Supplemental Material of
From Quasiperiodicity to a Complete Coloring of the Kohmoto Butterfly

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We add here further details about the calculations included in the main text.

A. Deriving Eq. (2) of the main text

Consider a rational number $\frac{p}{q}$ and an irrational α such that $\alpha_k = \frac{p}{q}$. Let v be a G-vertex representing a spectral gap of the spectrum of H_{α_k} with index $\mathfrak{c}_k(v)$. Then v corresponds to the n-th spectral gap in $\operatorname{Spec}(H_{\alpha_k})$ with $n = z_A(v) + z_B(v)$ and $N_{\alpha_k}(E) = \frac{n}{q}$.

By the theory of continued fractions, we have $p_k q_{k-1} - p_{k-1}q_k = (-1)^{k-1}$, equivalently, $p_k q_{k-1} = (-1)^{k-1} \mod q_k$. Since

$$q_{k-1} = Z_A(k-1) + Z_B(k-1) = Z_B(k)$$

and $Z_A(k) + Z_B(k) = q_k$ by the basic properties of the tree $\mathcal{T}_\alpha$, we get $p_k Z_B(k) = (-1)^{k-1} \mod q_k$ and $p_k Z_A(k) = (-1)^k \mod q_k$. Denoting $\mathfrak{i}_k(v) := (-1)^k \det Q_k(v)$ this implies

$$\begin{aligned}\mathfrak{i}_k(v)p_k &\equiv (-1)^k \left(Z_A(k)z_B(v) - Z_B(k)z_A(v)\right)p_k \mod q_k\\ &\equiv z_B(v) + z_A(v) \equiv n \mod q_k.\end{aligned}$$

By Eq. (7) of the main text, $\mathfrak{i}_k(v) = \mathfrak{c}_k(v) \mod q_k$ validating the Diophantine equation Eq. (2) of the main text.

We note that the identity $\left(p_k^{-1} \mod q_k\right) = (-1)^{k-1}q_{k-1}$ is a consequence of the above, and it may be used for explicitly solving Eq. (2) of the main text and coloring the Kohmoto butterfly.

B. Deriving Eq. (10) of the main text

Let v be a G-vertex at level k in a spectral α-tree. It is connected to a unique B-vertex v' at level $k+1$, with u and w the neighboring G-vertices. This immediately yields

$$Q_{k+1}(w) = Q_{k+1}(u) + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

For the second equality in Eq. (10) of the main text, start by justifying the left column, i.e.

$$\begin{pmatrix} Z_A(k+1) \\ Z_B(k+1) \end{pmatrix} = T_{k+1} \begin{pmatrix} Z_A(k) \\ Z_B(k) \end{pmatrix}. \tag{1}$$

To see this note that ***(i)*** all A-vertices at level $k+1$ emanate from either A or B vertices at level k, and their count is given by the branching degree (Eq. (5) of the main text); and ***(ii)*** there is a bijection between all B-vertices at level $k+1$ and all G-vertices at level k. The number of the latter equals the total number of both A and B vertices in level k. It is left to show

$$\begin{pmatrix} z_A(w) \\ z_B(w) \end{pmatrix} = T_{k+1} \begin{pmatrix} z_A(v) \\ z_B(v) \end{pmatrix} + \begin{pmatrix} 0 \\ k \mod 2 \end{pmatrix}. \tag{2}$$

The arguments are similar to those used for (1). The only difference is that the number of G-vertices to the left of v (and including v) equals the total number of A and B vertices to the left of v, plus ($k \mod 2$). This correction comes since the left most vertex at odd levels is always a G-vertex.

C. Index conservation

We establish here the index conservation along the paths $\gamma = (v_0, v_1, v_2, \ldots)$, described in the Letter. Explicitly, we show that $\mathfrak{c}_k(v_0) = \mathfrak{c}_{k+2m}(v_{2m})$ for all $m \in \mathbb{N}$. As a by-product, our computations justify the choice of the centered window $\left[-\frac{q}{2}, \frac{q}{2}\right)$ in the definition of $\mathfrak{c}_k$, Eq. (7) of the main text. We start by adopting (as in Sec. A) the notation $\mathfrak{i}_k(v) := (-1)^k \det Q_k(v)$, with which Eq. (7) of the main text reads $\mathfrak{c}_k(v) = \mathfrak{i}_k(v)\, mod^*\, q_k$.

Let v be a G-vertex at level k in a spectral α-tree. It connects to a unique B-vertex at level $k+1$, with u and w the neighboring G-vertices (see Fig. 1). We begin by showing the following identities: if k is even, then

$$\mathfrak{i}_k(v) = \mathfrak{i}_{k+1}(w) \quad \text{and} \quad \mathfrak{i}_k(v) - q_k = \mathfrak{i}_{k+1}(u) - q_{k+1}, \tag{3}$$

and if k is odd then

$$\mathfrak{i}_k(v) = \mathfrak{i}_{k+1}(u) \quad \text{and} \quad \mathfrak{i}_k(v) - q_k = \mathfrak{i}_{k+1}(w) - q_{k+1}. \tag{4}$$

Since $\det(T_k) = -1$, $\det\left(T_{k+1}Q_k(v)\right) = -\det\left(Q_k(v)\right)$. Together with Eq. (10) of the main text, this yields $\mathfrak{i}_k(v) = \mathfrak{i}_{k+1}(w)$ when k even, and $\mathfrak{i}_k(v) = \mathfrak{i}_{k+1}(u)$ when k is odd. This proofs the first parts of the equations (3) and (4).

For the second part, we compute

$$\begin{aligned}\mathfrak{i}_k(v) - q_k &= (-1)^k \det\left[Q_k(v) + \begin{pmatrix} 0 & (-1)^k \\ 0 & (-1)^{k+1} \end{pmatrix}\right]\\ &= (-1)^{k+1} \det\left[T_{k+1}\left(Q_k(v) + \begin{pmatrix} 0 & (-1)^k \\ 0 & (-1)^{k+1} \end{pmatrix}\right)\right]\\ &= (-1)^{k+1} \det\left[T_{k+1}Q_k(v) + \begin{pmatrix} 0 & (-1)^{k+1} \\ 0 & 0 \end{pmatrix}\right].\end{aligned}$$

Thus, Eq. (10) implies if k is even

$$\begin{aligned}\mathfrak{i}_k(v) - q_k &= (-1)\det\left[Q_{k+1}(u) + \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}\right] \\ &= \mathfrak{i}_{k+1}(u) - q_{k+1}\end{aligned}$$

and if k is odd

$$\begin{aligned}\mathfrak{i}_k(v) - q_k &= \det\left[Q_{k+1}(w) + \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\right] \\ &= \mathfrak{i}_{k+1}(w) - q_{k+1},\end{aligned}$$

which concludes the verification of (3) and (4)

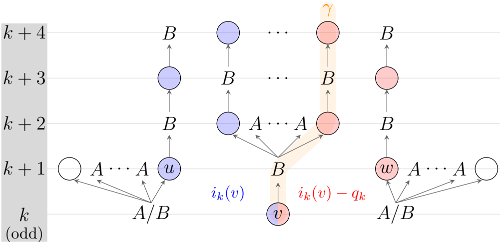

Figure 1. An illustration of the conservation deduced from (3) and (4) for odd k. For all blue vertices $i_*(*)$ is preserved and for all red vertices $i_*(*) - q_*$ is preserved. We indicate the path γ which is the one is chosen if $\mathfrak{c}_k(v) < 0$.

Equations (3) and (4) show that starting from a vertex v, there is a path along which i_k is conserved and another path along which $i_k - q_k$ is conserved. The left/right orientation of those paths depends on the parity of k (see Fig. 1). Consequently, it is natural to select the conserved quantity (either i_k or $i_k - q_k$) as the gap index, since this value remains invariant.

We proceed by induction to show that

$$i_k(v) \in [0, q_k) \tag{5}$$

for all G-vertices v. For $k = 0$ and $k = 1$ this follows directly from computation.

Suppose the claim holds up to level $k-1$, and let v be a G-vertex in level k. If v is a G-vertex with no neighbor on one side (either left or right), then $i_k(v) = 0$ by definition of the matrix $Q_k(v)$. For all other G-vertices, the construction of the spectral tree shows that either v has a neighboring B-vertex or both neighbors are A-vertices, Fig. 2. We treat each of these cases separately.

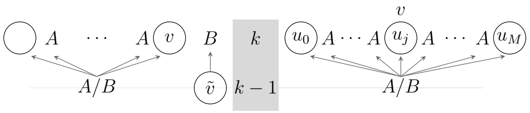

Figure 2. Illustration of different G-vertices: sandwiched between an A-vertex and a B-vertex (Left) or between two A-vertices (Right).

If, for example, there is a B-vertex to the right of v (Fig. 2, Left), then this B-vertex emanates from a G-vertex $\tilde{v}$ in level $k-1$. Since $i_k(\tilde{v}) \in [0, q_k)$, the relation (3) or (4) implies that $\mathfrak{i}_k(v) \in [0, q_k)$ using $q_{k+1} > q_k$. Similarly, one shows $\mathfrak{i}_k(v) \in [0, q_k)$ if there is a B-vertex to the left of v.

Now suppose both neighbors of v are A-vertices. Enumerate the G-vertices emanating from the same A/B-vertex as v, from left to right, by $u_0, \dots, u_M$, with $v = u_j$ for some j (Fig. 2, Right). Then either there exists a B-vertex to the left of u_0 (or symmetrically a B-vertex to the right of u_M), or u_0 is the left-most vertex in level k (symmetrically, u_M is the right-most vertex in level k). In either case we conclude from the previous considerations that $i_k(u_0), i_k(u_N) \in [0, q_k)$. Since

$$Q_k(u_j) = Q_k(u_0) + \begin{pmatrix} 0 & j \\ 0 & 0 \end{pmatrix},$$

the sequence $i_k(u_j)$ is monotone in j (either decreasing or increasing depending on the parity of k). Because both endpoints $i_k(u_0)$ and $i_k(u_N)$ lie in $[0, q_k)$, it follows that $i_k(u_j) \in [0, q_k)$ for all $0 \le j \le M$ and in particular for the vertex v. Therefore we have established (5).

By definition of $\mathfrak{c}_k$ (Eq. (7) of the main text and Eq. (5) here) we have

$$\mathfrak{c}_k(v) = \begin{cases} \mathfrak{i}_k(v) & 0 \le \mathfrak{i}_k(v) < \frac{q_k}{2}, \\ \mathfrak{i}_k(v) - q_k & \frac{q_k}{2} \le \mathfrak{i}_k(v) < q_k. \end{cases} \tag{6}$$

From (6) together with (3), (4), we can now show the conservation of $\mathfrak{c}_k$ along the paths γ, described in the Letter. We treat the case of odd k (the even case follows analogously), as illustrated in Fig. 1. Suppose first that $\mathfrak{c}_k(v) < 0$. Then the path $\gamma = (v_0, v_1, \dots)$ is defined by setting $v_0 = v$ and, at each branching, choosing the right-most descendant. Since $\mathfrak{c}_k(v) < 0$, (6) gives $\mathfrak{c}_k(v) = \mathfrak{i}_k(v) - q_k$. Applying (3) and (4) it inductively (see the red colored vertices, arranged in a zigzag pattern in Fig. 1) follows that $\mathfrak{c}_k(v) = \mathfrak{c}_{k+2m}(v_{2m})$ for all $m \in \mathbb{N}$.

Next suppose that $\mathfrak{c}_k(v) > 0$. Then the path $\eta = (u_0, u_1, \dots)$ is defined by setting $u_0 = v$ and, at each branching, choosing the left-most descendant. Since $\mathfrak{c}_k(v) > 0$, (6) gives $\mathfrak{c}_k(v) = \mathfrak{i}_k(v)$. Applying (3) and (4) it inductively (see the blue colored vertices, arranged in a zigzag pattern in Fig. 1) follows that $\mathfrak{c}_k(v) = \mathfrak{c}_{k+2m}(u_{2m})$ for all $m \in \mathbb{N}$.

D. Spectral gaps convergence along γ

Consider an irrational α with its spectral α-tree, $\mathcal{T}_\alpha$. Let v be a G-vertex with index $\mathfrak{c}_k(v)$ and let $\gamma = (v_0, v_1, \dots)$ be the infinite path starting at $v = v_0$, which is defined in this Letter, such that the indices along it are conserved, i.e., $\mathfrak{c}_k(v) = \mathfrak{c}_{k+2m}(v_{2m})$ for all $m \in \mathbb{N}$. For each v_{2m}, the open interval $I_m = (L_m, R_m)$ denotes the spectral gap of Spec $\left(H_{\alpha_{k+2m}}\right)$ represented by v_{2m}. Our goal is to verify that these spectral gaps I_m converge to the limiting spectral gap $I = (L, R)$ in Spec (H_α), which carries the same index $\mathfrak{c}_k(v)$.

We have two cases: either v_{2m} is the right-most vertex emanating from v_{2m-1} for all $m \in \mathbb{N}$ or it is always the left-most vertex. Without loss of generality we assume that v_{2m} is the right-most vertex for all $m \in \mathbb{N}$. We note that there is a neighboring path $\tilde{\gamma} = (w_0, w_1, \ldots)$ whose G-vertices share the same index values as the G-vertices of γ. For this path, w_0 is set to be the neighboring vertex at level $k+1$ to the right of the B-vertex emanating from v_0 (see Fig. 1 with $v_0 = v$ and $w_0 = w$). For the rest of the vertices of $\tilde{\gamma}$ we choose w_{2m} to be the left-most vertex emanating from w_{2m-1} for all $m \in \mathbb{N}$. By this construction we get that the vertex w_m is the right neighbor of the vertex v_{m+1} for all $m \in \mathbb{N}$.

Recall that the spectral gap associated with the G-vertex v_{2m} is $I_m = (L_m, R_m)$. By construction, the right endpoint $R_m \in \mathrm{Spec}\left(H_{\alpha_{k+2m}}\right)$ belongs to the spectral band associated with the B-vertex w_{2m+1}. These B-vertices correspond to spectral bands of the periodic operators, which form a decreasing nested sequence. Their intersection is a single point [1], lying in $\mathrm{Spec}\left(H_\alpha\right)$ and serving as the right endpoint R of the limiting gap I. Thus $R_m \to R$ as $m \to \infty$.

Moreover, $\mathrm{Spec}\left(H_{\alpha_{k+2m}}\right) \to \mathrm{Spec}\left(H_\alpha\right)$ as $m \to \infty$ [2], which implies that the left endpoints also converge, $L_m \to L$. Hence the gaps $I_m = (L_m, R_m)$ converge to the limiting gap $I = (L, R)$ of $\mathrm{Spec}\left(H_\alpha\right)$.

E. Negative indices versus positive indices

We comment here on the comparison between negative values of $\mathfrak{c}_k(v)$ versus positive values. For this discussion, recall the notation $\mathfrak{i}_k(v) := (-1)^k \det Q_k(v)$ (as in Sec. A and C) and the connection (6) between $i_k(v)$ and $\mathfrak{c}_k(v)$. We discuss to the particular case $i_k(v) = \frac{q_k}{2}$ for which one should determine the sign for $\mathfrak{c}_k(v) = \pm\frac{q_k}{2}$. This decision breaks the symmetry of the modulus window chosen for mod^*, i.e., whether the image of the modulus is $\left[-\frac{q_k}{2}, \frac{q_k}{2}\right)$ or $\left(-\frac{q_k}{2}, \frac{q_k}{2}\right]$. This decision was already made in Eq. (7) of the main text (see also Eq. (6) here) and we justify it here.

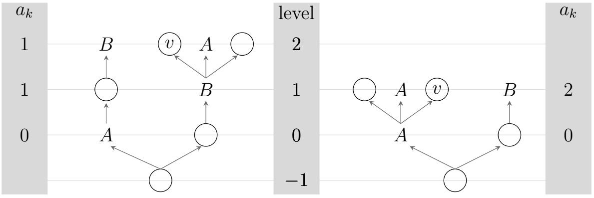

Figure 3. Two examples of spectral trees with a marked vertex v in level k which is either sandwiched between B and A vertices and k is even (Left) or v is sandwiched between A and B vertices and k is odd (Right).

As a guiding example we take $q_k = 2$. In this case, there are two spectral bands and only one bounded gap. We discuss the possible index value of that gap. To do so, consider a spectral tree $\mathcal{T}_\alpha$ with α having a continued fraction expansion starting with 2 (see Fig. 3, Right). In this tree there is a vertex v in level $k = 1$ which corresponds to the bounded gap we referred to, and indeed $q_1 = 2$. For this vertex $i_1(v) = 1 = \frac{q_1}{2}$, and we wish to explain the choice made in Eq. (7) of the main text for the modulus which gives $\mathfrak{c}_1(v) = -1$. Alternatively, $q_k = 2$ can be obtained from another spectral tree, $\mathcal{T}_\alpha$, where α has the continued fraction digits $1, 1, \ldots$ (see Fig. 3, Left). In this case there is a vertex v in level $k = 2$ for which $i_2(v) = 1 = \frac{q_2}{2}$.

In the first case above the vertex v is sandwiched between A and B vertices (in that order) and k is odd. In the second case, the vertex v is sandwiched between B and A vertices and k is even. These two cases belong to the same general class for all spectral trees: if either ***(i)*** v has A-vertex to its left and B-vertex to its right and k is odd or ***(ii)*** v has B-vertex to its left and A-vertex to its right and k is even, then $i_k(v) \geq \frac{q_k}{2}$. This justifies that the value $i_k(v) = \frac{q_k}{2}$ behaves under the modulus operation similarly to the values $i_k(v) \in (\frac{q_k}{2}, q_k)$, see Eq. (6), and hence a negative value for the index is obtained, $\mathfrak{c}_k(v) = -\frac{q_k}{2}$. We mention also the counterpart behavior: if either ***(iii)*** v has A-vertex to its left and B-vertex to its right and k is even or ***(iv)*** v has B-vertex to its left and A-vertex to its right and k is odd, then $i_k(v) < \frac{q_k}{2}$ (and $\mathfrak{c}_k(v)$ gets a positive value). The general statement above (with all of its parts (i)-(iv)) can be shown by induction, but we omit here the technical proof, and merely refer to Fig. 3 of the main text, which exemplifies it.

We complement this discussion with an additional viewpoint on negative versus positive index values for $\mathfrak{c}_k(v)$. One observes (main text, Fig. 1, Left) that the larger the absolute value of the index is, the smaller is the spectral gap and if the absolute value of two indices agree, then the one with negative index is dominant [3]. In particular, the gaps which correspond to the G-vertices whose index equals -1 are wider than those of index value 1 for all values of q_k, and hence more dominant and preferable in terms of index choice.

F. A worked example of the construction via Fig. 3 of the main text

We provide here a guided explanation of Fig. 3 of the main text to exemplify our construction. The figure contains an illustration of the tree for a continued fraction beginning with $0, 3, 2, 1, 2$. This tree fits any irrational α with such a prefix of its continued fraction expansion:

$$\alpha = 0 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cdots}}}}.$$

This determines the tree up to level $k = 4$, as drawn in Fig. 3, and the rest of the continued fraction expansion determines how the tree develops beyond that level. The vertices which correspond to spectral bands are indicated by A and B according to their type, whereas the G-vertices which correspond to spectral gaps are indicated by a circle. Inside each such circle, the value of the corresponding computed index $\mathfrak{c}_k(v)$ is written. The indices of all G-vertices which are to the extreme left or extreme right are zero. The corresponding gaps are the unbounded gaps at which the IDS attains either the value 0 (for gap on the left) or 1 (gap on the right).

We demonstrate the construction via a particular pair of paths marked by shaded blue and red on the right part of the tree. The vertex from which the red path begins is at level $k = 1$ (noting that the root of the tree is at level -1) and the corresponding Q matrix (see Eq. (6)) is

$$Q_1 = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}.$$

Indeed, in the first level there are two A-vertices and one B-vertex, which explains the left column of the matrix. To the left of that vertex there are two A-vertices and zero B-vertices, which explains the right column of the matrix. We get the corresponding index by following Eq. (7) and computing

$$\mathfrak{c}_k = (-1)^k \det Q_1 \, mod^* \, q_k = 2 \, mod^* \, 3 = -1,$$

which is indeed the index written in the circle of that vertex. To construct the infinite path which emanate from that vertex, we note that k is odd and $\mathfrak{c}_k$ is negative. This fits to case ***(ii)*** in the construction description (left column of page 3 in the main text). Hence, when building the path in the upward direction we always choose to branch towards the right-most emanating vertex. The result is indicated as the red shaded path in the figure. The indices of all the vertices along this path equal to -1, as we expect by the index conservation we have proven.

The second path in this pair (blue shaded) starts with a vertex on level $k = 2$ whose Q matrix is

$$Q_2 = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix},$$

and index is

$$\mathfrak{c}_k = (-1)^k \det Q_2 \, mod^* \, q_k = 6 \, mod^* \, 7 = -1.$$

The path emanating out of this vertex is constructed by always branching to the left-most vertex (the initial vertex this time fits case ***(iii)*** in the construction description: even k value and negative $\mathfrak{c}_k$). The indices along the blue path are also all equal to -1.

The vertices in this blue shaded alternate between B-type and G-type vertices. Each such B-type vertex corresponds to a spectral band. Taking the intersection of this infinitely many nested intervals gives a single point, which is an element in $\mathrm{Spec}\,(H_\alpha)$, [1] (this is also described in Sec. D above). Similarly, intersecting all the spectral bands corresponding to the B-vertices of the red path, yields another single point which also belongs to $\mathrm{Spec}\,(H_\alpha)$, which is schematically depicted at the top of Fig. 3 of the main text. These two elements of $\mathrm{Spec}\,(H_\alpha)$ are the the edge points of a spectral gap of $\mathrm{Spec}\,(H_\alpha)$, whose gap label (i.e., index) is -1. Indeed, it is proven in the main text (right column of page 3) that the index of such gap of $\mathrm{Spec}\,(H_\alpha)$ equals the common conserved index value of all vertices along the corresponding pair of paths. We end this description by referring to the colored Kohmoto butterfly (left of Fig. 1 in the main text). The index value -1 corresponds to the lightest blue color in the butterfly. The spectral gaps of $\mathrm{Spec}\,(H_{\alpha_k})$ indicated by the G-vertices mentioned above, correspond to a light-blue interval at frequency α_k. As $\alpha_k \to \alpha$, these intervals converge to a light-blue interval which is the mentioned spectral gap of $\mathrm{Spec}\,(H_\alpha)$.

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[1] R. Band, S. Beckus, and R. Loewy, The Dry Ten Martini Problem for Sturmian Hamiltonians, arXiv:2402.16703 (2024).

[2] J. Bellissard, B. Iochum, and D. Testard, Continuity properties of the electronic spectrum of 1d quasicrystals, Comm. Math. Phys. **141**, 353 (1991).

[3] This observation yet awaits a rigorous explanation.