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From quasiperiodicity to a complete coloring of the Kohmoto butterfly

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Abstract – The spectra of the Kohmoto model give rise to a fractal phase diagram, known as the Kohmoto butterfly. The butterfly encapsulates the spectra of all periodic Kohmoto Hamiltonians, whose index invariants are sought after. Topological methods are ill defined due to the discontinuous periodic potentials, and hence fail to provide index invariants. This letter overcomes that obstacle and provides a complete classification of the Kohmoto model indices —suggesting new physical invariants instead of Chern indices. Our approach encodes the Kohmoto butterfly as a spectral tree graph, reflecting the quasiperiodic nature via the periodic spectra. This yields a complete coloring of the phase diagram and a new perspective on other spectral butterflies.

Quasicrystals exhibit challenging spectral and topological properties. Their quasiperiodic order gives rise to fractal spectra with infinitely many spectral gaps, manifest in diverse wave systems [1–3].

Topology together with a variety of mathematical methods enables a classification of quasicrystals into equivalence classes governed by topological invariants [4–7]. Beyond the mathematical aspects, a wide range of phenomena arise and are studied across theoretical and experimental physics as well as engineering [8–16]. Nevertheless, there are topological shortcomings when trying to study Kohmoto model [17], a paradigmatic model of quasicrystals. The three main approaches towards topological indices are not applicable to the Kohmoto model: Chern numbers rely on differentiability of the spectral projections [5], and so are not defined; the bulk-boundary correspondence (Thouless pump [18–20]) breaks down; and the two-dimensional extension (via inverse Fourier transform) gives a parent Hamiltonian with a nonlocal slowly decaying potential, hence not possessing a Fredholm index [5,21]. This letter overcomes these obstacles and presents an approach that yields natural indices for the Kohmoto model. These indices consistently reflect the quasiperiodic limit, allow to color the corresponding phase diagram (Kohmoto butterfly) and suggest new physical invariants.

The Kohmoto model [17] is given by the Hamiltonians

$$(H_\alpha\psi)(n) = \psi(n+1) + \psi(n-1) + V_\alpha(n)\,\psi(n), \qquad (1)$$

where the potential $V_\alpha(n) = \lambda\chi_{[1-\alpha,1)}(n\alpha \mod 1)$ is determined by a frequency α, a coupling constant (also known as modulation amplitude) λ and $\chi_{[1-\alpha,1)}$ is the characteristic function of the interval $[1-\alpha, 1)$. It is well known that these quasiperiodic operators represent one-dimensional quasicrystals. For instance, the Fibonacci quasicrystal $\alpha = \frac{\sqrt{5}-1}{2}$ forms a prominent and well-studied example in this class [22,23].

For rational frequencies $\alpha = \frac{p}{q}$, the operator $H_{\frac{p}{q}}$ is q-periodic and its spectrum consists of q spectral bands. The integrated density of states (IDS), also known as electron density, is

$$N_\alpha(E) = \frac{n}{q} = \mathfrak{c}\alpha \mod 1 \qquad (2)$$

for energies E in the n-th gap. The index $\mathfrak{c}$ solving the Diophantine equation above is defined only modulo q. The same Diophantine equation appears in the Hofstadter model [24,25] of the quantum Hall effect. In that case, the modulo q ambiguity is resolved by identifying $\mathfrak{c}$ with a Chern number, which can be computed from Berry's

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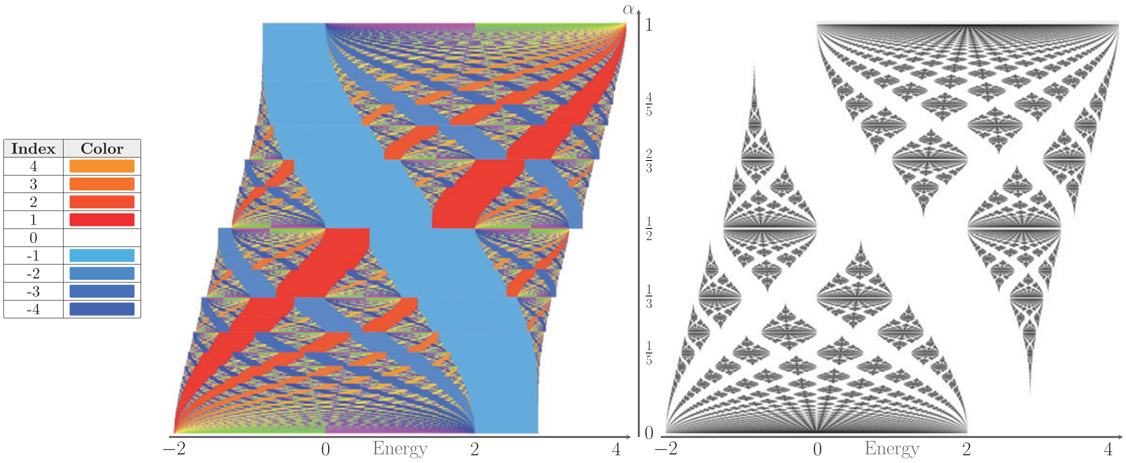

Fig. 1: The right panel shows the Kohmoto butterfly —the spectral bands are plotted for $\lambda = 2$ and rational frequencies $\alpha = p/q$ with $q \leq 250$. In the middle panel each spectral gap for a periodic Hamiltonian is colored according to its index —the color map for some lower indices is given on the left.

curvature of the corresponding spectral projection [26,27]. This endows $\mathfrak{c}$ with topological significance, as Chern numbers are invariant under variations that do not close the spectral gap. When one assigns a color to each integer value, this provides a consistent coloring of the phase diagram (Hofstadter butterfly) [28,29].

In the Kohmoto model $\mathfrak{c}$ cannot be identified with a Chern number, since the potential V_α is discontinuous, and Berry's phase needs the spectral projections to be differentiable[1]. Therefore, a coloring of the Kohmoto butterfly is not possible without further insights for resolving the modulo q ambiguity inherent in (2); see [32].

We provide here a consistent coloring (depicted in fig. 1, left) by resolving this ambiguity and determining the values of the index invariants. The explicit formula for the index $\mathfrak{c}$ is given in eq. (7); the required construction towards this formula is introduced next. We start by presenting the connection between the periodic Hamiltonians (with $\alpha \in \mathbb{Q}$) and the quasiperiodic ones ($\alpha \notin \mathbb{Q}$), where the former may be used to approximate the latter.

Let $\alpha \notin \mathbb{Q}$ be written in terms of its continued fraction expansion,

$$\alpha = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}}, \tag{3}$$

where $a_0 = 0$ and $a_n \in \mathbb{N}$ for all $n \in \mathbb{N}$. Truncating this expansion gives finite continued fraction expansions,

$$\alpha_k = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \frac{1}{a_k}}} = \frac{p_k}{q_k}, \qquad k \in \mathbb{N} \cup \{0\}, \tag{4}$$

where $p_k, q_k \in \mathbb{N}$ are chosen to be coprime. By convention, $\alpha_0 = \frac{p_0}{q_0} = \frac{0}{1}$ (as $a_0 = 0$).

[1] One could replace V_α by a smooth approximation, which was found in [30,31] to be useful for irrational frequencies. But it is not clear yet whether this fully resolves the problem for the periodic operators.

We construct an infinite directed tree graph $\mathcal{T}_\alpha$, and name it the spectral α-tree. This tree encodes the periodic approximations H_{α_k} of H_α. Specifically, for each k the vertices at level k represent the spectral bands and gaps of H_{α_k}. The tree is constructed recursively via the digits $\{a_1, a_2, a_3, \ldots\}$ of the continued fraction of α, as illustrated in fig. 2 and explained next (see also [33–35]).

We start by fixing a single vertex to be the root of the tree. We say that the root belongs to level $k = -1$ of the tree. Starting from the root, all other vertices belong to ascending levels k in the tree and they carry one of the three labels: A, B or G. The label tells whether the vertex represents a spectral gap (label G, appearing as a circle in fig. 2) or a spectral band (labels A, B, appearing as a corresponding interval in fig. 2). The root is connected to two vertices at level $k = 0$, the left has label A and the right has label G. The rest of the tree $\mathcal{T}_\alpha$ is constructed recursively: for every vertex v with label A or B in level $k \geq 0$, denote

$$M := \begin{cases} a_{k+1} - 1, & \text{if } v \text{ has the label } A, \\ a_{k+1}, & \text{if } v \text{ has the label } B, \end{cases} \tag{5}$$

and connect the vertex v to $2M + 1$ vertices in level $k + 1$. The labels of these vertices alternate between G and A, starting and ending with G, see fig. 2. For a G-vertex v in level k, connect it to a single B-vertex in level $k + 1$. This provides a complete description of $\mathcal{T}_\alpha$ (see footnote [2]).

Each vertex of the tree represents a spectral band or spectral gap of H_{α_k} as shown in fig. 2 (the spectral bands are depicted as the corresponding intervals). The ordering of the vertices within a certain level k corresponds to the spectral order [34]. An import property is that if two vertices of spectral bands (*i.e.*, of labels A/B) are connected

[2] The description here is for $\lambda > 0$. The tree for $\lambda < 0$ is a vertical reflected image of the tree described above, see details in [34].

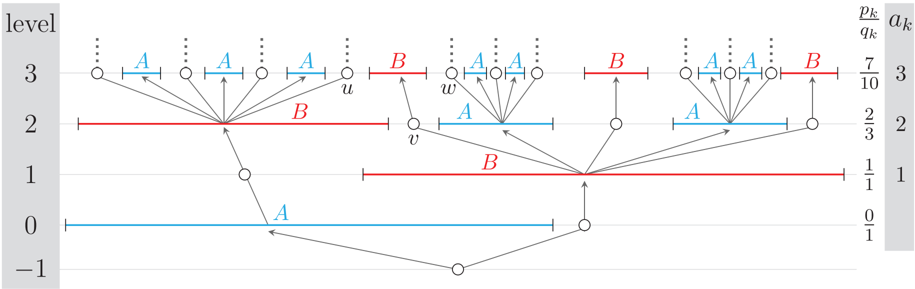

Fig. 2: An example of a spectral α-tree is sketched if α has continued fraction expansion $(a_k)_{k=0}^{\infty}$ starting with $0, 1, 2, 3$. The vertices of the graph are drawn as the spectral bands to which they correspond; their labels (A/B) are indicated. Vertices representing gaps G are indicated by circles.

by a directed path, it means that the upper band is fully contained in the lower, as is demonstrated in fig. 2.

We now use the tree $\mathcal{T}_\alpha$ to assign indices to the spectral gaps (*i.e.*, to the G-vertices). For each level k, count the number of A-vertices at level k and denote this number by $Z_A(k)$. Similarly, denote the number of B-vertices in level k by $Z_B(k)$. For each G-vertex v in level k, count how many A and B vertices there are at level k to the left of v and denote these numbers by $z_A(v)$ and $z_B(v)$. Combine this information to form the matrix

$$Q_k(v) = \begin{pmatrix} Z_A(k) & z_A(v) \\ Z_B(k) & z_B(v) \end{pmatrix}. \qquad (6)$$

See fig. 3 for examples of $Q_k(v)$ for two vertices. Using this matrix we assign the following index to the spectral gap represented by v:

$$\mathfrak{c}_k(v) = (-1)^k \det Q_k(v) \ \mathit{mod}^* \ q_k. \qquad (7)$$

The notation *mod** stands for the centered modulo (also known as symmetric modulo) which is defined as

$$x \ \mathit{mod}^* \ q := x - q\left\lfloor \frac{x}{q} + \frac{1}{2} \right\rfloor \in \left[-\frac{q}{2}, \frac{q}{2}\right) \cap \mathbb{Z}, \qquad (8)$$

where $\lfloor\cdot\rfloor$ is the floor function. Note that we have slightly changed here the conventional definition of *mod**, by including $-\frac{q}{2}$, rather than $\frac{q}{2}$, as is usually done. See the Supplementary Material `Supplementarymaterial.pdf` (SM), sect. E for an explanation of the rationale behind this choice. Figure 3 demonstrates the $\mathfrak{c}_k(v)$ values which are assigned to vertices in the few first levels of a particular tree $\mathcal{T}_\alpha$.

In the sequel we provide the justification of the index formula (7). To do so, we employ the tree construction presented above; we show a conservation of indices along suitable paths of the tree, and furthermore that this index agrees with the quasiperiodic limit ($k \to \infty$). The discussion in what follows may be read alongside fig. 3, which illustrates the fundamental properties of the tree construction; see also sect. F of the SM for a guided explanation of the figure. First, note that the index $\mathfrak{c}_k(v)$ in (7) is indeed a solution for the Diophantine equation (2), see SM, sect. A. We proceed to further demonstrate that $\mathfrak{c}_k(v)$ is actually the natural solution for the modulo q ambiguity when taking into account the governing quasiperiodic structure.

Fix a rational number $\frac{p}{q} \in \mathbb{Q}$ and take a spectral gap of $H_{\frac{p}{q}}$ to which one wants to assign an index. Choose a finite continued fraction which represents $\frac{p}{q}$ as in (4). Extend it arbitrarily to obtain an irrational $\alpha \notin \mathbb{Q}$. This means that there is a $k \in \mathbb{N}_0$ such that $\frac{p}{q} = \alpha_k$, so $\frac{p}{q}$ is a rational approximation of α. Now, consider the tree $\mathcal{T}_\alpha$ and let v be the vertex representing the chosen spectral gap of $H_{\frac{p}{q}}$. This G-vertex v with index $\mathfrak{c}_k(v)$ can be seen as an approximation of a particular gap of the quasiperiodic Hamiltonian H_α, as is explained below. This spectral gap of H_α has a well-defined (*i.e.*, nonambiguous) index $\mathfrak{c} \in \mathbb{Z}$, such that the IDS satisfies

$$N_\alpha(E) = \mathfrak{c}\alpha \mod 1, \qquad \mathfrak{c} \in \mathbb{Z}, \qquad (9)$$

for energies E in that gap. Equation (9) results from the gap labelling theorem [36,37], which determines the set of allowed values that the IDS may attain at spectral gaps (see also the recent work [38]). Note that (9) has the same form as (2), however, it does not carry the same modulo ambiguity, since here α is irrational. Concretely, when α is irrational there is at most one $\mathfrak{c} \in \mathbb{Z}$ solution to (9), given some value for $N_\alpha(E)$. This substantial difference between rational and irrational α values is a key ingredient in the ambiguity resolution. We proceed to show that the index of the gap of $H_{\frac{p}{q}}$ coincides with the index of the corresponding gap of H_α, *i.e.*, $\mathfrak{c}_k(v) = \mathfrak{c}$. This is independent of how we choose α. Therefore, $\mathfrak{c}_k(v)$ reflects the quasiperiodic structure and our index choice resolves the bespoken modulo ambiguity.

To determine the mentioned spectral gap in H_α, we construct (sketched in fig. 3) an infinite path γ in the tree $\mathcal{T}_\alpha$ starting from v, such that all the G-vertices of γ have the index $\mathfrak{c} = \mathfrak{c}_k(v)$ as well.

The path construction depends on the sign of $\mathfrak{c}_k(v)$ and the parity of k. Assume first that either i) k is even and $\mathfrak{c}_k(v)$ is positive, or ii) k is odd and $\mathfrak{c}_k(v)$ is negative. We set the first vertex of γ to be v; the second vertex

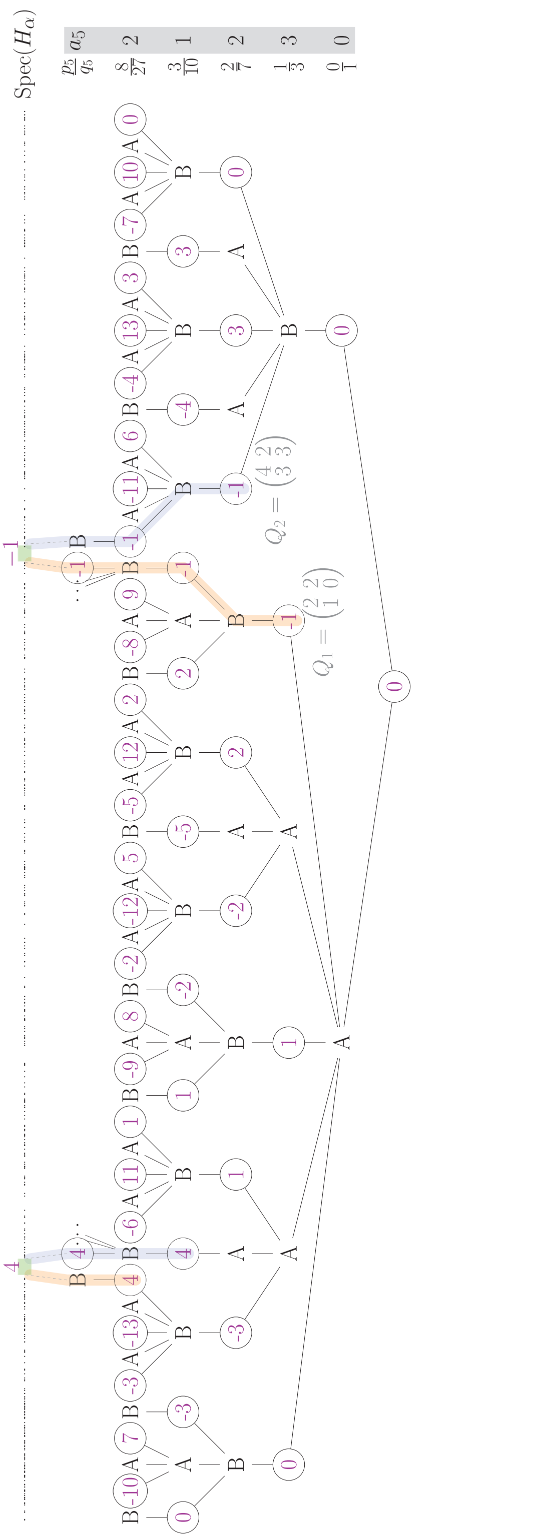

Fig. 3: Illustration of the tree for a continued fraction beginning with $0, 3, 2, 1, 2$ (this tree corresponds to any irrational α with such a prefix of its continued fraction expansion). The index is marked within the circle of the G-vertex. Four G-vertices are highlighted with the corresponding infinite paths having index $\mathfrak{c}_k(v) = -1$ respectively $\mathfrak{c}_k(v) = 4$. The spectrum $\mathrm{Spec}(H_\alpha)$ is schematically depicted to illustrate its connection to particular pairs of paths in the tree.

is the single B vertex at level $k+1$ which emanates from v. Afterwards in each level $k+m+1$ (for $m \geq 1$), γ is defined by choosing the rightmost vertex which emanates from its vertex at level $k+m$. This uniquely determines a path starting at v where the vertex labels alternate between G and B. The complementary scenario is either iii) k is even and $\mathfrak{c}_k(v)$ is negative or iv) k is odd and $\mathfrak{c}_k(v)$ is positive. Then, we act in a reverse manner: start from v and construct γ by always picking the leftmost vertex when branching, see an example in fig. 3.

We continue to verify that all G-vertices in γ have the same index. Let v be a G-vertex at level k. It is connected to a unique B-vertex at level $k+1$ and its neighboring G-vertices are denoted by u and w (u on the left, w on the right, as exemplified in fig. 2). The Q matrices of these vertices are related as follows:

$$
\begin{aligned}
Q_{k+1}(w) &= Q_{k+1}(u) + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&= T_{k+1} Q_k(v) + \begin{pmatrix} 0 & 0 \\ 0 & k \mod 2 \end{pmatrix},
\end{aligned} \tag{10}
$$

where $T_k = \begin{pmatrix} a_k - 1 & a_k \\ 1 & 1 \end{pmatrix}$ with a_k being the digits of the continued fraction expansion of α. Validating (10) is straightforward given the tree branching structure and that in each level we alter between G-vertices and A/B-vertices. Now, let $\gamma = (v_0, v_1, v_2, \ldots)$ be an infinite path, as described above, starting from $v = v_0$ at level k. Then all the even vertices v_{2m} are G-vertices. Using $\det T_k = -1$ for all k, we conclude $\mathfrak{c}_k(v_0) = \mathfrak{c}_{k+2m}(v_{2m})$ for all $m \in \mathbb{N}$ from (7) and (10). More details are provided in the SM, sect. C.

We proceed to describe the gap of H_α which corresponds to γ and show that its index equals the common value $\mathfrak{c}_k(v_0) = \mathfrak{c}_{k+2m}(v_{2m})$ of all G-vertices of γ, as mentioned above. The vertices of γ alternate between the labels G and B (the first vertex v_0 has label G). The G-vertices represent spectral gaps of the periodic operators; these spectral gaps converge to a spectral gap of H_α. On an intuitive level, the spectral convergence stems from the tree construction —for a path containing infinitely many B-vertices, the corresponding spectral bands are nested, and the intersection of all of them contains a single point belonging to $\mathrm{Spec}(H_\alpha)$. A complete proof appears in [39]; see also sect. D of the SM. Furthermore, the infinite paths γ described above come in pairs, with each path representing one boundary (left/right) of a spectral gap of H_α [34,40], see fig. 3. The conservation law holds for all indices $\mathfrak{c}_k$ on both paths.

Pick a spectral gap I of H_α and let $\gamma = (v_0, v_1, v_2, \ldots)$ be the left path whose gaps (represented by the vertices $\{v_{2m}\}$) converge to the chosen gap I of H_α; we choose the left path as an example, but all arguments work as well for the right path with suitable right/left switches. For each $m \geq 0$ denote by E_m the right boundary of the gap represented by v_{2m}. The energies $\{E_m\}$ converge to the right boundary of I (sect. D of the SM), which we denote by E.

The IDS values of the periodic operators at these energies satisfy $N_{\alpha_{k+2m}}(E_m) = \mathfrak{c}_{k+2m}(v_{2m})\alpha_k \mod 1$, since the index $\mathfrak{c}_{k+2m}(v_{2m})$ was shown to satisfy the Diophantine equation (2). By the definition of the IDS [34] together with the convergence $E_m \to E$ and $\alpha_{k+2m} \to \alpha$ we get $N_{\alpha_{k+2m}}(E_m) \to N_\alpha(E)$ as $m \to \infty$. The gap labelling theorem [36] yields $N_\alpha(E) = \mathfrak{c}\alpha \mod 1$ for some value $\mathfrak{c} \in \mathbb{Z}$, as in (9). Hence, by the conservation $\mathfrak{c}_k(v_0) = \mathfrak{c}_{k+2m}(v_{2m})$ shown above and the convergence of the IDS values, we get that this conserved index equals the index $\mathfrak{c}$ of the quasiperiodic operator H_α, *i.e.*, $\mathfrak{c}_{k+2m}(v_{2m}) = \mathfrak{c}$ for all $m \geq 0$. This concludes the arguments justifying that the Kohmoto model indices are given by (7).

To summarize, this letter establishes a full classification of the indices of the Kohmoto model, and while doing so four additional goals are reached. First, the tree-based description via $\mathcal{T}_\alpha$ shows that the spectra of all periodic Kohmoto Hamiltonians are intrinsically connected, collectively forming the Kohmoto butterfly (fig. 1, right). Furthermore, it provides a structural framework to investigate related quasiperiodic models and to deepen the understanding of their indices.

Second, the tree structure exposes that the quasiperiodicity is reflected in the periodic approximations. By the recent resolution of the dry ten Martini problem [33,34] it is known that all integer values show up as an index value in (9). Explicitly, when fixing $\alpha \notin \mathbb{Q}$, we know that every integer value $\mathfrak{c} \in \mathbb{Z}$ appears as the index value of some open gap of H_α. The current letter identifies the minimal periodic (finite-size) approximations that realize every such possible index, and specifies the energy gaps in which that value occurs. The ability to exactly specify for each index in which finite-size system it appears is substantial for experimental realizations. To state it precisely, all irrational α values with the same prefix of length k would share the same indices up to level k. Using the notation $\alpha_k = \frac{p_k}{q_k}$, this ought to be revealed in finite-size (or periodic) systems of length q_k, and the corresponding indices would be in the range $[-q_k/2, q_k/2)$, as in (8).

Third, for the Kohmoto model we settle the ambiguity problem highlighted in [32] by confirming and sharpening the conjecture posed there. Not only that the index values offered in (7) resolve the ambiguity problem, but they are also invariant, as they are shown to be conserved in the quasiperiodic limit ($k \to \infty$). The full physical meaning of this invariant is left as an open challenge.

Fourth, the index values lead to a full coloring of the topological phase diagram of the Kohmoto model (fig. 1, left), in a direct analogy to Hofstadter's colored butterfly [28,29]. Nevertheless, the two models are substantially different. In contrast to the Hofstadter butterfly, the complement of the Kohmoto butterfly consists of a single connected component. Due to this, the gap indices are not restricted to connected components of the phase diagram (fig. 1) as opposed to the Hofstadter butterfly. For example, one can see in the colored Kohmoto butterfly that the red phases terminate without a gap continuously shrinking and closing. Indeed, gaps do not necessarily continuously deform with α, but rather split into two at rational values of α [41]. This is against the folk wisdom, grounded in smooth models. On the other hand, the color ordering in the phase diagram is identical for both butterflies, highlighting a similarity between the two models. This provides another perspective on the question of topological equivalence between the Kohmoto and Hofstadter models; a question raised in [42], gained a substantial progress in [30,43] and even more recently in [6], but not yet conclusively resolved.

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REFERENCES

[1] Ozawa T., Price H. M., Amo A., Goldman N., Hafezi M., Lu L., Rechtsman M. C., Schuster D., Simon J., Zilberberg O. and Carusotto I., *Rev. Mod. Phys.*, **91** (2019) 015006.

[2] Cherkaev E., Vasquez F. G., Mauck C., Prisbrey M. and Raeymaekers B., *Phys. Rev. Lett.*, **126** (2021) 145501.

[3] Lesser O. and Lifshitz R., *Phys. Rev. Res.*, **4** (2022) 013226.

[4] Bellissard J., *K-theory of C*-algebras in solid state physics*, in *Statistical Mechanics and Field Theory: Mathematical Aspects*, *Lect. Notes Phys.*, Vol. **257**, edited by Dorlas T. C., Hugenholtz N. M. and Winnink M. (Springer, Berlin, Heidelberg) 1986, pp. 99–156.

[5] Prodan E. and Schulz-Baldes H., *Bulk and Boundary Invariants for Complex Topological Insulators: From K-Theory to Physics*, *Mathematical Physics Studies* (Springer International Publishing, Cham) 2016.

[6] Kellendonk J., *Israel J. Chem.*, **64** (2024) e202400027.

[7] Doll N., Loring T. and Schulz-Baldes H., *Math. Phys. Anal. Geom.*, **28** (2025) 13.

[8] Verbin M., Zilberberg O., Kraus Y. E., Lahini Y. and Silberberg Y., *Phys. Rev. Lett.*, **110** (2013) 076403.

[9] Dareau A., Levy E., Aguilera M. B., Bouganne R., Akkermans E., Gerbier F. and Beugnon J., *Phys. Rev. Lett.*, **119** (2017) 215304.

[10] Baboux F., Levy E., Lemaître A., Gómez C., Galopin E., Gratiet L. L., Sagnes I., Amo A., Bloch J. and Akkermans E., *Phys. Rev. B*, **95** (2017) 161114.

[11] Stepanov P., Xie M., Taniguchi T., Watanabe K., Lu X., MacDonald A. H., Bernevig B. A. and Efetov D. K., *Phys. Rev. Lett.*, **127** (2021) 197701.

[12] Phong V. T. and Mele E. J., *Phys. Rev. Lett.*, **128** (2022) 176406.

[13] Wang P., Zheng Y., Chen X., Huang C., Kartashov Y. V., Torner L., Konotop V. V. and Ye F., *Nature*, **577** (2020) 42.

[14] Nuckolls K., Scheer M., Wong D., Oh M., Lee R., Herzog-Arbeitman J., Watanabe K., Taniguchi T., Lian B. and Yazdani A., *Nature*, **639** (2025) 60.

[15] Becker S., Ge L. and Wittsten J., *Ann. Henri Poincaré*, **26** (2025) 3103.

[16] Zilberberg O., *Opt. Mater. Express*, **11** (2021) 1143.

[17] Kohmoto M., Kadanoff L. P. and Tang C., *Phys. Rev. Lett.*, **50** (1983) 1870.

[18] Thouless D. J., *Phys. Rev. B*, **27** (1983) 6083.

[19] Niu Q. and Thouless D. J., *J. Phys. A: Math. Gen.*, **17** (1984) 2453.

[20] Kraus Y. E., Lahini Y., Ringel Z., Verbin M. and Zilberberg O., *Phys. Rev. Lett.*, **109** (2012) 106402.

[21] Chung J.-H. and Shapiro J., *Adv. Math.*, **480** (2025) 110486.

[22] Tanese D., Gurevich E., Baboux F., Jacqmin T., Lemaître A., Galopin E., Sagnes I., Amo A., Bloch J. and Akkermans E., *Phys. Rev. Lett.*, **112** (2014) 146404.

[23] Damanik D., Gorodetski A. and Yessen W., *Invent. Math.*, **206** (2016) 629.

[24] Hofstadter D. R., *Phys. Rev. B*, **14** (1976) 2239.

[25] Jitomirskaya S., *One-dimensional quasiperiodic operators: global theory, duality, and sharp analysis of small denominators*, in *Proceedings of the International Congress of Mathematicians 2022*, Vol. **2** (EMS Press) 2023, pp. 1090–1120.

[26] Thouless D. J., Kohmoto M., Nightingale M. P. and den Nijs M., *Phys. Rev. Lett.*, **49** (1982) 405.

[27] Dana I., Avron Y. and Zak J., *J. Phys. C: Solid State Phys.*, **18** (1985) L679.

[28] Osadchy D. and Avron J. E., *J. Math. Phys.*, **42** (2001) 5665.

[29] Avron J. E., Osadchy D. and Seiler R., *Phys. Today*, **56**, issue No. 8 (2003) 38.

[30] Kellendonk J. and Prodan E., *Ann. Henri Poincaré*, **20** (2019) 2039.

[31] Kellendonk J. and Scaglione L., *Augmentation and bulk edge correspondence for one dimensional aperiodic tight binding operators*, arXiv:2512.01555.

[32] Avron J. E., Kenneth O. and Yehoshua G., *J. Phys. A*, **47** (2014) 185202.

[33] Band R., Beckus S. and Loewy R., *MFO Report: The Dry Ten Martini problem for Sturmian dynamical systems*, arXiv:2309.04351 (2023).

[34] Band R., Beckus S. and Loewy R., *The Dry Ten Martini Problem for Sturmian Hamiltonians*, arXiv:2402.16703 (2024).

[35] Band R., Beckus S. and Loewy R., *Complete Hierarchical Coding of the Spectral Bands in the Kohmoto Model*, in preparation (2026).

[36] Bellissard J., Bovier A. and Ghez J.-M., *Rev. Math. Phys.*, **4** (1992) 1.

[37] Pimsner M. and Voiculescu D., *J. Operator Theory*, **4** (1980) 93.

[38] Damanik D. and Fillman J., *Gap labelling for discrete one-dimensional ergodic Schrödinger operators*, in *From Complex Analysis to Operator Theory—A Panorama*, *Oper. Theory Adv. Appl.*, Vol. **291** (Birkhäuser/Springer, Cham) 2023, pp. 341–404.

[39] Bellissard J., Iochum B. and Testard D., *Commun. Math. Phys.*, **141** (1991) 353.

[40] Band R., Beckus S., Biber B., Raymond L. and Thomas Y., *A review of a work by L. Raymond: Sturmian Hamiltonians with a Large Coupling Constant—Periodic Approximations and Gap Labels*, in *Mathematical Models for Interacting Dynamics on Networks*, edited by Čolić M., Giesselmann J., Glück J., Kramar Fijavž M., Mauroy A. and Mugnolo D. (Springer Nature Switzerland, Cham) 2026, pp. 1–83.

[41] Beckus S., Bellissard J. and Thomas Y., *Ann. Henri Poincaré*, (2025) doi:10.1007/s00023-025-01578-8.

[42] Hiramoto H. and Kohmoto M., *Phys. Rev. Lett.*, **62** (1989) 2714.

[43] Kraus Y. E. and Zilberberg O., *Phys. Rev. Lett.*, **109** (2012) 116404.