f'\right|^2 dx \;+ \sum_{u\in s_\omega(t)\cup o(\Gamma_\omega)} \alpha(u)\left|f(u)\right|^2 \\ &+ m_a(E)|f(v_a^{\mathrm{hor}})|^2 - m_a(E)|f(v_a^{\mathrm{dec}})|^2 \\ &+ \cot\left(\tau/2\right)|f(v_a^{\mathrm{hor}}) - f(v_a^{\mathrm{dec}})|^2, \end{aligned} \tag{B.5}$$

where the Robin parameter $m_a\left(E\right)$ is fixed as above and the domain of Q_τ is taken as all functions in $H^1\left(\Gamma_\omega\left(t\right)\right)$ which are continuous on Γ_a and continuous on $\Gamma_\omega\left(t\right)\backslash\Gamma_a$ (but without requiring continuity at v_a, i.e., that $f(v_a^{\mathrm{hor}}) = f(v_a^{\mathrm{dec}})$). For $\tau = 0$, the domain further restricts to functions satisfying $f(v_a^{\mathrm{hor}}) = f(v_a^{\mathrm{dec}})$ as well. Thus at $\tau = 0$ the operator satisfies also Kirchhoff conditions at v_a (without splitting it into v_a^{dec} and v_a^{hor}).

Thus the family $(H_\tau)_{\tau\in[0,\pi]}$ continuously interpolates between an operator on the full graph $\Gamma_\omega\left(t\right)$ (at $\tau = 0$) and an operator on the cut graph $\Gamma_\omega\left(t\right)\backslash\Gamma_a$ (at $\tau = \pi$). Now, we consider all the other decorations (in addition to Γ_a discussed above) which are connected to $\Gamma_\omega(t)$. We redefine the operator family $(H_\tau)_{\tau\in[0,\pi]}$ such that the vertex conditions at all vertices where the decorations are attached are changed simultaneously in the same manner as for v_a. Namely, the vertex conditions (B.3), (B.4) are imposed at all these decoration attachment vertices. The effect of this redefined operator family $(H_\tau)_{\tau\in[0,\pi]}$ is equivalent to disconnecting $\Gamma_\omega\left(t\right)$ at all these vertices simultaneously at $\tau = \pi$. At $\tau = 0$, we get the operator $H_\omega|_{\Gamma_\omega(t)}$, namely, the vertex conditions at the vertices of $\Gamma_\omega\left(t\right)$ are all Kirchhoff, except for the boundary vertices $s_\omega(t)\cup o(\Gamma_\omega)$, where the Robin conditions (B.2) are imposed. What is important to emphasize is that exactly as above $f_{\omega,E}|_{\Gamma_\omega(t)}$ is an eigenfunction of H_τ for all $\tau \in [0,\pi]$.

By standard methods (cf. [BK13, thm. 1.4.4]), H_τ are all self-adjoint with compact resolvents. Noticing that the map $\tau \mapsto Q_\tau\left[f\right]$ is piecewise analytic with non-positive derivative for all fixed f, a standard Kato-type [Kat76] argument (see e.g. [Sof22, BPS] for detailed proofs in similar systems) can be used to prove the following:

Lemma B.1. *The eigenvalue branches $\left(\lambda_n\left(\tau\right)\right)_{n\in\mathbb{N}}$ of $(H_\tau)_{\tau\in[0,\pi]}$ are piecewise real-analytic, and monotone non-increasing in any interval where they are differentiable.*

We have collected all that is needed.

Proof of Lemma 2.1. The lemma considers the operators $H_\omega|_{\Gamma_\omega(t)}$, $H_\omega|_{[0,tL]}$ and $H_\omega|_{\Gamma_a}$. Their spectral counting functions are denoted by $n_{\omega,t}$, $n_{\omega,t}^{\mathrm{horiz}}$ and $n^{(a)}$, respectively.

We note that the operator $H_\omega|_{\Gamma_\omega(t)}$ is exactly H_0 of the operator family $(H_\tau)_{\tau\in[0,\pi]}$ defined above. Denoting the spectral counting functions of this family by $n^{(\tau)}_{\omega,t}(E)$ (i.e., $n^{(0)}_{\omega,t} = n_{\omega,t}$), the statement of Lemma 2.1 translates to

$$n^{(0)}_{\omega,t}(E) = n^{\text{horiz}}_{\omega,t}(E) + \sum_{a\in\mathcal{A}} \#^t_a(\omega)\left(n^{(a)}(E) - 1\right). \tag{B.6}$$

Now, consider the operator H_π. It is an operator on the cut version of $\Gamma_\omega(t)$, namely the disjoint union of the horizontal graph with the individual decorations. As such, its spectral counting function equals the sum of the individual counting functions,

$$n^{(\pi)}_{\omega,t}(E) = n^{\text{horiz}}_{\omega,t}(E) + \sum_{a\in\mathcal{A}} \#^t_a(\omega)\, n^{(a)}(E). \tag{B.7}$$

Given (B.7), we can prove (B.6) by showing the following properties on the eigenvalue curves:

(1) The operator family $(H_\tau)_{\tau\in[0,\pi]}$ is uniformly bounded from below.
(2) For $\varepsilon > 0$ small enough, there are exactly $\lfloor t\rfloor$ crossings of the eigenvalue curves with the horizontal line $\lambda = E + \varepsilon$ in the interval $\tau \in [0, \pi]$.

We first explain why the two properties above together with (B.7) imply (B.6), and then prove these properties. Consider the rectangle bounded by $\tau = 0$, $\tau = \pi$, $\lambda = E+\varepsilon$ and $\lambda = -C$, where $-C$ is a uniform lower bound of the family $(H_\tau)_{\tau\in[0,\pi]}$ (see Figure B.2). It is clear that $n^{(0)}_{\omega,t}(E)$ and $n^{(\pi)}_{\omega,t}(E)$ equal the number of intersections of the eigenvalue curves with the left and right sides of the rectangle respectively. Due to property (1) there are no intersections with the lower side of the rectangle. By Lemma B.1 the eigenvalue curves are monotone non-increasing, and hence the number of intersections with the top side equals $n^{(\pi)}_{\omega,t}(E) - n^{(0)}_{\omega,t}(E)$. By property (2) the number of these intersections is $\lfloor t\rfloor$, so that

$$n^{(\pi)}_{\omega,t}(E) - n^{(0)}_{\omega,t}(E) = \lfloor t\rfloor = \sum_{a\in\mathcal{A}} \#^t_a(\omega), \tag{B.8}$$

which combined with (B.7) proves (B.6).

Next, we prove the two properties mentioned above. Property (1) follows from the quadratic form (B.5): $(H_\tau)_{\tau\in[0,\pi]}$ is uniformly bounded from below since $\cot(\tau/2)$ is bounded from below there.

For property (2), first recall that E is as in the proof of Theorem 1.7 (and was used to define the family $(H_\tau)_{\tau\in[0,\pi]}$). We have already observed that the function $f_{\omega,E}$ is an eigenfunction of H_τ for all $\tau \in [0, \pi]$. Hence, there exists a 'flat' eigenvalue branch at the constant value of $\lambda = E$. Denote by M the multiplicity of E as an eigenvalue of H_0 (we know that $M \geq 1$, since $f_{\omega,E}$ is an eigenfunction). Next, we will show the following two statements: (a) The multiplicity of E as an eigenvalue of H_π is $M + \lfloor t\rfloor$ and (b) The multiplicity of E as an eigenvalue of H_τ for any $\tau \in [0, \pi)$ is M. From this we would get that there are exactly $\lfloor t\rfloor$ eigenvalue branch which cross $\lambda = E$ and

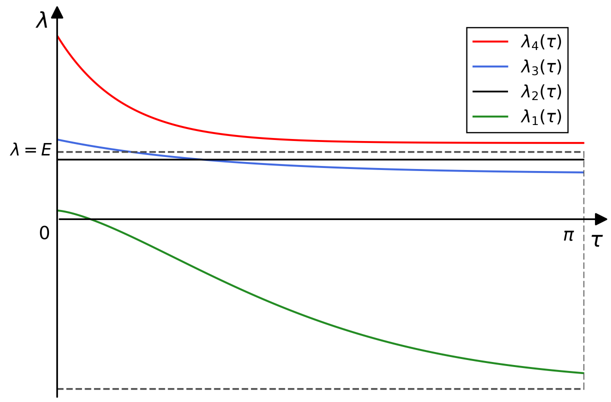

FIGURE B.2. Demonstration of the rectangle argument: the number of intersections through the top side is equal to the spectral shift.

these crossings occur at $\tau = \pi$. This immediately yields property (2) which finishes the proof.

(a) Denote by $\{f^{(j)}\}_{j=1}^{M}$ a basis to the E-eigenspace of H_0. We use the functions $\{f^{(j)}\}_{j=1}^{M}$ to construct $M + \lfloor t \rfloor$ functions on $\Gamma_\omega(t)$. Note that given the value E, at each of the $\lfloor t \rfloor$ decorations there is a unique (up to scalar) function which is a solution of the ODE (2.1) with Kirchhoff conditions imposed at all vertices, except at the base vertex where only a continuity condition is imposed. The uniqueness is guaranteed since we demanded that for all decorations Γ_a, E is not in the spectrum of the decoration with Dirichlet condition (if uniqueness is violated, one may construct such a function which vanishes at the base vertex). We denote these unique functions on the decorations by $\{f_a\}_{a\in\mathcal{A}}$. We thus get that for each $f^{(j)}$ $(1 \le j \le M)$, its restriction to each of the decorations either equals the corresponding f_a, or identically vanishes at the decoration (in the case where $f^{(j)}(v_a) = 0$).

Given the above observations, we construct $M + \lfloor t \rfloor$ functions on $\Gamma_\omega(t)$ as follows. For each of the $\lfloor t \rfloor$ decorations construct a function which is supported only at this decoration and vanishes everywhere else (i.e., it vanishes at all the other decorations and at the horizontal line). This gives $\lfloor t \rfloor$ eigenfunctions of H_π. We additionally take $\{ f^{(j)}\big|_{[0,tL]}\}_{j=1}^{M}$, which are also eigenfunctions of H_π. We thus get $M + \lfloor t \rfloor$ E-eigenfunctions of H_π. Clearly these functions are linearly independent and we now show that there are no other E-eigenfunctions of H_π. Assume by contradiction that there is another E-eigenfunction of H_π, denoted by g which is not a linear combination of the $M + \lfloor t \rfloor$ functions mentioned above. To get a contradiction, we construct a function h on $\Gamma_\omega(t)$, such that $h|_{[0,tL]} = g|_{[0,tL]}$ and at each decoration Γ_a which is included in $\Gamma_\omega(t)$ we set $h|_{\Gamma_a}$ to equal f_a up to a scalar multiple which is chosen to guarantee that h is continuous at the base vertex of Γ_a. This way, we obtain that h is an E-eigenfunction of H_0. At every decoration Γ_a, either $g|_{\Gamma_a}$ equals f_a up to a scalar multiple, or $g|_{\Gamma_a} \equiv 0$ (by the uniqueness mentioned above) . Therefore, g is a linear combination of h and the $\lfloor t \rfloor$ functions which are supported solely at the decorations of $\Gamma_\omega(t)$, but this is a contradiction.

(b) Let $\tau \neq \pi$. Note that $\{f^{(j)}\}_{j=1}^{M}$ are E-eigenfunctions of H_τ. We should only show that there are no additional eigenfunctions. Assume by contradiction that there is an eigenfunction g of H_τ which is linearly independent of $\{f^{(j)}\}_{j=1}^{M}$. In particular, on every decoration Γ_a which is included in $\Gamma_\omega(t)$ we have that g is a solution of the same ODE as $f_{\omega,E}|_{\Gamma_a}$ (and as all of $\{f^{(j)}|_{\Gamma_a}\}_{j=1}^{M}$). This implies that at the base vertex v_a of the decoration we get $\sum_{e\sim v_a^{\mathrm{dec}}} g'|_e\left(v_a^{\mathrm{dec}}\right) = m_a(E)\, g\left(v_a^{\mathrm{dec}}\right)$. Comparing this to the vertex condition (B.3) and using $\cot(\tau/2) \neq 0$ we get that $g(v_a^{\mathrm{dec}}) = g(v_a^{\mathrm{hor}})$, i.e., that g is continuous at the base vertex of the decoration. Since this is valid for all the decorations contained in $\Gamma_\omega(t)$ we get that g is an E-eigenfunction of H_0 which is linearly independent of $\{f^{(j)}\}_{j=1}^{M}$. A contradiction. □

Remark. The proof of Lemma 2.1 is based on the notion of spectral flow. The spectral flow of an operator family such as $(H_\tau)_{\tau\in[0,\pi]}$ is informally defined as the number of oriented intersections of the eigenvalue curves of H_τ with some horizontal line $E =$ const. The spectral flow is a topological invariant and an interesting framework in its own right. To keep the proof self-contained, we instead used a direct argument. We refer to [BBLP05, BBZ13, BBZ18] and references therein for a thorough background about the spectral flow, and also to [BPS, Pro, LS20] for applications of the spectral flow specifically in the context of quantum graphs.

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