STURM-HURWITZ THEOREM FOR QUANTUM GRAPHS

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Abstract. We prove upper and lower bounds for the number of zeroes of linear combinations of Schrödinger eigenfunctions on metric (quantum) graphs. These bounds for general graphs are distinct from the bounds for both the interval and manifolds. We complement these bounds by giving non-trivial examples for the lower bound as well as sharp examples for the upper bound. Furthermore, we show that every tree graph differs from the interval with respect to the nodal count of linear combinations of eigenfunctions. This stands in distinction to previous results which show that all tree graphs have the same eigenfunction nodal count as the interval.

1. Introduction

1.1. Historical background.

The rigorous study of the zero set of eigenfunctions of second-order differential operators originated in the 19th century, with Sturm's oscillation theorem on the interval being the first major result. The subject now encompasses graphs and manifolds, and while certain results hold true for all these objects, other properties of the nodal set are highly dependant on the dimension.

For instance, Sturm's theorem asserts that the n-th eigenfunction of any Sturm-Liouville operator on an interval has exactly $n-1$ zeroes [26].

For quantum graphs, the n-th eigenfunction of the Laplacian with Neumann-Kirchhhoff continuity conditions has between $n-1$ and $n-1+\beta$ zeroes[1] [7, 14], where β is the first Betti number, i.e. the number of graph cycles. In particular, the n-th eigenfunction of a tree graph (which is a graph with $\beta = 0$) has exactly $n-1$ zeroes, exactly as the interval. Furthermore, tree graphs are the only graphs such that the n-th eigenfunction has exactly $n-1$ zeroes for all n [2].

On manifolds, Courant's theorem [12] states that when we remove the zero set of the n-th eigenfunction of the Laplace-Beltrami operator on a connected smooth manifold M, we are left with at most n connected components (also known as nodal domains). This implies that the zero set has at most $n-1$ connected components. However, there are examples of eigenfunctions on the sphere and the square where the nodal set has exactly one connected component [25].

A lesser-known generalization of Sturm's result was published in the same year [27]: let $F := \sum_{i=j}^{k} a_i f_i$, where f_i is the i-th Sturm-Liouville eigenfunction on an interval $[a, b]$ with Dirichlet boundary conditions. Then, F has at least $j-1$ and at most $k-1$ zeroes in (a, b). We refer to this as the Sturm-Hurwitz theorem. An interesting survey on the story of this result and its proofs can be found in [5, 6].

[1]Under the assumption that λ_n is simple and the eigenfunction is non-zero at interior vertices.

In the case of manifolds, no such bounds can be found in full generality: there are metrics on the torus and the sphere and linear combinations of eigenfunctions of the corresponding Laplace-Beltrami operator such that their nodal set has infinitely many connected components [4, 11]. We also note that the Sturm-Hurwitz theorem is true for the isotropic quantum harmonic operator in dimension two [4, proposition $B.1$].

1.2. **Definitions.**

The modern interest in quantum graphs (first, as a paradigm of quantum chaos) arose in [17, 18]. Since then, some reviews and books on quantum graphs have been published, for example [8, 15, 19, 20]. The reader can find there many relevant aspects of the spectral theory of quantum graphs. Here, we only provide the concise definitions which will be relevant to this work.

- We denote by $\Gamma = \Gamma(V, E)$ a graph with vertices V and edges E. We will assume throughout the paper that the graph Γ is connected and has a finite number of edges, each of which has finite length.
- Each edge $e \in E$ is identified with the interval $[0, l_e]$, where l_e is the length of the edge.
- We set $\mathcal{E}_v$ as the multi-set of edges e which are connected to a vertex v, where each loop appears twice (once per direction).
- The degree of a vertex v is defined as $|\mathcal{E}_v|$, and will be denoted by $deg(v)$.
- Vertices of degree one are boundary vertices. The set of such vertices will be denoted V_b.
- Vertices of degree two or higher are inner vertices. The set of such vertices will be denoted V_i.
- For any $x, y \in \Gamma$, we set $d(x, y)$ as the usual shortest path distance between x and y, which is always well-defined since Γ is connected, making it a complete metric space.
- β denotes be the first Betti number of the graph, so that $\beta = |E| - |V| + 1$ since Γ is connected. It also represents the number of independant cycles, or the number of edges that one has to cut to turn the graph into a tree.
- $L^2(\Gamma) := \bigoplus_{e\in E} L^2(e)$, $C^1(\Gamma) := \bigoplus_{e\in E} C^1(e)$ and $\mathcal{H}^2(\Gamma) := \bigoplus_{e\in E} \mathcal{H}^2(e)$
- Let $f \in C^1(\Gamma)$. If f is continuous at inner vertices, for any $v \in V$ we define $f(v)$ as the common limit of $f(x)$ as x approaches v on any edge $e \in \mathcal{E}_v$.
- The normal derivative of a function f at a vertex v in the direction of $e \in \mathcal{E}_v$ will be denoted by $\partial_e f(v)$ and defined by taking the right limit at 0^+ of f' when the edge e is identified with $[0, l_e]$ and v is mapped to zero.
- We set $\mathcal{H}_\Gamma$ as the space of functions $f \in \mathcal{H}^2(\Gamma)$ with the following continuity conditions at the vertices:
  (1) If $v \in V_b$, $f(v) = 0$ (Dirichlet boundary condition).
  (2) If $v \in V_i$,
    (2a) f is continuous at v.
    (2b) $\sum_{e\in\mathcal{E}_v} \partial_e f(v) = 0$.

  Conditions (2a) and (2b) together are called Neumann-Kirchhoff continuity conditions.
- We say that a function has a degenerate edge if it is identically zero on that edge.
- Let $W : \Gamma \to \mathbb{R}$ be a continuous function. We define the Schrödinger operator $H_W : \mathcal{H}^2(\Gamma) \to L^2(\Gamma)$, $H_W := -\frac{\partial^2}{\partial x^2} + W$.

- The operator H_W restricted to $\mathcal{H}_\Gamma$ is self-adjoint and has an increasing sequence of eigenvalues $\lambda_1 \leq \lambda_2 \ldots$, numbered with multiplicity.
- To each eigenvalue λ_i we associate an eigenfunction f_i such that any two different f_i are orthogonal in $L^2(\Gamma)$.
- The number of zeroes of a function f which are distinct from the boundary vertices will be denoted by $N(f)$.
- We say that a graph Γ is W-generic if all eigenfunctions of H_W do not vanish at any inner vertex. We note that by continuity of eigenfunctions, this assumption implies that any eigenfunction of H_W does not have a degenerate edge and that every eigenvalue of H_W is simple. Indeed, if there is a multiple eigenvalue, given any vertex $v \in V_i$ it is always possible to choose a function in the linear space of eigenfunctions associated to this eigenvalue that is zero at v.

1.3. **Acknowledgements.**

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2. Results

Our main result is a two-sided bound on $N(F)$.

Theorem 2.1. *Let Γ be a W-generic graph with first Betti number β. Let f_k be the eigenfunctions of $H_W = -\frac{\partial^2}{\partial x^2} + W$ with Dirichlet boundary conditions on V_b and Neumann-Kirchhoff continuity conditions on V_i. Let k_i be a strictly increasing sequence and $F(x) = \sum_{i=1}^M a_i f_{k_i}(x)$ where each a_i is not zero. We have the following bounds:*

$$k_1 - 1 - (M-1)\left(|V_b| + 2\beta - 2\right) \leq N(F) \leq k_M - 1 + \beta + (M-1)\left(|V_b| + 2\beta - 2\right). \tag{2.1}$$

Remark 2.2. *By setting Γ as the unit interval, we recover Sturm's original theorem.*

On the interval, $N(f_1 + f_2) = 0$ or 1. On a 0-generic tree, $N(f_k) = k - 1$. On any tree graph, we construct examples of linear combinations of f_1 and f_2 with more zeroes that f_2.

Theorem 2.3. *Let Γ be a 0-generic tree that is not an interval. Let s be the highest degree of any vertex of Γ. Then, there exists $a(\Gamma)$ such that $N(f_1 + a(\Gamma)f_2) \geq s - 1$.*

We claim that there exist tree graphs that saturate the upper bound in Theorem 2.1.

Theorem 2.4. *For any $M > 0$ and $s \geq 3$, there exists a 0-generic star graph with s edges and a linear combination of the first M eigenfunctions of $-\frac{\partial^2}{\partial x^2}$ with $M - 1 + (M-1)(s-2)$ zeroes.*

Remark 2.5. *Theorem 2.3 shows that all tree graphs are different from the interval in terms of the nodal count of linear combinations of eigenfunctions. This behaviour is completely different than the nodal count of individual eigenfunctions - all tree graphs have exactly the same eigenfunction*[2] *nodal count as the interval* [2, 21, 24]*. Also, Theorem 2.4 shows that certain star graphs (with appropriately chosen edge lengths) can possess linear combinations whose nodal count is substantially higher than the number of zeroes of the highest eigenfunction.*

We now give examples of graphs with linear combinations that have much less zeroes than the lowest eigenfunction in the linear combination.

Theorem 2.6. *For any $m \geq 2$, there exists a 0-generic graph with $\beta = m$ and $a(m) \in \mathbb{R}$ such that $N(f_2) = m$, $N(f_3) = 2$ and $N(f_2 + a(m)f_3) = 2$.*

While this is not a proof that the lower bound in Theorem 2.1 is sharp, it shows that a linear combination F can have much less zeroes than f_{k_1}, unlike for the interval.

3. Proof of Theorem 2.1

3.1. Extension in two variables.

Let Γ be a W-generic graph, $H_W = -\frac{\partial^2}{\partial x^2} + W$ and f_k be the eigenfunctions of H_W. Let $F(x) = \sum_{i=1}^{M} a_i f_{k_i}(x)$. We define $g : \Gamma \times \mathbb{R} \to \mathbb{R}$ as $g(x,y) := \sum_{i=1}^{M} a_i e^{-\lambda_{k_i} y} f_{k_i}(x)$. The function g is a solution to $\frac{\partial g}{\partial y} = \frac{\partial^2 g}{\partial x^2} - W(x)g$.

For fixed y, we will denote $g_y(x) := g(x,y)$. Note that $g_0 = F$.

Since the eigenvalues are simple, $\lim_{y\to-\infty} g_y e^{\lambda_{k_M} y} = a_M f_{k_M}$ and $\lim_{y\to+\infty} g_y e^{\lambda_{k_1} y} = a_1 f_{k_1}$. Also, since each eigenfunction is non-zero at inner vertices, as $y \to -\infty$ the zeroes of g_y will converge to the zeroes of f_{k_M} and as $y \to +\infty$ the zeroes of g_y will converge to the zeroes of f_{k_1}.

3.2. Strategy of the proof of Theorem 2.1.

First, we will describe in section 3.3 the local behaviour of the zero set of g. Then, starting at $y = -\infty$, we will follow $N(g_y)$ as y increases. We will look in section 3.4 at all possible local behaviours of the zero set of g which would cause $N(g_y)$ to change as y increases. Finally, we will combine all these observations in section 3.5 to complete the proof.

3.3. Local behaviour of g_y near a zero.

First, we state the following fact about the zeroes of g:

Lemma 3.1. *The function g does not have an isolated zero.*

[2]Here, we mean the nodal count of individual eigenfunctions rather than the nodal count of linear combinations.

The proof of Lemma 3.1 will be postponed to Appendix A.

Now, the behaviour of zeroes of solutions of parabolic equations is well-understood. However, since on each edge, the function g does not obey homogeneous boundary conditions, we will extend it to the whole real line so we can use already-known results.

For an edge $e \in E$ (remember that e is identified as $[0, l_e]$), we define W^e as

$$W^e(x) := \begin{cases} W(x) & x \in [0, l_e] \\ W(0) & x < 0 \\ W(l_e) & x > l_e \end{cases} \tag{3.1}$$

Now, for $n \in \mathbb{N}$, we define f_n^e as the C^2 solution of $-(f_n^e)'' + W^e f_n^e = \lambda_n f_n^e$ on $\mathbb{R}$ such that $f_n^e \equiv f_n$ on $[0, l_e]$. By our choice of W^e, we know that on $(l_e, +\infty)$, $(f_n^e)'' = (W(l_e) - \lambda_n) f_n^e$. Therefore, we have three options: if $(W(l_e) - \lambda_n) > 0$, f_n^e is a linear combination of $e^{-\sqrt{W(l_e)-\lambda_n}x}$ and $e^{\sqrt{W(l_e)-\lambda_n}x}$, if $(W(l_e) - \lambda_n) = 0$, f_n^e is a linear function and if $(W(l_e) - \lambda_n) < 0$, f_n^e is a linear combination of $\cos(\sqrt{W(l_e) - \lambda_n}x)$ and $\sin(\sqrt{W(l_e) - \lambda_n}x)$.

Let $c_{e,n} := \sqrt{|W(l_e) - \lambda_n| + |W(0) - \lambda_n| + 1}$.

By the preceding discussion, there is a constant C_n such that for any $x \in \mathbb{R}$,

$$|f_n^e(x)| \leq C_n e^{c_{e,n}|x|} . \tag{3.2}$$

Remark 3.2. *Note that since Γ is W-generic, any f_n^e cannot be identically zero on either $(-\infty, 0)$ or $(l_e, +\infty)$.*

We now define $g^e(x, y) := \sum a_i e^{-\lambda_{k_i} y} f_{k_i}^e(x)$. The function g^e is defined on $\mathbb{R} \times \mathbb{R}$ and coincides with g on $[0, l_e] \times \mathbb{R}$. It is C^2 and solves

$$\frac{\partial g^e}{\partial y} = \frac{\partial^2 g^e}{\partial x^2} - W^e g^e \tag{3.3}$$

on $\mathbb{R} \times \mathbb{R}$. Also, the bound (3.2) guarantees that f_n^e grows at most exponentially in $|x|$. This ensures that for each y, g_y^e grows at most exponentially. Finally, by Remark 3.2, as $y \to -\infty$, on any compact set the zeroes of g_y^e will converge to the zeroes of $f_{k_M}^e$.

The following Theorem summarizes theorems A, B, 5.3, 5.5 and 5.6 from [1]:

Theorem 3.3. [1] *Let $g^e : \mathbb{R} \times [0, T]$ be a non identically zero C^2 solution to $\frac{\partial g^e}{\partial y} = \frac{\partial^2 g^e}{\partial x^2} + W^e(x, y) g$ with $W^e \in L^\infty$ and $|g^e(x, y)| \leq A e^{Bx^2}$ for some constants A and B. Then, g^e has the following properties:*

1. *For each $y \in (0, T)$, for each $x \in \mathbb{R}$, there is a neighbourhood of x where g_y^e has finitely many zeroes[3].*
2. *For each zero (x_0, y_0) of g^e, there is at least one continuous curve $(x(y), y)$ of zeroes of g^e for $y \in [0, y_0]$ such that $x(y_0) = x_0$.*
3. *If $g^e(x_0, y_0) = \frac{\partial g^e}{\partial x}(x_0, y_0) = 0$, then there exists δ_0 such that for any $\delta < \delta_0$, $g_{y_0+\delta}^e$ has at most one zero in $[x_0 - \delta_0, x_0 + \delta_0]$ and $g_{y_0-\delta}^e$ has at least two zeroes in $[x_0 - \delta_0, x_0 + \delta_0]$.*

[3]This statement is a by-product of the proof of Theorem 5.6 in [1].

(4) *Let* $x_1(y)$ *and* $x_2(y)$ *be two continuous curves of zeroes of* g^e *for* $y \in [0, y_1]$. *If* $x_1(y_1) < x_2(y_1)$, *then* $x_1(y) < x_2(y)$ *for all* $y \in [0, y_1]$.

From this result, we can deduce several properties of the zero set of g. Let Z be the zero set of g, S the set of singular zeroes of g (where g and $\frac{\partial g}{\partial x}$ are zero) and $S' := \{(v, y) \,|\, g(v, y) = 0 \quad \text{and} \quad v \in V\}$.

First, for any $y \in \mathbb{R}$, g_y has finitely many zeroes in $[0, l_e]$. Indeed, for any y, we have a covering of $\mathbb{R}$ with open sets where g_y^e has finitely many zeroes. By compactness, there is a finite subcover of $[0, l_e]$ by such open sets, which implies that g_y^e has finitely many zeroes on $[0, l_e]$.

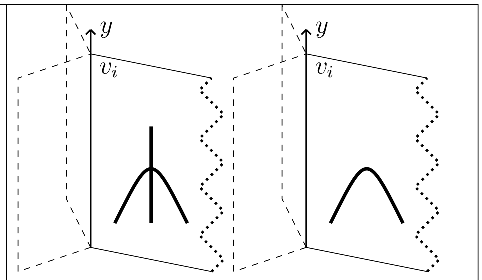
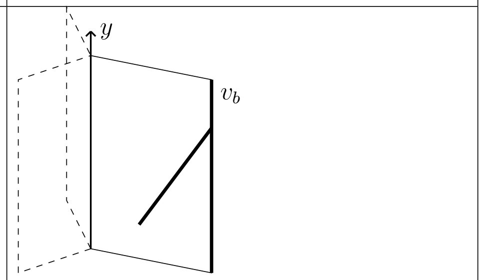
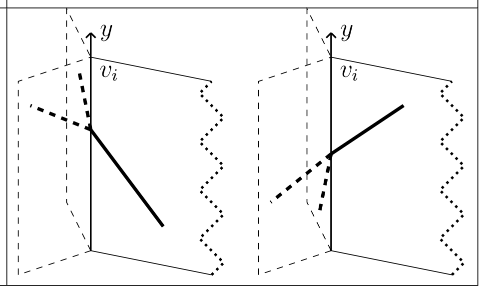
Second, since equation (3.3) does not change under a shift in the y-direction, every zero in Z is included in a continuous curve of zeroes with three possible types of endpoints : at $y \to \pm\infty$, at a point in S or at a point in S'. We will call a nodal line the restriction of a connected component of $Z \backslash (S \cup S')$.

Third, if (x, y) is a regular zero of g, then by the implicit function theorem there is exactly one nodal line of g that intersects (x, y). Also, g has to change sign when we go across a nodal line.

Finally, if $e \in \mathcal{E}_v$, we say that m nodal lines come into (v, y) on e if there is a δ_0 such that for all $\delta < \delta_0$ and any small enough neighbourhood O of v in Γ, exactly m nodal lines intersect the set $\{O \cap e\} \times (y - \delta, y)$. Similarly, we say that m nodal lines come out of (v, y) on e if there is a δ_0 such that for all $\delta < \delta_0$ and any small enough neighbourhood O of v in Γ, exactly m nodal lines intersect the set $\{O \cap e\} \times (y + \delta, y)$. By the fourth statement in Theorem 3.3 at most one nodal line can come out of (v, y) on each edge in $\mathcal{E}_v$.

3.4. **Possible events.**

By the results of the previous section, since there are no isolated zeroes the only values of y such that $N(g_y)$ may change is when there is a singular zero (x, y), or when a nodal line hits a vertex at (v, y). We will call either of these occurences an event. By Theorem 3.3, here are all the possible behaviours of the zero set of g around singular zeroes or vertices:

(E1) $x \notin V$	
(E2) $v \in V_b$	
(E3) $v \in V_i$	

By the results of the previous section, events $E1$ and $E2$ must decrease $N(g_y)$ at y_0. We now examine what happens when one (or more) nodal lines comes into an inner vertex. Let us fix an inner vertex v and set $g_v(y) := g(v, y)$. This is well-defined since g is continuous at inner vertices. There exist coefficients $a_i(v)$ such that $g_v(y) = \sum_{i=1}^{M} a_i(v) e^{-\lambda_i y}$ (specifically, $a_i(v) := a_i f_{k_i}(v)$). A nodal line (one or more) comes into (v, y) when $g_v(y) = 0$. To count the number of y values for which a nodal line can come into (v, y), we will use the following bound on the number of zeroes of a linear combination of exponentials:

Theorem 3.4. [22, part V, Chapter 1, problem 77] *Let $h(x) = \sum_{i=1}^{n} a_i e^{b_i x}$ with $a_i \neq 0$ and $b_{i+1} > b_i$ for any $i \leq n$. Let $C(h)$ be the number of times that a_{i+1} and a_i have different signs. Then, the number of zeroes of h is less or equal than $C(h)$.*

This implies that for any $v \in V_i$, g_v has at most $M - 1$ zeroes. We now look at what can happen to $N(g_y)$ at each of these zeros when y increases.

Lemma 3.5. *When a nodal line comes into an inner vertex v, the nodal count can increase by at most $deg(v) - 2$ as y increases.*

Proof. Assume that $g(v, y') = 0$. For each $e \in \mathcal{E}_v$ we now define $In(v, y', e)$ as the number of nodal lines that come into (v, y') on e and $Out(v, y', e)$ as the number of nodal lines that come out of (v, y') on e. By the fourth statement in Theorem 3.3, $Out(v, y', e) \leq 1$.

We will treat two cases separately. If g_v changes sign at y', since g changes sign once when going across a nodal line then for all $e \in \mathcal{E}_v$, $In(v, y', e) + Out(v, y', e)$ is odd. This means that for each $e \in \mathcal{E}_v$ there is $m_e \geq 0$ such that either $2m_e$ nodal lines come into (v, y') from e and one comes out, or $2m_e + 1$ nodal lines come into (v, y') and none come out. Therefore, the maximum increase of the nodal count happens if one nodal line comes into (v, y') from a single edge in $\mathcal{E}_v$ and it comes out on every other edge in $\mathcal{E}_v$. This increases the nodal count by $deg(v) - 2$.

Similarly, if g_v does not change sign at y' then for all $e \in \mathcal{E}_v$, $In(v, y', e) + Out(v, y', e)$ is even. Therefore, for each $e \in \mathcal{E}_v$ there is $m_e \geq 0$ such that either $2m_e + 1$ nodal lines come into (v, y') from e and one comes out, or $2m_e$ nodal lines come into (v, y') and none come out. In both of these cases, the nodal count does not increase. This completes the proof of Lemma 3.5.

□

Combining Theorem 3.4 and Lemma 3.5, we get the following characterization of nodal lines coming into inner vertices:

Lemma 3.6. *Event E3 can happen at most $M-1$ times at each inner vertex, and each time it happens the nodal count can increase by at most* $\deg(v) - 2$.

3.5. **Global bounds.**

We will combine the lemmas of the previous subsection in order to prove Theorem 2.1.

We start with $N(f_{k_M})$ zeroes at $y = -\infty$. As y increases to zero, the only event which increases $N(g_y)$ is a nodal line coming into an inner vertex. By Lemma 3.6, for each inner vertex v this can happen at most $M-1$ times and each time it happens it can increase $N(g_y)$ by at most $deg(v) - 2$.

Similarly, when y goes from zero to $+\infty$, the maximum increase in $N(g_y)$ is at most $(M-1)(deg(v) - 2)$ for each inner vertex v. As $y \to +\infty$, g_y has $N(f_{k_1})$ zeroes. This gives us the following two-sided bound:

$$N(f_{k_1}) - (M-1)\sum_{v \in V_i}(\deg(v) - 2) \leq N(F) \leq N(f_{k_M}) + (M-1)\sum_{v \in V_i}(\deg(v) - 2)\,. \tag{3.4}$$

We are ready to finish the proof of Theorem 2.1.

Proof of Theorem 2.1. The nodal count of any eigenfunction f_k of H_W satisfies the bounds (see [8, Theorem 5.2.8]):

$$k - 1 \leq N(f_k) \leq k - 1 + \beta\,. \tag{3.5}$$

Now, recalling that $\beta = |E| - |V| + 1$ and $\sum_{v \in V} deg(v) = 2|E|$, we have the following:

$$\begin{aligned}\sum_{v \in V_i}(deg(v) - 2) &= \sum_{v \in V} deg(v) - |V_b| - 2|V_i|\,,\\ &= |V_b| + (2|E| - 2|V|)\,,\\ &= |V_b| + 2\beta - 2\,. \end{aligned} \tag{3.6}$$

Combining (3.4), (3.5) and (3.6) completes the proof of Theorem 2.1.

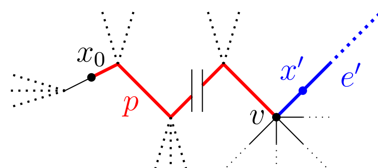
□

Remark 3.7. *It is clear from the proof that the condition that Γ is W-generic can be replaced by the condition that the eigenfunctions in the linear combination (rather than all eigenfunctions in the spectrum) are non-zero at inner vertices. Furthermore, one can see how the proof could be adjusted to allow for Neumann or mixed Dirichlet-Neumann boundary conditions. First, the bounds* (3.5) *also hold. Second, one could reflect the potential and eigenfunctions across boundary vertices and then apply a suitable extension step as in Section 3.3. Finally, by a suitable application of Theorem 3.3 to that extension, one could show that events E2 can only decrease the nodal count, just like in the Dirichlet case.*

4. Proof of Theorem 2.3

Let Γ be a 0-generic tree and s be the highest degree of any vertex of Γ. Recall that we want to construct a linear combination of f_1 and f_2 with at least $s-1$ zeroes.

Proof of Theorem 2.3. By [7, 21, 24], $N(f_1) = 0$ and $N(f_2) = 1$. Let x_0 be the position of the single zero of f_2 and v be any vertex of degree s. Let p be the path connecting x_0 to v without self-intersections. Let $e' \in \mathcal{E}_v$ such that $p \cap e' = v$ and x' be any point in the interior of e' such that $f_2(x') \neq 0$.



Since $f_2(x') \neq 0$, there exists $a \in \mathbb{R}$ such that $(f_1 + af_2)(x') = 0$.

Consider the function $g(x,y) := f_1(x)e^{-\lambda_1 y} + af_2(x)e^{-\lambda_2 y}$. We have that $g(x',0) = 0$ and for y' large enough, $g_{-y'}$ only has one zero on the same edge as x_0. By Theorem 3.3, the zero set of g is connected. This implies that there exists $y_0 \in (-y', 0)$ such that $g(v, y_0) = 0$.

Since Γ is a tree, at $y = y_0$ a single nodal line comes into v and then comes out of every other edge in $\mathcal{E}_v$ (see an argument in the proof of Lemma 3.5). Since $deg(v) = s$, we get that $s-1$ nodal lines come out of (v, y_0). Taking ϵ small enough, this means that $g_{y_0+\epsilon}$ has at least $s-1$ zeroes.

□

5. Saturating examples for the upper bound - proof of Theorem 2.4

Let $G(s,\varepsilon)$ be a star graph with s edges - one edge of length 1 and $s-1$ edges of length ε.

Figure 5.1. $G(s,\varepsilon)$ for $s = 8$

This graph has some interesting properties if ε is taken small enough:

Lemma 5.1. *For any s and M we can take ε small enough such that the following occurs:*

(a) The first M eigenvalues of $G(s,\varepsilon)$ are all simple.

(b) *The first M eigenfunctions are all invariant with respect to permutations of the small edges.*
(c) *For any $n \leq M$, each eigenfunction f_n of $H_0 = -\frac{\partial^2}{\partial x^2}$ has exactly $n-1$ zeroes on the long edge and no zeroes on the small edges or on the inner vertex.*

Proof. By [9, Theorem 4.5], as $\varepsilon \to 0$ the eigenvalues of $G(s,\varepsilon)$ converge to those of the unit interval with Dirichlet boundary conditions. Therefore, for any $M \in \mathbb{N}$ and $\delta > 0$ small, there exists $\varepsilon_0 > 0$ small enough such that for any $0 < \varepsilon < \varepsilon_0$ and $1 \leq n \leq M+1$, $|\lambda_n(G(s,\varepsilon)) - \pi^2 n^2| < \delta$. Hence, the first M eigenvalues are simple for any $\varepsilon < \varepsilon_0$. As a consequence, for any $1 \leq n \leq M$ and any permutation P of the small edges, $f_n \circ P = \pm f_n$ (since $f_n \circ P$ is an eigenfunction with the same eigenvalue as f_n). Fix $\varepsilon < \varepsilon_0$ such that for $1 \leq n \leq M$, $\varepsilon\sqrt{\lambda_n} < \pi/2$. This implies that for $1 \leq n \leq M$, f_n does not have a zero inside any small edge or at the inner vertex. Consequently, for any permutation P of the small edges, $f_n \circ P = f_n$ (as $f_n \circ P = -f_n$ implies a zero at the inner vertex). Finally, by Courant's theorem for nodal domains [16], since λ_n is simple f_n has at most $n-1$ zeroes. These zeros must be on the long edge, by the argument above. □

We use Lemma 5.1 to construct linear combinations of eigenfunctions of $G(s,\varepsilon)$ with a high nodal count:

Proposition 5.2.
For any $M, s > 0$, there exists $\varepsilon_1(M,s)$ small enough such that for any $\varepsilon < \varepsilon_1(M,s)$ and any $L \leq M$, there exist linear combinations of the first L eigenfunctions of $G(s,\varepsilon)$ with exactly $L-1+(L-1)(s-2)$ zeroes on the small edges.

Proof. We choose $\varepsilon_1(M,s)$ such that Lemma 5.1 applies and $\varepsilon < \varepsilon_1(M,s)$. For any $L \leq M$ we can choose $F := \sum_{n \leq L} a_n f_n$ such that F has $L-1$ zeroes on a chosen small edge of $G(s,\varepsilon)$[4]. Since these f_n are symmetric with respect to permutations of the small edges, so is F. Therefore, F has exactly $(L-1)(s-1) = L-1+(L-1)(s-2)$ zeroes on the small edges. □

We note that the graph $G(s,\varepsilon)$ is not 0-generic, since it is possible to find eigenfunctions which are zero at the inner vertex (for instance by choosing $\lambda = \pi^2\varepsilon^{-2}$). We take care of the genericity by constructing a perturbation of $G(s,\varepsilon)$ and linear combinations of eigenfunctions that have the same behaviour as in Proposition 5.2:

Lemma 5.3. *There exists a star graph $G_\delta(s,\varepsilon)$, which has s edges and satisfies the following properties:*

- *$G_\delta(s,\varepsilon)$ is obtained by perturbing the edge lengths of $G(s,\varepsilon)$ by at most δ.*
- *$G_\delta(s,\varepsilon)$ is 0-generic.*
- *For any $L \leq M$, there exists a linear combination F_δ of the first L eigenfunctions on $G_\delta(s,\varepsilon)$ such that $N(F_\delta) = L-1+(L-1)(s-2)$.*

This construction immediately implies Theorem 2.4.

[4]This is always possible since by Lemma 5.1, the first M eigenfunctions are not identically zero on the small edges, and their restriction to small edges are sine functions with different frequencies.

Proof. First, we choose $\varepsilon > 0$ such that Lemma 5.1 applies to $G(s, \varepsilon)$. For $1 \le L \le M$, take a linear combination $F = \sum_{n \le L} a_n f_n$ with $L - 1$ zeroes on each small edge of $G(s, \varepsilon)$, as is done in Proposition 5.2.

Now, we use the fact that for a given graph, there exists an arbitrarily small perturbation of the edge lengths such that the spectrum of the Laplacian is simple and no eigenfunction vanishes on a vertex [10, Theorem 3.6]. Hence, for any $\delta > 0$, it is possible to find a graph $G_\delta(s, \varepsilon)$ with one edge of length 1 and s edges e_i of length $l_{e_i} \in (\varepsilon, \varepsilon + \delta)$ such that $G_\delta(s, \varepsilon)$ is 0-generic. Furthermore, as $\delta \to 0$, the eigenvalues of $G_\delta(s, \varepsilon)$ converge to those of $G(s, \varepsilon)$ (see for instance [3, Appendix A], [9, Theorem 3.6] or [23, Theorem 4.15]).

We define the map $\phi_\delta : G_\delta(s, \varepsilon) \to G(s, \varepsilon)$ that fixes the long edge and sends $x \in e_i$ to $(\varepsilon / l_{e_i})x$. Let $f_{\delta,n}$ be the n-th eigenfunction on $G_\delta(s, \varepsilon)$. We know from [8, Theorem 3.1.4] that eigenfunctions depend analytically on perturbations of edge lengths. Therefore, since the first M eigenvalues of $G(s, \varepsilon)$ are simple, as δ goes to zero, $\epsilon_n(\delta) := \sup_{x \in G_\delta(s,\varepsilon)} |f_n(\phi_\delta(x)) - f_{\delta,n}(x)|$ will go to zero for any $1 \le n \le M$.

Let $F_\delta : G_\delta(s, \varepsilon) \to \mathbb{R}$, $F_\delta := \sum_{n \le L} a_n f_{\delta,n}$ with the same coefficients a_n as F. Since F is continuous and its zeroes are discrete, if we take δ small enough, F_δ will have at least as many sign changes as F on each small edge. By the mean value theorem, $N(F_\delta) \ge N(F)$. Also, since $G_\delta(s, \varepsilon)$ is 0-generic, $N(F_\delta) \le L - 1 + (L - 1)(s - 2)$ by Theorem 2.1. Hence, $N(F_\delta) = L - 1 + (L - 1)(s - 2)$, which completes the proof of Lemma 5.3 and of Theorem 2.4.

□

6. Examples of non-trivial lower bounds - proof of Theorem 2.6

Now, let $I(m, \varepsilon)$ be the following graph: start with two edges of length 1/2 and connect them with m parallel edges of length ε. This graph has 2 boundary vertices and 2 inner vertices of degree $m + 1$.

We will define the involution $\psi : I(m, \varepsilon) \to I(m, \varepsilon)$ as the reflection across the dotted line in figure 6.1.

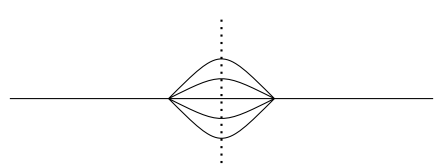


FIGURE 6.1. $I(m, \varepsilon)$ for $m = 5$

Lemma 6.1. *For any m, there exists ε small enough such that $I(m, \varepsilon)$ has the following properties:*

(a) The first three eigenvalues are simple.
(b) $N(f_2) = m$.
(c) $N(f_3) = 2$.

Proof. As in the proof of Lemma 5.1, by [3], [9] and [23], when ε goes to zero the eigenvalues of $I(m,\varepsilon)$ converge to the eigenvalues of the unit interval with Dirichlet boundary conditions. Since all the eigenvalues on the interval are simple, by taking ε small enough the first three eigenvalues of $I(m,\varepsilon)$ are simple. The simplicity of the eigenvalues implies that for $i=1,2,3$, $f_i\circ\psi=\pm f_i$. By Courant's theorem [16], f_2 has two nodal domains. Therefore, $f_2\circ\psi=-f_2$ and the only zeroes of f_2 are at the center of the small edges. Hence, $N(f_2)=m$. Finally, since as $\varepsilon\to 0$, $\lambda_3\to 9\pi^2$, for ε small enough f_3 has a zero on each long edge. Again, by Courant's theorem, f_3 has at most three nodal domains, which implies that it does not have any other zero. As a consequence, $N(f_3)=2$. □

As the zeroes of f_2 and f_3 are away from the vertices, there exists $a(m)$ small enough such that $N\left(a(m)f_2+f_3\right)=N(f_3)$. Since the graph $I(m,\varepsilon)$ is not 0-generic, we will slightly perturb it without changing the nodal count to complete the proof of Theorem 2.6.

Proof of Theorem 2.6. As in the proof of Lemma 5.3, we can construct a δ-small perturbation $I_\delta(m,\varepsilon)$ of $I(m,\varepsilon)$ which is 0-generic. Let $f_{\delta,n}$ be the n-th eigenfunction on $I_\delta(m,\varepsilon)$. By a similar argument to the one in the proof of Lemma 5.3, we can choose δ small enough such that $N(f_{\delta,2})=N(f_2)=m$, $N(f_{\delta,3})=N(f_3)=2$ and $N(a(m)f_{\delta,2}+f_{\delta,3})=N(a(m)f_2+f_3)=2$, which proves Theorem 2.6. □

Appendix A. Proof of Lemma 3.1

We wish to show that the function $g(x,y)=\sum_{i=m}^{M}a_ie^{-\lambda_{k_i}y}f_{k_i}(x)$ does not have isolated zeroes.

The maximum principle for parabolic equations (see for instance [13, Section 7, Theorem 8]) implies that g cannot have an isolated zero inside an edge.

The proof of the maximum principle for solutions of parabolic equations in the plane hinges on three arguments.

- First, if we assume that there is an isolated minimum (x',y') of g in the interior of a domain of the form $(a,b)\times(c,d)$, then there is a neighbourhood Y of y' with the following property: if $y\in Y$, then the minimum of $g(\cdot,y)$ is attained in (a,b). This ensures that at any global minima (x'',y'') of $g(\cdot,y)$, $\partial_x g(x'',y'')=0$ and $\partial_{xx}g(x'',y'')\geq 0$.
- Second, if a local minimum of $g(\cdot,y)$ is attained at $(x'',y'')\in(a,b)\times Y$, then $\partial_y g(x,y'')>0$ for any x in a neighbourhood of x''.
- Finally, by a covering argument, this implies that the minimum value of $g(\cdot,y)$ on (a,b) has to increase as $y\in Y$ increases, which contradicts the fact that the minimum of $g(\cdot,y)$ in (a,b) goes to 0 as y increases to y'.

We will follow similar steps for the proof in the case of graphs, accounting for the behaviour of g at inner vertices.

Proof of Lemma 3.1. Let v be any inner vertex and assume that (v,y_0) is an isolated zero of g. Without loss of generality, assume that g is positive in a punctured neighbourhood of (v,y_0).

Let $m:=\max_\Gamma W$ and $G(x,y):=g(x,y)e^{(m+1)y}$. The function G satisfies

$$\partial_y G = \partial_{xx} G + (-W(x) + m + 1)G\,. \tag{A.1}$$

Since (v, y_0) is an isolated zero of g, it is an isolated zero of G as well.

Now, choose $\delta_1 > 0$ such that $G(x, y) > 0$ if $d(x, v) \le 2\delta_1$ and $|y - y_0| \le \delta_1$.

Let $A := \min_{\substack{\frac{\delta_1}{2} \le d(x,v) \le \frac{3\delta_1}{2} \\ |y-y_0| \le \delta_1}} G(x, y)$.

By the continuity of G, we can choose $\delta_2 < \delta_1/2$ such that $\max_{\substack{d(x,v) \le \delta_2 \\ |y-y_0| \le \delta_2}} G(x, y) \le A/2$.

Therefore, for any $y \in (y_0 - \delta_2, y_0 + \delta_2)$, the minimum of G_y on the set $\Gamma' := \{x | d(x, v) < \delta_1\}$ is attained inside the smaller set $\Gamma'' := \{x | d(x, v) \le \delta_1/2\}$. This completes the first part of the proof.

Now, let $H(y)$ be the minimum of G_y on Γ'. By assumption, $H(y_0 - \delta_2) > 0$ and $H(y_0) = 0$. Our goal is to show that $H(y)$ is increasing in $(y_0 - \delta_2, y_0)$.

First, we will show the following simple fact:

Lemma A.1. *If $y' \in (y_0 - \delta_2, y_0)$ and if $x'' \in \Gamma'$ is any local minima of $G_{y'}$, then $\partial_y G(x'', y') \ge H(y')$.*

Proof. First, assume that x'' is not a vertex. Then, since $x'' \in \Gamma'$ is a local minima of $G_{y'}$, $\partial_x G_{y'}(x'') = 0$ and $\partial_{xx} G_{y'}(x'') \ge 0$. By equation (A.1), $\partial_y G(x'', y') = \partial_{xx} G(x'', y') + (-W(x') + m + 1)G(x'', y') \ge H(y')$ since $-W(x) + m + 1 \ge 1$ and $G(x'', y') \ge \min_{x \in \Gamma'} G(x, y') = H(y')$.

Now, assume that x'' is a vertex. In order for (x'', y') to be a local minima of $G_{y'}$, then for each $e \in \mathcal{E}_{x''}$, $\partial_e G(x'', y') = 0$.

Also, since for each eigenfunction f_n, f_n'' is continuous, then $\partial_{xx} G_y$ is also continuous. This implies that $\lim_{x' \to x''} \partial_{xx} G(x', y') \ge 0$. Finally, $\partial_y G$ is continuous on $\Gamma \times \mathbb{R}$.

This gives us the following:

$$\begin{aligned}
\lim_{x' \to x''} \partial_y G(x', y') &= \lim_{x' \to x''} \partial_{xx} G(x', y') + (-W(x') + m + 1)G(x', y')\,, \\
&\ge 0 + (-W(x'') + m + 1)G(x, y')\,, \\
&\ge H(y')\,.
\end{aligned}$$

Since, $\partial_y G$ is continous, this implies that $\partial_y G(x'', y') \ge H(y')$, which completes the proof of Lemma A.1.

□

Now, at each (x, y') such that $G(x, y') = H(y')$, $\partial_y G(x, y') > H(y')$. Since $\partial_y G$ is continuous, for each such x there exists an open neighborhood of x where $\partial_y G > H(y')/2$. Let $U(y')$ be the union of all these open neighbourhoods.

Since $\overline{\Gamma'}$ is closed and bounded, $V(y') := \overline{\Gamma'} \backslash U(y')$ is compact. Therefore, by the definition of A and δ_2 and the continuity of G, there exists $\epsilon > 0$ such that $G(x, y') \ge H(y') + \epsilon$ for any $x \in V(y')$. Furthermore, there exists $\delta_3 > 0$ such that $G(x, y) \ge H(y') + \epsilon/2$ if $x \in V(y')$ and $0 \le y - y' \le \delta_3$.

Also, since $\partial_y G$ is continuous, there is a $\delta_4 > 0$ such that $\partial_y G(x, y) > H(y')/4$ if $x \in U(y')$ and $0 \le y - y' \le \delta_4$. Hence, for such (x, y),

$$G(x, y) \ge G(x, y') + (y - y')H(y')/4 \ge H(y') + (y - y')H(y')/4 .$$

By taking $\delta_5 := \min(\delta_3, \delta_4)$, if $y \in (y', y' + \delta_5)$,

$$H(y) \ge \min(H(y') + \epsilon/2, H(y') + (y - y')H(y')/4) > H(y') .$$

This implies that if $y' \in (y_0 - \delta_2, y_0)$ and $H(y') > 0$ then H is locally increasing. To complete the proof of Lemma 3.1, we note that $H(y) > 0$ if $y \in (y_0 - \delta_2, y_0)$ and $H(y_0) = 0$, a contradiction.

$\square$

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