SPECTRAL FLOW AND ROBIN DOMAINS ON METRIC GRAPHS

RAM BAND, MARINA PROKHOROVA, AND GILAD SOFER

Abstract. We introduce the Robin count, a generalization of the nodal and Neumann counts of eigenfunctions. This counts the number of points where f'/f takes a prescribed value, known as the Robin parameter, delta coupling, or cotangent of Prüfer angle. Correspondingly, we introduce the Robin map (a generalization of the Dirichlet-to-Neumann map) and prove an index theorem which relates the Morse index of the Robin map with the Robin count deficiency. In this context, we present some basic properties of the spectral flow for families of Schrödinger operators on metric graphs. It is shown that the spectral flow may be used to express topological information of the graph and its operator – the graph's first Betti number, the number of interaction vertices and their positions with respect to the graph cycles.

1. Introduction and main results

By Courant's nodal line theorem [29],

$$n - \nu(f_n) \geq 0, \tag{1.1}$$

where f_n is the n-th eigenfunction of the Laplacian (or Schrödinger operator) on a manifold and $\nu(f_n)$ is the number of nodal domains of f_n. The expression on the left hand side of (1.1) is frequently called the *nodal deficiency*. In recent years, the nodal deficiency has gained several interesting interpretations via index theorems, expressing it as a stability index (Morse index) of various operators. These theorems can be categorized into three classes, based on the relevant operator: (1) the two-sided Dirichlet-to-Neumann map [18, 30], (2) an energy functional on the space of partitions [8, 21, 23, 25, 39, 41, 42, 43], and (3) an eigenvalue of the magnetic Laplacian [24, 28, 14]. Additional insights may then be obtained by drawing connections between these three types of index theorems [16, 17, 20, 40]. These index theorems may also be classified according to the space on which the Laplacian acts: a manifold, a metric graph, or a discrete graph. Experience shows that results obtained for one of these structures often lead to progress in studying the others. Nevertheless, there are currently two significant gaps within the theory: the lack of index theorems for the nodal deficiency via the magnetic Laplacians on manifolds, and the absence of Dirichlet-to-Neumann index theorems for the nodal deficiency on graphs. The current work addresses this second gap, by providing such an index theorem (Theorem 1.5), and doing so in a more general form than the existing results for manifolds. Specifically, we show that on metric graphs, one may count not only eigenfunction zeros (or nodal domains), but rather any arbitrary value of the Prüfer angle. This generalization is achieved by considering a Robin map (Section 1.5), which itself serves as a generalization of the Dirichlet-to Neumann map. Our notion of Robin domains is also related to the recent progress made in the study of Robin partitions [43].

The proof of our index theorem is based on analyzing the spectral flow of an appropriately defined operator family. The spectral flow has been used for proving index theorems in a variety of different settings over past years, and was first introduced for quantum graphs in [47]. This leads to the second aspect of the current work, which involves proving several basic and useful properties for the spectral flow on metric graphs. The connection between the spectral flow and the index theorem is bi-directional. First, the spectral flow approach provides a natural setting for the proof of our main index theorem (Theorem 1.5). Second, this index theorem serves in turn to prove a basic property of the spectral flow (Theorem 1.8). In addition, we show that the spectral flow provides topological information of the graph, and may be considered under the perspective of inverse spectral geometric problems (Propositions 1.9 and 1.10).

1.1. **Quantum graph preliminaries.** A *metric graph* is a pair $\Gamma = \left(G, \vec{\ell}\right)$, where $G = (\mathcal{V}, \mathcal{E})$ is a combinatorial graph and $\vec{\ell} \in \mathbb{R}_+^{|\mathcal{E}|}$ is a vector of positive lengths associated with the edges in $\mathcal{E}$. Each $e \in \mathcal{E}$ is identified with the interval $[0, \ell_e]$, so that Γ inherits a natural structure of a metric space. We denote the set of edges connected to a vertex $v \in \mathcal{V}$ by $\mathcal{E}_v$. The degree of a vertex, denoted $\deg(v)$, is the number of edges incident to v, i.e., $\deg(v) = |\mathcal{E}_v|$. In the case of loop edges (edges that connect v to itself), the associated loop edges should be counted twice in $\deg(v)$.

A *Quantum Graph* is a metric graph Γ equipped with a self-adjoint differential operator H acting on the Sobolev space $H^2(\Gamma) := \oplus_{e\in\mathcal{E}} H^2(0, \ell_e)$. It is common to take H as the Schrödinger operator $H = -\frac{d^2}{dx^2} + q(x)$, where $q \in L^\infty(\Gamma)$ is a real-valued potential, along with vertex conditions which render H self-adjoint. The most common vertex conditions are known as the Neumann-Kirchhoff (or standard) conditions:

$$f \text{ is continuous at } v \text{ - } f|_e(v) = f|_{e'}(v), \forall e, e' \in \mathcal{E}_v, \tag{1.2}$$

$$\text{Current conservation - } \sum_{e\in\mathcal{E}_v} f'|_e(v) = 0, \tag{1.3}$$

where the derivatives are taken in the outwards pointing direction from the vertex.

Throughout this work, we take Γ to be a compact, connected metric graph, equipped with the Laplacian $H = -\frac{d^2}{dx^2}$. We generally impose Neumann-Kirchhoff conditions at the vertices, with a few important exceptions discussed later. The resulting operator H is self-adjoint and bounded from below with compact resolvent, and its spectrum is thus an infinite, discrete subset of $\mathbb{R}$, consisting of eigenvalues of finite multiplicity, denoted $\lambda_1 < \lambda_2 \leq ... \nearrow \infty$. The corresponding complete orthonormal set of eigenfunctions is denoted by $\{f_n\}_{n=1}^{\infty}$. Some introductory texts for quantum graphs are [12, 15, 19, 38, 46].

1.2. **Vertex trace maps.** For the remainder of this work, we fix an orientation on the graph (i.e., each edge is directed). This orientation is arbitrary, up to the constraint that each vertex of degree two has one edge directed towards it, and the other edge directed away from it (see Figure 1.1).

Given a function $f \in H^2(\Gamma)$ and a vertex $v \in \mathcal{V}$ of degree one, we denote by $f(v)$ and $f'(v)$ the value and derivative of f at v. The derivative f' is taken according to direction

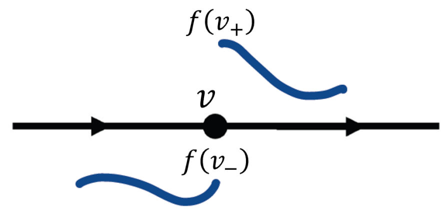

FIGURE 1.1. The orientation on the edges determines $f(v_\pm)$. $f'(v_-)$ is directed into the vertex, while $f'(v_+)$ is directed into the edge.

of the edge connected to v. These values and derivatives are commonly referred to as Dirichlet and Neumann trace maps; we generalize those in the following.

Let $\alpha \in S^1 = [0, 2\pi] / \{0, 2\pi\}$ be an angle parameter, and let v be a vertex of degree one. We define the *mixed (or generalized) trace maps* τ_α and τ'_α at v by

$$\left(\begin{array}{c} \tau_\alpha f \\ \tau'_\alpha f \end{array}\right)(v) := \left(\begin{array}{cc} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{array}\right)\left(\begin{array}{c} f(v) \\ f'(v) \end{array}\right). \tag{1.4}$$

Namely, the vector $(\tau_\alpha f, \tau'_\alpha f)(v)$ is merely a rotation of the 'usual' Dirichlet and Neumann traces, $(f(v), f'(v))$. For example if $\alpha = 0$, then (τ_0, τ'_0) are respectively the standard Dirichlet and Neumann traces. For $\alpha = \frac{\pi}{2}$, the mixed traces maps $\left(\tau_{\pi/2}, \tau'_{\pi/2}\right)$ are respectively the (negative) Neumann trace and the Dirichlet trace.

The definition of the mixed trace maps extends naturally to vertices of degree two. Given a vertex v of degree two, we may refer to the values and derivatives that a function f attains at both its sides and denote those by $f(v_-), f(v_+)$ and $f'(v_-), f'(v_+)$, see Figure 1.1. Hence, τ_α may be evaluated at both v_- and v_+ and thus extends to $\tau_\alpha \colon H^2(\Gamma) \to \mathbb{C}^2$ (and similarly for τ'_α). Now, if some vertex subset $\mathcal{B} \subset \mathcal{V}$ is such that every $v \in \mathcal{B}$ is of degree two, then we extend the mixed trace maps so that

$$\tau_\alpha, \tau'_\alpha \colon H^2(\Gamma) \to \mathbb{C}^{2|\mathcal{B}|}, \tag{1.5}$$

noting that we take the same α parameter at all vertices of $\mathcal{B}$.

1.3. **The δ_α vertex conditions and the Schrödinger operator $H_\alpha(t)$.** Next, we introduce a generalization of the delta (or Robin) vertex condition, which is based on the mixed trace maps. Fix $\alpha \in [0, \pi)$, and let $t \in \overline{\mathbb{R}}$ (where $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\} \cong S^1$ is the one point compactification of $\mathbb{R}$). Let $v \in \mathcal{V}$ be a degree two vertex. We define the $\delta_\alpha(t)$ *vertex condition* at v by

$$\tau_\alpha f(v_+) = \tau_\alpha f(v_-), \tag{1.6}$$

$$\tau'_\alpha f(v_+) - \tau'_\alpha f(v_-) = t \cdot \tau_\alpha f(v_-), \tag{1.7}$$

where in the case $t = \infty$ both conditions above are interpreted as

$$\tau_\alpha f(v_+) = \tau_\alpha f(v_-) = 0. \tag{1.8}$$

Heuristically, the case $t = \infty$ can be viewed as disconnecting the graph Γ at v and imposing the Robin condition $\tau_\alpha f(v_\pm) = 0$ at the obtained two cut points, v_- and

v_+. We note that the $\delta_\alpha(t)$ condition depends on the choice of edge orientation, simply because the generalized trace maps $\tau_\alpha, \tau'_\alpha$ depend on the edge orientation.

As an example, if $\alpha = 0$, then the $\delta_0(t)$ condition is the well-known δ type condition, with a coupling coefficient t (it is sometimes also called Robin condition). For $\alpha = {}^{\pi}\!/_{2}$, the $\delta_{\pi/2}$ condition is known as the non-symmetric δ' ('delta-prime') type condition[1], as presented in [1, 7, 33].

Fix a subset $\mathcal{B} \subset \mathcal{V}$ of vertices of degrees two and let $\alpha \in [0, \pi)$, $t \in \overline{\mathbb{R}}$. Define a Schrödinger operator $H_\alpha^{\mathcal{B}}(t)$ by imposing the $\delta_\alpha(t)$ condition at every $v \in \mathcal{B}$, and the Neumann-Kirchhoff condition at all other vertices. We sometimes omit the set $\mathcal{B}$ and write $H_\alpha(t)$ when this vertex set is irrelevant or clear from context. It is important to observe that $H_\alpha(0)$ is the Neumann-Kirchhoff Laplacian H, regardless of the α value. In the following, we will consider one-parameter families of these Schrödinger operators by taking $(H_\alpha(t))_{t\in[a,b]}$ or even $(H_\alpha(t))_{t\in\overline{\mathbb{R}}}$. We shall consider the family $(H_\alpha(t))_{t\in\overline{\mathbb{R}}}$ as a continuous loop of operators, in the sense explained in Section 4.

1.4. **Robin points and Robin domains.** We provide here a generalization of the well-known notions of nodal points and Neumann domains.

Definition 1.1. Let Γ be a quantum graph equipped with the Neumann-Kirchhoff Laplacian H. Let $\alpha \in [0, \pi)$, $f \in \operatorname{dom}(H)$ and x be some point at an edge $e \in \mathcal{E}$ such that

$$\sin(\alpha) f'(x) = \cos(\alpha) f(x), \tag{1.9}$$

where the derivative f' is taken with respect to the given orientation on e. We say that x is an α-Robin point of f (or simply a *Robin point*). The set of all these points is denoted by $\mathcal{P}_\alpha(f)$ or just by $\mathcal{P}_\alpha$. The value α is also known as the *Prüfer angle* of $f(x)$.

Throughout the paper, for convenience, we tend to introduce additional degree-two vertices to Γ at all α-Robin points of an eigenfunction for a given α. If Neumann-Kirchhoff conditions are imposed at these new vertices, then this procedure does not affect the spectrum or the eigenfunctions of the operator. Introducing such a new vertex, v, allows us to use vertex trace maps and express condition (1.9) as $\tau_\alpha f(v_-) = \tau_\alpha f(v_+) = 0$, where we recall that by the Neumann-Kirchhoff condition, $f(v_-) = f(v_+)$ and $f'(v_-) = f'(v_+)$.

A Robin point with $\alpha = 0$ is a nodal point and a Robin point with $\alpha = {}^{\pi}\!/_{2}$ is called a Neumann point (see [5, 4]).Note that for $\alpha \notin \{0, {}^{\pi}\!/_{2}\}$ the position and number of the α-Robin points depend on the chosen edge orientation.

Definition 1.2. Let Γ be a quantum graph, $\alpha \in [0, \pi)$ and $f \in \operatorname{dom}(H)$. The connected components (i.e., sub-graphs) of $\Gamma \backslash \mathcal{P}_\alpha(f)$ are called Robin domains, and their number is denoted by $\nu_\alpha(f)$. These connected components are called nodal domains if $\alpha = 0$, and they are called Neumann domains if $\alpha = {}^{\pi}\!/_{2}$.

[1] Noting that since $\tau_{\pi/2}$ is the negative Neumann trace, we get a delta-prime with the reversed sign parameter t.

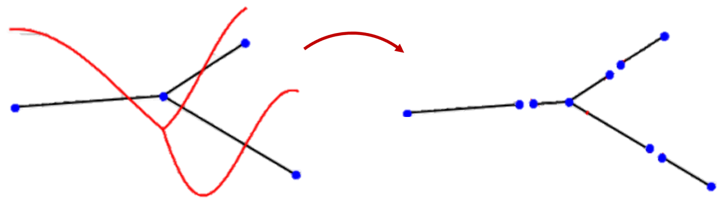

FIGURE 1.2. The eigenfunction f_4 of the Neumann-Kirchhoff Laplacian on a star graph, partitioning it into $\nu_0(f_4) = 4$ nodal domains.

1.5. **The Robin map.** In this subsection we define the *Robin map*, which is both an adaptation to graphs and a generalization of the two-sided Dirichlet-to-Neumann map presented in [16, 18].

Let Γ be a metric graph and let $\mathcal{B} \subset \mathcal{V}$ be a set of degree two vertices, which we will refer to as boundary vertices. The Robin map $\Lambda = \Lambda_\alpha^{\mathcal{B}}(\lambda)$ is an operator on $\mathbb{C}^{\mathcal{B}}$ depending on the parameters $\alpha \in [0, \pi)$ and $\lambda \in \mathbb{R}$. For $v \in \mathcal{B}$ and a vector $w \in \mathbb{C}^{\mathcal{B}}$, denote by w_v its v-coordinate. Consider the following Robin-type boundary value problem on Γ:

$$\begin{aligned} -\frac{d^2 f}{dx^2} &= \lambda f \quad \text{on } \Gamma, \\ \tau_\alpha f(v_+) = \tau_\alpha f(v_-) &= w_v \quad \text{for all } v \in \mathcal{B}, \\ f \text{ satisfies Neumann-Kirchhoff conditions} &\quad \text{for all } v \in \mathcal{V}\backslash\mathcal{B}. \end{aligned} \tag{1.10}$$

Suppose that this boundary value problem does not have nontrivial solutions for $w = 0$ (equivalently, that $\lambda \notin \mathrm{Spec}\left(H_\alpha^{\mathcal{B}}(\infty)\right)$). Then it has a unique solution f^w for every $w \in \mathbb{C}^{\mathcal{B}}$ (see e.g., [19, sec. 3.5]), and we define the Robin map $\Lambda : \mathbb{C}^{\mathcal{B}} \to \mathbb{C}^{\mathcal{B}}$ by the formula

$$(\Lambda w)_v = \tau'_\alpha f^w(v_-) - \tau'_\alpha f^w(v_+) \text{ for } v \in \mathcal{B}. \tag{1.11}$$

Remark 1.3. There is an alternative (and equivalent) description of this Robin map using similar 'local' Robin maps. Specifically, we may consider the connected components $\{\Gamma_i\}$ of $\Gamma\backslash\mathcal{B}$ and assign a local Robin map to each such Γ_i. The global Robin map $\Lambda_\alpha^{\mathcal{B}}(\lambda)$ described above is merely a signed sum of such local Robin maps. A similar approach of local maps (for the specific case of Dirichlet-to-Neumann maps on manifolds) appears in [16, 18, 30]. For instance, for $\alpha = 0$ the local Robin map on each sub-graph Γ_i is exactly the Dirichlet-to-Neumann map of a quantum graph, whereas for $\alpha = \pi/2$ it corresponds to its inverse – the Neumann-to-Dirichlet map (up to a minus sign).

We note that there are other generalizations of the Dirichlet-to-Neumann map on manifolds to Robin boundary value problems, see e.g. [37, 31] for an introduction of such Robin-to-Robin maps.

1.6. **An index theorem for the number of Robin domains.** We denote the eigenvalues of the Neumann-Kirchhoff Laplacian H by $\{\lambda_n\}_{n=1}^{\infty}$ with eigenfunctions $\{f_n\}_{n=1}^{\infty}$. Throughout the paper, we will assume the following:

Assumption 1.4. *Let λ be an eigenvalue of H. We assume that λ is a simple eigenvalue and that the corresponding eigenfunction f does not vanish at a vertex of degree larger than two.*

This assumption is common within the study of nodal domains and it is a generic one [2, 34, 22, 3].

Theorem 1.5. *Let $\alpha \in [0,\pi)$ and let (λ_n, f_n) be an eigenpair of H which satisfies Assumption 1.4. Further assume that $\lambda_n > \left(\frac{\pi}{\ell_{min}}\right)^2$, where ℓ_{min} denotes the shortest edge length in the graph. Let $\mathcal{B} = \mathcal{P}_\alpha(f_n)$ the set of α-Robin points of f_n. Then for $\epsilon > 0$ small enough,*

$$n - \nu_\alpha(f_n) = |\mathcal{P}_0(f_n)| - \mathrm{Pos}\left(\Lambda_\alpha^{\mathcal{B}}(\lambda_n+\epsilon)\right), \tag{1.12}$$

where Pos *denotes the number of positive eigenvalues of a matrix.*
In addition,

$$n - \nu_0(f_n) = \mathrm{Mor}\left(\Lambda_\alpha^{\mathcal{B}}(\lambda_n+\epsilon)\right), \tag{1.13}$$

where Mor *is the Morse index, i.e., the number of negative eigenvalues.*

The theorem serves as both a metric graph analogue of the index formula in [18, 30], and as a generalization of it, by counting Robin domains via the Robin map.

Remark 1.6. We emphasize that the statement (1.13) is not a trivial implication of (1.12). It is obtained if one substitutes $\alpha = 0$ *only* in the left hand side of (1.12), which is justified in the proof of the theorem. When substituting $\alpha = 0$ also in the right hand side of (1.13), we obtain the metric graph analogue of the index formula from [18, 30].

Remark 1.7. We may deduce from (1.13) that the Morse index of the Robin map, $\mathrm{Mor}\left(\Lambda_\alpha^{\mathcal{B}}(\lambda_n+\epsilon)\right)$, remains unchanged as one simultaneously changes α and (consequently) the positions of the α-Robin points, $\mathcal{B} = \mathcal{P}_\alpha(f_n)$. Nevertheless, such a change of α might also change the total number of the α-Robin points which is also the size of the matrix $\Lambda_\alpha^{\mathcal{B}}(\lambda_n+\epsilon)$. Hence, such a change may only affect the positive index. This agrees with (1.12): a change of the number of the α-Robin points results with the change of the Robin deficiency which is indeed determined by the positive index according to (1.12). More about such variation of the α value appears in [54, sec. 8.3].

1.7. **The spectral flow.** We finish the first section by presenting the notion of the spectral flow, first introduced by Atiyah, Patodi, and Singer in [6]. It has a two-fold purpose in the current work: as a natural setting for the proof of Theorem 1.5, and as a framework for obtaining further spectral geometric results.

The spectrum of $H_\alpha(t)$ consists of analytic eigenvalue branches in $t \in \overline{\mathbb{R}}$ (see Lemma 2.1), which we call *spectral curves* (see Figure 1.3). This allows to define a spectral flow as follows. Let $I \subset \overline{\mathbb{R}}$ be some closed interval, i.e., $I = [a,b]$ or $I = \overline{\mathbb{R}}$. Colloquially, the

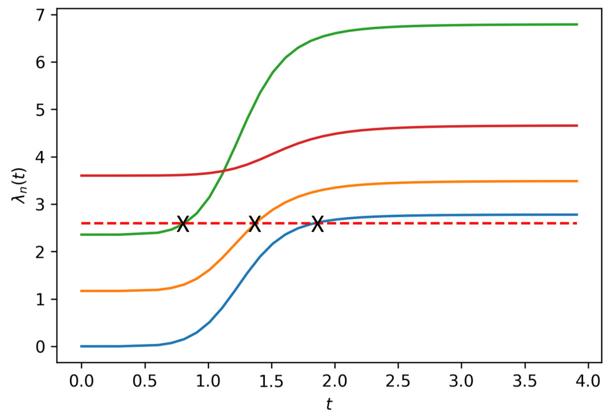

FIGURE 1.3. Illustration of the spectral curves of some operator family $(H(t))_{t\in I}$ and the spectral flow for a given horizontal cross-section, λ = constant. There are three (positively directed) intersections of the spectral curves with the dashed line, so that $\mathrm{Sf}_\lambda\,(H(t))_{t\in I} = 3$.

spectral flow of a continuous path of operators $(H(t))_{t\in I}$ is the total number of oriented intersections of the spectral curves with a given horizontal cross-section, $\lambda = \text{const}$.

For the precise definition we fix $\lambda \in \mathbb{R}$, and choose an arbitrary partition $a = t_0 < t_1 < \ldots < t_N = b$, and positive numbers $(\varepsilon_j)_{j=1}^N$ such that

$$\forall\ 1 \le j \le N, \quad \forall t \in [t_{j-1}, t_j]\,, \quad \lambda \pm \varepsilon_j \notin \mathrm{Spec}\,(H(t))\,, \tag{1.14}$$

where $\mathrm{Spec}\,(H(t))$ denoted the spectrum of the operator $H(t)$. If $I = \overline{\mathbb{R}}$ then we may choose arbitrary values for $a = b$ and take the partition above to (cyclically) cover $\overline{\mathbb{R}}$. The spectral flow of $(H(t))_{t\in I}$ through λ is defined by

$$\mathrm{Sf}_\lambda\,(H(t))_{t\in I} := \sum_{j=1}^{N} \left(\mathrm{rank} E_{[\lambda,\lambda+\varepsilon_j]}\,(H(t_j)) - \mathrm{rank} E_{[\lambda,\lambda+\varepsilon_j]}\,(H(t_{j-1}))\right), \tag{1.15}$$

where $E_{[\lambda,\lambda+\varepsilon_j]}(H)$ is the spectral projection of the operator H on $[\lambda, \lambda+\varepsilon_j]$.

One can show that the spectral flow does not depend on the choice of partition, see e.g. [27]. In the case where the family $(H(t))_{t\in I}$ is a loop (i.e., if $I = \overline{\mathbb{R}}$), the spectral flow is independent of the choice of λ. In this case, we tend to omit the subscript λ and simply write $\mathrm{Sf}\,(H(t))_{t\in I}$. For more details, see [51] for norm continuous families of bounded operators and [26] for graph continuous families of unbounded operators.

1.8. **Main results for the spectral flow.** The next results show that the spectral flow contains some topological information about the graph.

Theorem 1.8. *Let Γ be a quantum graph and let $\mathcal{B} \subset \mathcal{V}$ be a set of degree two vertices. Then for all $\alpha \in [0, \pi)$ and for all $\lambda \in \mathbb{R}$*

$$\mathrm{Sf}_\lambda\left(H_\alpha^{\mathcal{B}}(t)\right)_{t\in\overline{\mathbb{R}}} = |\mathcal{B}|. \tag{1.16}$$

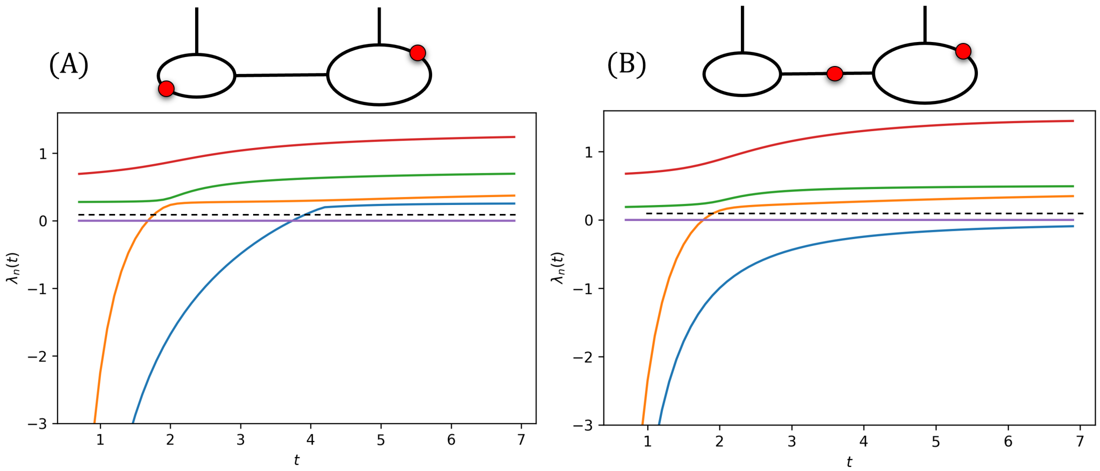

FIGURE 1.4. Demonstration of Proposition 1.9 for a two cycle graph ($\beta_\Gamma = 2$) and two different choices of the set $\mathcal{B}$. In (A), the set $\mathcal{B}$ is chosen such that $\beta_{\Gamma_{\rm cut}} = 0$, and so the spectral flow through the dashed line $\lambda = \epsilon$ is $\beta_\Gamma - \beta_{\Gamma_{\rm cut}} = 2$. In (B), the set $\mathcal{B}$ is chosen such that $\beta_{\Gamma_{\rm cut}} = 1$, and so the spectral flow through the dashed line $\lambda = \epsilon$ is $\beta_\Gamma - \beta_{\Gamma_{\rm cut}} = 1$.

Proposition 1.9. *Let Γ be a connected quantum graph and let $\mathcal{B} \subset \mathcal{V}$ be a set of degree two vertices. For the operator family $\left(H^{\mathcal{B}}_{\pi/2}(t)\right)_{t\in[0,\infty]}$ and $\epsilon > 0$ small enough, we have*

$$\mathrm{Sf}_\epsilon \left(H^{\mathcal{B}}_{\pi/2}(t)\right)_{t\in[0,\infty]} = \beta_\Gamma - \beta_{\Gamma_{cut}}, \tag{1.17}$$

where Γ_{cut} is the metric graph obtained by cutting Γ at the vertices in $\mathcal{B}$, and β_Γ, $\beta_{\Gamma_{cut}}$ are the first Betti numbers of the graphs Γ, Γ_{cut}.

Proposition 1.9 is interesting as it provide the following basic insight: when probing the graph by placing (non-symmetric) delta-prime condition at certain points, the spectral flow tells the number of graph cycles which are 'broken' by these points.

The next proposition provides another connection between the spectral flow and the number of cycles.For our operator family $(H_\alpha(t))_{t\in\overline{\mathbb{R}}}$, all the spectral curves are monotonically increasing (see Lemma 2.1) and hence the spectral flow for the loop $\overline{\mathbb{R}}$ can be decomposed into two summands – one consisting of the intersections of the spectral curves with λ at $t = \infty$, and the other consisting of the intersections along $(-\infty, \infty)$. The second summand is related to the graph's first Betti number, as is stated next.

Proposition 1.10. *Let Γ be a quantum graph and let (λ, f) be an eigenpair of the Neumann-Kirchhoff Laplacian on Γ, which satisfies Assumption 1.4. Assume that $\lambda > \left(\frac{\pi}{\ell_{min}}\right)^2$, where ℓ_{min} is the length of the shortest edge of the graph. Let $\alpha \in [0, \pi)$ and $\mathcal{B} = \mathcal{P}_\alpha(f)$. Then,*

$$\lim_{T\to\infty} \mathrm{Sf}_\lambda \left(H^{\mathcal{P}_\alpha}_\alpha(t)\right)_{t\in[-T,T]} = \beta_\Gamma. \tag{1.18}$$

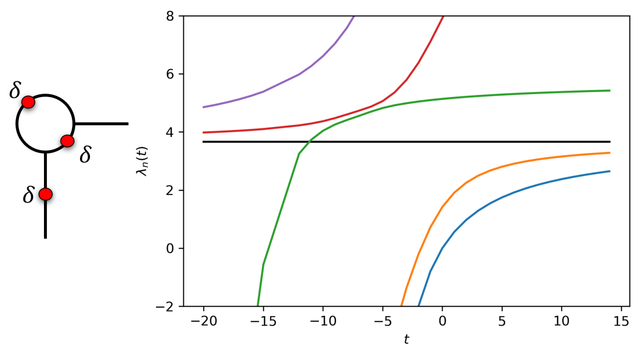

FIGURE 1.5. Demonstration of Theorem 1.8 and Proposition 1.10 on a graph with $\beta_\Gamma = 1$ and $\alpha = 0$. We choose an eigenfunction f and for which $|\mathcal{P}_0(f)| = 3$. Taking $\mathcal{B} = \mathcal{P}_0(f)$ (indicated by the red points), we see that within the spectral curves of $H_0^{\mathcal{P}_0}(t)$ only one crosses the λ horizontal cross-section (marked in black) at a finite t value, corresponding to Proposition 1.10. The total spectral flow is $\mathrm{Sf}_\lambda \left(H_\alpha^{\mathcal{B}}(t)\right)_{t\in\overline{\mathbb{R}}} = 3$, as is stated in Theorem 1.5.

The rest of the paper is organized as follows. In Section 2 we describe some basic tools in the spectral analysis of quantum graphs. These tools are then used in Section 3 to prove Theorem 1.5. This concludes the first part of the paper. The second part is focused on the spectral flow. In Section 4 we describe a general framework for the spectral flow via the Maslov index, scattering matrices and their winding numbers, specialized for quantum graphs. These tools are used in Section 5 to prove Theorem 1.8. The other results on the spectral flow (Propositions 1.9, 1.10, and 6.1), which are of spectral geometric nature, are obtained in Section 6 as corollaries of Theorem 1.8.

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2. Basic properties of the H_α family

We denote the sesquilinear form corresponding to $H_\alpha^{\mathcal{B}}(t)$ by $\mathcal{Q}_\alpha^{(t)}(\cdot,\cdot)$ (for brevity, we omit the set $\mathcal{B}$ from the notation). In all cases, the form domain of $\mathcal{Q}_\alpha^{(t)}$ consists of functions in $H^1(\Gamma)$ which are continuous at all vertices in $\mathcal{V}\backslash\mathcal{B}$.

For $\alpha \neq 0$, the sesquilinear form is

$$\mathcal{Q}_\alpha^{(t)}(f,g) = \int_\Gamma \frac{df}{dx}\overline{\frac{dg}{dx}}dx + \sum_{v\in\mathcal{B}} \left\langle \begin{pmatrix} f(v_-) \\ f(v_+) \end{pmatrix}, Q_\alpha^{(t)} \begin{pmatrix} g(v_-) \\ g(v_+) \end{pmatrix} \right\rangle, \tag{2.1}$$

where τ_0 is the Dirichlet trace (see (1.4)) and the matrix $Q_\alpha^{(t)}$ is given by

$$Q_\alpha^{(t)} = \begin{cases} 0, & t=0, \\ \cot(\alpha)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, & t=\infty, \\ \cot(\alpha)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{\sin^2(\alpha)t}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, & \text{otherwise}, \end{cases} \tag{2.2}$$

and in the case $t=0$, the functions in the form domain are also required to be continuous at $\mathcal{B}$.

For the case $\alpha = 0$ and $t \neq \infty$ the form domain consists of all continuous functions in $H^1(\Gamma)$, and the form $\mathcal{Q}_0^{(t)}$ is given by

$$\mathcal{Q}_0^{(t)}(f,g) = \int_\Gamma \frac{df}{dx}\overline{\frac{dg}{dx}}dx + t\sum_{v\in\mathcal{B}} f(v)\overline{g(v)}. \tag{2.3}$$

For $\alpha = 0$ and $t=\infty$ we just take $\mathcal{Q}_0^{(\infty)}(f,g) = \int_\Gamma \frac{df}{dx}\overline{\frac{dg}{dx}}dx$ with the form domain being all continuous functions in $H^1(\Gamma)$ vanishing at $\mathcal{B}$. For additional details and derivation of the sesquilinear form, see [54, Appendix A] and [19, Sec. 1.4.4].

We now employ the sesquilinear form for proving two useful lemmas.

Lemma 2.1. *The family $H_\alpha^{\mathcal{B}}(t)$ admits a family of real-analytic eigenvalue branches. Every such spectral curve $\lambda^\alpha(t)$ is monotone increasing with t. A spectral curve is horizontal $\lambda^\alpha(t) \equiv \lambda$ if and only if $\lambda \in \operatorname{Spec}\left(H_\alpha^{\mathcal{B}}(\infty)\right)$.*

Proof. Analyticity of the eigenvalue curves follows from standard Kato type arguments (see for instance [19, thm. 2.5.4], [54, lem. 6.1.], or [48, thm. 2.10-2.11]).

To show the monotonicity of the spectral curves we follow the methods presented in [18, lem. 2], where additional details may be found. For a spectral curve $\lambda^\alpha(t)$ we denote by $f_n^\alpha(t)$ the associated analytic branch of L^2-normalized eigenfunctions.

By definition of the sesquilinear form $\mathcal{Q}_\alpha^{(t)}$, we have that

$$\mathcal{Q}_\alpha^{(t)}(f^\alpha(t), g) = \lambda^\alpha(t)\langle f^\alpha(t), g\rangle, \ \forall g \in \operatorname{dom}\left(\mathcal{Q}_\alpha^{(t)}\right). \tag{2.4}$$

Differentiating both sides with respect to t, we obtain

$$\frac{\partial \mathcal{Q}_\alpha^{(t)}(f^\alpha, g)}{\partial t} + \mathcal{Q}_\alpha^{(t)}\left(\frac{\partial f^\alpha}{\partial t}, g\right) = \frac{\partial \lambda^\alpha}{\partial t}\langle f^\alpha, g\rangle + \lambda^\alpha \left\langle \frac{\partial f^\alpha}{\partial t}, g\right\rangle. \tag{2.5}$$

Taking $g = f^{\alpha}$, a straightforward computation gives

$$\frac{\partial \lambda^{\alpha}(t)}{\partial t} = \frac{\partial \mathcal{Q}_{\alpha}^{(t)}(f^{\alpha}(t), f^{\alpha}(t))}{\partial t} = \begin{cases} \sum_{v\in\mathcal{B}} \frac{1}{\sin^2(\alpha)t^2} \left|f^{\alpha}(v_+) - f^{\alpha}(v_-)\right|^2, & \alpha \neq 0, \\ \sum_{v\in\mathcal{B}} \left|f(v)\right|^2, & \alpha = 0. \end{cases} \tag{2.6}$$

The expression above is non-negative, which proves that the spectral curves are monotone increasing.

To show the last part of the lemma, assume first that $\alpha \neq 0$. Then by (2.6) we get that $\frac{\partial\lambda^{\alpha}(t)}{\partial t} \equiv 0$ if and only if $f^{\alpha}(v_+) = f^{\alpha}(v_-)$ for all $v \in \mathcal{B}$. By (1.6) this is equivalent to $\frac{df^{\alpha}(v_+)}{dx} = \frac{df^{\alpha}(v_-)}{dx}$ for all $v \in \mathcal{B}$. Equivalently, the left hand side of (1.7) vanishes for all $v \in \mathcal{B}$, regardless of the value of t. This corresponds to f^{α} satisfying the $\delta_{\alpha}(\infty)$ conditions at all $v \in \mathcal{B}$ (as is explained after Equations (1.6),(1.7)), i.e., $\lambda^{\alpha} \in \operatorname{Spec}\left(H_{\alpha}^{\mathcal{B}}(\infty)\right)$. The equivalence for the case $\alpha = 0$ is verified similarly. □

Lemma 2.2. (1) *For $\alpha \neq 0$ and for every $\epsilon > 0$, the family $H_{\alpha}^{\mathcal{B}}(t)$ is uniformly bounded from below on $\overline{\mathbb{R}}\setminus(0,\epsilon)$.*

(2) *For $\alpha = 0$ and for every $M > 0$, the family $H_0^{\mathcal{B}}(t)$ is uniformly bounded from below on $[-M, \infty)$.*

Proof. To prove the uniform boundedness from below, we first note that that each operator of the type $H_{\alpha}^{\mathcal{B}}(t)$ (regardless of the values of α and t) is bounded from below (see, e.g., [19, thm. 1.4.19]). Therefore, to show a uniform boundedness from below we will show that there exists a lower bound on the quadratic form $\mathcal{Q}_{\alpha}^{(t)}$, which is independent of t. We start with the case $\alpha \neq 0$ and $t \in \overline{\mathbb{R}}\setminus(0,\epsilon)$. The quadratic form associated with the operator $H_{\alpha}^{\mathcal{B}}(t)$ is given by

$$\mathcal{Q}_{\alpha}^{(t)}(f,f) = \int_{\Gamma} \left|\frac{df}{dx}\right|^2 dx + \sum_{v\in\mathcal{B}} \begin{pmatrix} f(v_-) \\ f(v_+) \end{pmatrix}^T \left[\cot(\alpha)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{\sin^2(\alpha)\, t}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\right] \begin{pmatrix} \overline{f(v_-)} \\ \overline{f(v_+)} \end{pmatrix}. \tag{2.7}$$

The only t-dependence of the quadratic form is in the term

$$\sum_{v\in\mathcal{B}} \begin{pmatrix} f(v_-) \\ f(v_+) \end{pmatrix}^T \frac{1}{\sin^2(\alpha)\, t}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \overline{f(v_-)} \\ \overline{f(v_+)} \end{pmatrix},$$

which is positive if $t < 0$ and is bounded from below by

$$-\frac{2}{\sin^2(\alpha)\epsilon} \sum_{v\in\mathcal{B}} \left\| \begin{matrix} f(v_-) \\ f(v_+) \end{matrix} \right\|^2$$

if $t > \epsilon$. Hence, there is a lower bound for the quadratic form which is independent of $t \in \overline{\mathbb{R}}\setminus(0,\epsilon)$. We conclude that in this case the operator family is indeed uniformly bounded from below.

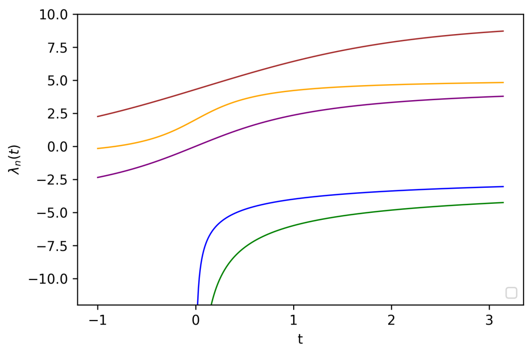

FIGURE 2.1. The lowest analytic eigenvalue branches of $H_{\alpha}^{\mathcal{B}}(t)$ with $|\mathcal{B}|=2$, where two of the eigenvalue branches tend to $-\infty$ as $t\to 0^{+}$, showing that the family is not uniformly bounded from below.

In the case of $\alpha=0$ and $t\in[-M,\infty)$ the quadratic form is

$$\mathcal{Q}_{0}^{(t)}\left(f,g\right)=\int_{\Gamma}\frac{df}{dx}\overline{\frac{dg}{dx}}dx+t\sum_{v\in\mathcal{B}}f\left(v\right)\overline{g\left(v\right)}\geq\int_{\Gamma}\frac{df}{dx}\overline{\frac{dg}{dx}}dx-M\sum_{v\in\mathcal{B}}f\left(v\right)\overline{g\left(v\right)},$$

and the bound is independent on t. This shows a uniform lower bound in this case as well. $\square$

Remark 2.3. One may substitute the quadratic form (2.7) in the Rayleigh quotient characterization of the lowest eigenvalue. This can be used to show that for $\alpha\neq 0$, the family $H_{\alpha}\left(t\right)$ is not uniformly bounded from below on $(0,\epsilon)$, as $\lim_{t\to 0^{+}}\lambda_{\min}\left(H_{\alpha}\left(t\right)\right)=-\infty$ (confer Figure 2.1). One can similarly show that for $\alpha=0$, $\lim_{t\to-\infty}\lambda_{\min}\left(H_{0}\left(t\right)\right)=-\infty$, and so the family $H_{0}\left(t\right)$ is not uniformly bounded from below on $(-\infty,-M)$. In general, Theorem 1.8 implies that exactly $|\mathcal{B}|$ of the spectral curves tend to $-\infty$ for these singular t values. We briefly mention that the existence of such spectral curves is related to the presence of a nontrivial Dirichlet part in the boundary conditions. This phenomenon is further discussed in [17, 52].

3. PROOF OF THEOREM 1.5

We start by establishing a useful spectral connection between the Robin map and the $H_{\alpha}^{\mathcal{B}}$ family for an arbitrary set $\mathcal{B}$ of degree two vertices. This connection forms a key argument in the proof of Theorem 1.5.

Lemma 3.1. *Let $\lambda\notin\operatorname{Spec}\left(H_{\alpha}^{\mathcal{B}}\left(\infty\right)\right)$. Then $-t\in\mathbb{R}$ is an eigenvalue of $\Lambda_{\alpha}^{\mathcal{B}}\left(\lambda\right)$ if and only if λ is an eigenvalue of $H_{\alpha}^{\mathcal{B}}\left(t\right)$, and both eigenvalues have the same multiplicity. In other words,*

$$\forall t\in\mathbb{R},\qquad \dim\ker\left(H_{\alpha}^{\mathcal{B}}\left(t\right)-\lambda\right)=\dim\ker\left(\Lambda_{\alpha}^{\mathcal{B}}\left(\lambda\right)+t\right). \tag{3.1}$$

Proof. First, assume that $-t$ is an eigenvalue of $\Lambda_{\alpha}^{\mathcal{B}}\left(\lambda\right)$ with eigenvector $w\in\mathbb{C}^{\mathcal{B}}$:

$$\Lambda_{\alpha}^{\mathcal{B}}\left(\lambda\right)w=-tw. \tag{3.2}$$

Then the solution u to the boundary value problem (1.10) satisfies

$$-\frac{d^2u}{dx^2} = \lambda u, \tag{3.3}$$

and on each $v \in \mathcal{B}$ it satisfies

$$\tau_\alpha u(v_+) = \tau_\alpha u(v_-) = w, \tag{3.4}$$

$$\tau'_\alpha u(v_-) - \tau'_\alpha u(v_+) = \Lambda_\alpha(\lambda)\, w = -tw = -t\tau_\alpha u(v_-), \tag{3.5}$$

which means that u is an eigenfunction of $H_\alpha^{\mathcal{B}}(t)$ with eigenvalue λ.

For the contrary, if λ is an eigenvalue of $H_\alpha^{\mathcal{B}}(t)$ with eigenfunction u, then u must satisfy the $\delta_\alpha(t)$ vertex condition at every $v \in \mathcal{B}$:

$$\tau'_\alpha u(v_+) - \tau'_\alpha u(v_-) = t \cdot \tau_\alpha u(v_-). \tag{3.6}$$

Choosing $w = \tau_\alpha u$ as input to $\Lambda_\alpha^{\mathcal{B}}(\lambda)$ gives

$$\Lambda_\alpha^{\mathcal{B}}(\lambda)\, w = \tau'_\alpha u(v_-) - \tau'_\alpha u(v_+) = -t \cdot \tau_\alpha u(v_-) = -tw, \tag{3.7}$$

and so $-t$ is an eigenvalue of $\Lambda_\alpha^{\mathcal{B}}(\lambda)$ with eigenvector w, assuming that $w \neq 0$. Assume by contradiction that $w = 0$. Then the associated solution u to the boundary value problem (1.10) satisfies the α-Robin vertex condition $\tau_\alpha(u) = 0$ on $\mathcal{B}$. We thus conclude that u is an eigenfunction of $H_\alpha^{\mathcal{B}}(\infty)$ with the same eigenvalue λ. This contradicts the assumption that $\lambda \notin \operatorname{Spec}\left(H_\alpha^{\mathcal{B}}(\infty)\right)$. □

Remark 3.2. One can find similar correspondence results between eigenvalues of a Dirichlet-to-Neumann map and eigenvalues of another related operator in [16, prop. 4.1], [18, lem. 1], [13, cor. 4.2-4.3], [31, thm. 5.1].

Towards the proof of Theorem 1.5, fix an eigenpair (λ_n, f_n) of the Neumann-Kirchhoff Laplacian H. Consider the family $\left(H_\alpha^{\mathcal{B}}(t)\right)_{t\in\overline{\mathbb{R}}}$ where $\mathcal{B} = \mathcal{P}_\alpha(f_n)$ is the set of α-Robin points of f_n. Note that for every $t \in \overline{\mathbb{R}}$, this fixed f_n is an eigenfunction of $H_\alpha^{\mathcal{B}}(t)$ with the same eigenvalue λ_n (for additional details, see [54, lem. 7.1]). We assume (as in the statement of Theorem 1.5) that $\lambda_n > \left(\frac{\pi}{\ell_{\min}}\right)^2$, where $\ell_{\min}$ is the minimal edge length of Γ. With this setting we prove two additional lemmas, and immediately use those (as well as Lemma 3.1) to prove Theorem 1.5.

Lemma 3.3. *Denote the multiplicity of the eigenvalue λ_n in* $\operatorname{Spec}\left(H_\alpha^{\mathcal{B}}(t)\right)$ *by* $\operatorname{Mult}_{\lambda_n}(t)$. *Then,*

$$\operatorname{Mult}_{\lambda_n}(\infty) = \nu_\alpha(f_n). \tag{3.8}$$

Proof. Denote the Robin domains of f_n by $(\Gamma_i)_{i=1}^{\nu_\alpha(f_n)}$. First, note that $\operatorname{Mult}_{\lambda_n}(\infty) \geq \nu_\alpha(f_n)$. This is true since to each Robin domain Γ_i we can match an eigenfunction of $H_\alpha(\infty)$, by choosing $\chi_{\Gamma_i} f_n$ (where χ_{Γ_i} is the characteristic function of Γ_i). We remind the reader that domain of the operator $H_\alpha^{\mathcal{B}}(\infty)$ consists of functions which satisfy the Robin condition at the selected subset of points $\mathcal{B}$:

$$\sin(\alpha)\, f'(x) = \cos(\alpha)\, f(x). \tag{3.9}$$

Since in our case $\mathcal{B} = \mathcal{P}_\alpha(f_n)$, we see that by construction, $\chi_{\Gamma_i} f_n$ are all indeed eigenfunctions of $H_\alpha(\infty)$ with eigenvalue λ_n (with disjoint supports), and there are thus at least $\nu_\alpha(f_n)$ such linearly independent eigenfunctions.

Assume by contradiction that $\mathrm{Mult}_{\lambda_n}(\infty) > \nu_\alpha(f_n)$. Then there is an additional eigenfunction $g \notin \mathrm{span}\left\{\chi_{\Gamma_1} f_n, ..., \chi_{\Gamma_{\nu_\alpha(f_n)}} f_n\right\}$. In particular, there is some Robin domain Γ_i such that $g|_{\Gamma_i} \not\equiv 0$ and $g|_{\Gamma_i} \not\propto f_n|_{\Gamma_i}$. This gives two linearly independent eigenfunctions for the Laplacian on the sub-graph Γ_i, $f_n|_{\Gamma_i}$ and $g|_{\Gamma_i}$, which means that λ_n is not a simple eigenvalue of H on Γ_i. Since $\lambda_n > \left(\frac{\pi}{\ell_{\min}}\right)^2$, Γ_i is a star graph (see, e.g., [4, prop. 3.9]). By a simple adaptation of [19, cor. 3.1.9], since λ_n is a multiple eigenvalue of the tree graph Γ_i, then there exists an internal vertex $v \in \Gamma_i$ such that $f_n(v) = 0$. This contradicts Assumption 1.4. □

Lemma 3.4. *Let $p(\lambda_n)$ denote the spectral position of λ_n at $t = \infty$:*

$$p(\lambda_n) := 1 + \#\{\lambda \in \mathrm{Spec}(H_\alpha(\infty)) : \lambda < \lambda_n\}.$$

Then,

$$p(\lambda_n) = \begin{cases} 1, & \alpha = 0, \\ 1 + |\mathcal{P}_0(f_n)|, & \alpha \neq 0, \end{cases} \tag{3.10}$$

where $|\mathcal{P}_0(f_n)|$ is the number of zeros (nodal points) of f_n.

Proof. The case $\alpha = 0$, which corresponds to nodal domains, is a well known consequence of Courant's nodal theorem (See, e.g., [19, 54, rem. 5.2.9] for more details).

We now assume $\alpha \neq 0$. Our strategy for counting the eigenvalues $\lambda \in \mathrm{Spec}\left(H_\alpha^{\mathcal{B}}(\infty)\right)$ below λ_n will be to partition Γ into the Robin domains $(\Gamma_i)_{i=1}^{\nu_\alpha(f_n)}$ of f_n, and count the eigenfunctions of each such sub-graph.

Denoting the spectral position of λ_n as an eigenvalue of the sub-graph Γ_i by p_i, we have the following formula for the spectral position $p(\lambda_n)$:

$$p(\lambda_n) = 1 + \sum_{i=1}^{\nu_\alpha(f_n)} (p_i - 1). \tag{3.11}$$

The formula above is obtained by using the eigenfunctions of $H_\alpha(\infty)$ on each sub-graph to define an eigenfunction of $H_\alpha(\infty)$ on the entire graph (similar to Lemma 3.3). On each Robin domain Γ_i, there are $p_i - 1$ eigenvalues of $H_\alpha(\infty)$ smaller than λ_n, denoted by $\left(\lambda_k^{\Gamma_i}\right)_{k=1}^{p_i-1}$ with eigenfunctions $\left(\phi_k^{\Gamma_i}\right)_{k=1}^{p_i-1}$. Note that by the same argument as in Lemma 3.3, each $\lambda_k^{\Gamma_i}$ is also an eigenvalue of $H_\alpha(\infty)$ on Γ with eigenfunction $\chi_{\Gamma_i}\phi_k^{\Gamma_i}$. This overall gives us $\sum_{i=1}^{\nu_\alpha(f_n)} (p_i - 1)$ linearly independent eigenfunctions for eigenvalues of $H_\alpha(\infty)$ which are smaller than λ_n. This procedure also exhausts all possible (linearly independent) eigenfunctions for Γ, which proves (3.11).

Since $\lambda_n > \left(\frac{\pi}{\ell_{\min}}\right)^2$, then on each Robin domain Γ_i (which is a star sub-graph of Γ), the spectral position p_i of λ_n is equal to $|\mathcal{P}_0(f_n|_{\Gamma_i})| + 1$ (see [4, lem. 3.1]). Plugging this into (3.11) we get

$$p(\lambda_n) = 1 + \sum_{i=1}^{\nu_\alpha(f_n)} (p_i - 1) = 1 + \sum_{i=1}^{\nu_\alpha(f_n)} \left|\mathcal{P}_0\left(f_n|_{\Gamma_i}\right)\right| = 1 + \left|\mathcal{P}_0(f_n)\right|, \tag{3.12}$$

which completes the proof. □

Proof of Theorem 1.5. We first write the proof for $\alpha \neq 0$. Consider the H_α family placed at the set of α-Robin points of f_n. The idea of the proof will be to compute the spectral flow for this family with $\lambda_n + \epsilon$ along $[-\infty, 0]$.

By Lemmas 3.3 and 3.4, at $t = \pm\infty$ exactly $\nu_\alpha(f_n) + |\mathcal{P}_0(f_n)|$ spectral curves lie below $\lambda_n + \epsilon$. On the other hand, there are exactly n spectral curves below $\lambda_n + \epsilon$ at $t = 0$. Since the spectral curves are monotone increasing, uniformly bounded from below and continuous on all of $(-\infty, 0]$ (Lemmas 2.1 and 2.2), we conclude that along $[-\infty, 0]$, exactly $\nu_\alpha(f_n) + |\mathcal{P}_0(f_n)| - n$ spectral curves intersect $\lambda_n + \epsilon$, giving rise to the same number of positive eigenvalues of $\Lambda_\alpha(\lambda_n + \epsilon)$ in the process (Lemma 3.1). Thus,

$$\operatorname{Pos}(\Lambda_\alpha(\lambda_n + \epsilon)) = \operatorname{Sf}_{\lambda_n+\epsilon}(H_\alpha(t))_{t\in[-\infty,0]} = \nu_\alpha(f_n) + |\mathcal{P}_0(f_n)| - n, \tag{3.13}$$

$$\Rightarrow \quad n - \nu_\alpha(f_n) = |\mathcal{P}_0(f_n)| - \operatorname{Pos}(\Lambda_\alpha(\lambda_n + \epsilon)), \tag{3.14}$$

proving (1.12).

For $\alpha = 0$, we repeat the same procedure, counting the spectral flow through $\lambda_n + \epsilon$ along $[0, \infty]$ instead of $[-\infty, 0]$ (this is done in order to have the boundedness from below of the spectral curves, as in Lemma 2.2). This time, Lemma 3.4 gives that λ_n is the lowest eigenvalue of $H_0(\infty)$. The same intersection counting argument as before now gives

$$n - \nu_0(f_n) = \operatorname{Mor}(\Lambda_0(\lambda_n + \epsilon)). \tag{3.15}$$

In order to obtain (1.12) for $\alpha = 0$, we use the fact that

$$\forall \alpha \in [0, \pi), \quad \operatorname{Mor}(\Lambda_\alpha(\lambda_n + \epsilon)) + \operatorname{Pos}(\Lambda_\alpha(\lambda_n + \epsilon)) = |\mathcal{P}_\alpha(f_n)|, \tag{3.16}$$

which holds since $\Lambda_\alpha(\lambda_n + \epsilon)$ is a self-adjoint matrix (see, e.g. [19, (3.5.10)] and [35]) and it is of dimension $|\mathcal{P}_\alpha(f_n)|$. For this argument we also use that one may choose ϵ small enough so that $\Lambda_\alpha(\lambda_n + \epsilon)$ does not have zero as an eigenvalue.

To complete the proof of (1.13), we first note that the assumption $\lambda_n > \frac{\pi}{\ell_{\min}}$ implies that all Robin domains are star graphs and hence the connection between the number of Robin domains to the number of Robin points is simply given by $\nu_\alpha(f_n) = |\mathcal{P}_\alpha(f_n)| - \beta_\Gamma + 1$. This connection is independent of the value of α and hence

$$\nu_\alpha(f_n) - |\mathcal{P}_\alpha(f_n)| = \nu_0(f_n) - |\mathcal{P}_0(f_n)|. \tag{3.17}$$

Plugging (3.17) and (3.16) into (3.14) gives the required identity

$$n - \nu_0(f_n) = \operatorname{Mor}(\Lambda_\alpha(\lambda_n + \epsilon)). \tag{3.18}$$

□

Remark 3.5. If λ_n is not assumed to be a simple eigenvalue, then by the same proof as above, Formulas (3.14) and (3.15) easily generalize to

$$n - \nu_\alpha(f_n) = |\mathcal{P}_0(f_n)| + 1 - \dim(\ker(H - \lambda_n)) - \mathrm{Pos}(\Lambda_\alpha(\lambda_n + \epsilon)), \tag{3.19}$$

$$n - \nu_0(f_n) = \mathrm{Mor}(\Lambda_\alpha(\lambda_n + \epsilon)) + 1 - \dim(\ker(H - \lambda_n)). \tag{3.20}$$

4. Basic properties of the spectral flow

4.1. **Trace maps, boundary conditions, and the Lagrangian Grassmannian.**

We introduce the standard Dirichlet and Neumann traces as:

$$\gamma, \gamma' \colon H^2(\Gamma) \to \mathbb{V} := \bigoplus_{e \in \mathcal{E}} \mathbb{C}^2 \cong \bigoplus_{v \in \mathcal{V}} \mathbb{C}^{\deg(v)}.$$

where γ and γ' stand for evaluation of the function and its derivative at all the end points of the graph edges[2]. The derivative in the Neumann trace γ' is taken in the direction from the end point of an edge into its interior (as in (1.3), for example). Taken together, the Dirichlet and Neumann traces define the full trace map

$$\hat{\gamma} = \gamma \oplus \gamma' \colon H^2(\Gamma) \to \hat{\mathbb{V}} = \mathbb{V} \oplus \mathbb{V} \cong \mathbb{C}^{4|\mathcal{E}|}. \tag{4.1}$$

The space $\hat{\mathbb{V}}$ is called the *space of boundary values* for the Laplacian on Γ.

By the classical von Neumann theory of self-adjoint extensions of symmetric operators, self-adjoint boundary conditions for the Laplacian $H = -\frac{d^2}{dx^2}$ on Γ can be described in terms of Lagrangian subspaces in the space of boundary values. Let us describe briefly how this theory is applied to our context. See, e.g., [53, Section 14.2] for details.

Integrating by parts over each edge, one obtains Green's identity,

$$\langle Hf, g\rangle - \langle f, Hg\rangle = \langle \gamma' f, \gamma g\rangle - \langle \gamma f, \gamma' g\rangle \text{ for every } f, g \in H^2(\Gamma). \tag{4.2}$$

The right hand side of (4.2) can be written equivalently as $\omega(\hat{\gamma} f, \hat{\gamma} g)$, where ω is the symplectic form on $\hat{\mathbb{V}}$ given by the formula

$$\omega(\xi, \eta) = \langle J\xi, \eta\rangle, \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{4.3}$$

We note that J is a skew-adjoint unitary operator acting on $\hat{\mathbb{V}} = \mathbb{V} \oplus \mathbb{V}$. Recall that a sesquilinear form ω on a complex Hilbert space $\hat{\mathbb{V}}$ is called symplectic if $\omega(\eta, \xi) = -\overline{\omega(\xi, \eta)}$ for every $\xi, \eta \in \hat{\mathbb{V}}$.

A subspace $\mathcal{L}$ of a symplectic space $(\hat{\mathbb{V}}, \omega)$ is called *Lagrangian* if $\mathcal{L}$ coincides with its skew-adjoint complement

$$\mathcal{L}^\omega = \left\{\xi \in \hat{\mathbb{V}} \;:\; \omega(\xi, \eta) = 0 \text{ for every } \eta \in \mathcal{L}\right\}.$$

[2]The traces γ, γ' are obviously connected to the traces τ_0, τ_0' introduced in Section 1.2. But the ranges of these maps are different.

The set of all Lagrangian subspaces in $\hat{\mathbb{V}}$ is called the *Lagrangian Grassmannian* of $\hat{\mathbb{V}}$ and denoted $\mathrm{LGr}(\hat{\mathbb{V}})$. It has a natural structure of a smooth manifold. The von Neumann theory of self-adjoint extensions applied to the operator H provides the following description.

Proposition 4.1 ([53, Prop. 14.7(v)]). *Consider the symmetric operators $H_{\min}, H_{max}$ acting as $-\frac{d^2}{dx^2}$ with the following domains:*

$$\mathrm{dom}(H_{\min}) = H_0^2(\Gamma), \tag{4.4}$$

$$\mathrm{dom}(H_{\max}) = H^2(\Gamma). \tag{4.5}$$

Then the self-adjoint extensions of $H_{\min}$ are in one-to-one correspondence with Lagrangian subspaces of $\mathbb{V} \oplus \mathbb{V}$. This correspondence takes a Lagrangian subspace $\mathcal{L}$ to the restriction $H_{\mathcal{L}}$ of $H_{\max}$ to the domain

$$\mathrm{dom}(H_{\mathcal{L}}) = \left\{ f \in H^2(\Gamma) \ : \ \hat{\gamma}(f) \in \mathcal{L} \right\}. \tag{4.6}$$

Such a Lagrangian subspace $\mathcal{L}$ is also called a *boundary* condition for H.

4.2. **Scattering matrix.** Equivalently, the boundary conditions can be determined by unitary operators on $\mathbb{V}$.

Proposition 4.2 ([53, Prop. 14.4]). *Lagrangian subspaces of $\mathbb{V} \oplus \mathbb{V}$ are in one-to-one correspondence with unitary operators on $\mathbb{V}$. This correspondence takes a unitary operator U to the Lagrangian subspace*

$$\mathcal{L} = \{\xi \oplus \eta \in \mathbb{V} \oplus \mathbb{V} \ : \ (\mathbf{1} + U)\eta = i(\mathbf{1} - U)\xi\}\,; \tag{4.7}$$

Thus, self-adjoint extensions of $H_{\min}$ are parameterized by the unitary group $\mathcal{U}(\mathbb{V})$; the domain of the corresponding extension is given by the formula

$$\mathrm{dom}(H_U) = \left\{ f \in H^2(\Gamma) \ : \ (\mathbf{1} + U)\gamma' f = i(\mathbf{1} - U)\gamma f \right\}. \tag{4.8}$$

The matrix U is actually the well-known bond scattering matrix $\sigma(-1)$ commonly used in spectral analysis of quantum graphs, see e.g. [19, sec. 2.1.2].

Remark 4.3. In [53] the correspondence $\mathrm{LGr}(\hat{\mathbb{V}}) \cong \mathcal{U}(\mathbb{V})$ is written in terms of a different unitary operator $V = -U$, the so called *Cayley transform* of $\mathcal{L}$. We choose to use U instead in order to match notations with [19] and other standard sources in quantum graph theory.

4.3. **The Maslov index and winding number for loops of boundary conditions.** Let $U(t) \in \mathcal{U}(\mathbb{V})$, $t \in \overline{\mathbb{R}}$ be a continuous loop of unitary operators. The degree of the determinant map

$$t \mapsto \det U(t)\,, \quad \overline{\mathbb{R}} \to \mathcal{U}(\mathbb{C}) \cong S^1, \tag{4.9}$$

is called the *winding number* of the loop $U(t)$ and denoted $\mathrm{wind}\, U(t)$.

A loop $\mathcal{L}(t)$, $t \in \overline{\mathbb{R}}$ of Lagrangian subspaces in the finite-dimensional symplectic space has an integer-valued homotopy invariant $\mathrm{Mas}(\mathcal{L}(t))$ called the *Maslov index*. For the symplectic space $(\mathbb{V} \oplus \mathbb{V}, \omega)$, the Maslov index can be defined as the spectral flow through $\lambda = 1$ of the unitary loop $U(t)\,V$, where $U(t)$ is the unitary corresponding to $\mathcal{L}(t)$, and the reference unitary $V \in \mathcal{U}(\mathbb{V})$ may be chosen arbitrarily, see [47, 36]. In other words, for an arbitrary partition $t_0 < t_1 < ... < t_N$ of $\overline{\mathbb{R}}$ as in Subsection 1.7, and positive numbers $(\varepsilon_j)_{j=1}^N$ with $\varepsilon_j \in (0, \pi)$ such that

$$\forall\ 1 \leq j \leq N, \quad \forall t \in [t_{j-1}, t_j], \quad e^{\pm i\varepsilon_j} \notin \mathrm{Spec}\,(U(t)), \tag{4.10}$$

the Maslov index is defined as:

$$\mathrm{Mas}(\mathcal{L}(t)) = \sum_{j=1}^{N} \Big(\mathrm{rank} E_{S_{\varepsilon_j}}(U(t_j)V) - \mathrm{rank} E_{S_{\varepsilon_j}}(U(t_{j-1})V) \Big), \tag{4.11}$$

where

$$S_{\varepsilon_j} = \left\{ e^{i\phi} \in S^1 : \phi \in [0, 0 + \varepsilon_j] \right\}. \tag{4.12}$$

The Maslov index and winding number are related by the following result (see [32, prop. 1.5.12], [55]):

Proposition 4.4. *We have that*

$$\mathrm{Mas}(\mathcal{L}(t)) = \mathrm{wind}(U(t)). \tag{4.13}$$

The following fundamental relation then holds [52]:

Theorem 4.5. *Let $H(t)$, $t \in \overline{\mathbb{R}}$ be a loop of self-adjoint extensions of H, parameterized by a continuous loop of boundary conditions $U(t) \in \mathcal{U}(\mathbb{V})$, resp. $\mathcal{L}(t) \in \mathrm{LGr}(\hat{\mathbb{V}})$. Then for every $\lambda \in \mathbb{R}$,*

$$\mathrm{Sf}_\lambda(H(t)) = \mathrm{Mas}(\mathcal{L}(t)) = \mathrm{wind}(U(t)). \tag{4.14}$$

Remark 4.6. The relation $\mathrm{Sf}_\lambda(H(t)) = \mathrm{Mas}(\mathcal{L}(t))$ was first proven for continuously differentiably paths of quantum graph operators in [47, thm 3.3], and later again in [49, sec. 4]. However, these proofs are for a slightly different setting and require different assumptions. The proof appearing in [52] is for more general graph continuous paths of operators, which fits better with our setting.

In this paper we consider quantum graphs with local vertex conditions. This means that the conditions at each vertex involve only the values and derivatives of functions at that vertex (the Neumann-Kirchhoff, (1.2),(1.3) and the δ_α vertex conditions (1.6),(1.7) are such examples). For a quantum graph with local vertex conditions, the scattering matrix U is given by a tuple $U^v \in \mathcal{U}(\mathbb{C}^{\deg v})$ of vertex scattering matrices, such that $U = \oplus_{v\in\mathcal{V}} U^v$. Similarly, a loop $U(t)$ is decomposed into the direct sum of loops $U^v(t) \in \mathcal{U}(\mathbb{C}^{\deg v})$. Since the determinant is multiplicative with respect to the direct sum of operators, the winding number is additive, and we get the following immediate corollary of Theorem 4.5.

Corollary 4.7. *Let $U^v(t)$ (resp. $\mathcal{L}^v(t)$), $t \in S^1$, $v \in \mathcal{V}$ be loops of vertex boundary conditions. Let $H(t)$ be the corresponding loop of self-adjoint extensions of H. Then for every $\lambda \in \mathbb{R}$,*

$$\mathrm{Sf}_\lambda(H(t)) = \sum_{v\in\mathcal{V}} \mathrm{wind}(U^v(t)) = \sum_{v\in\mathcal{V}} \mathrm{Mas}(\mathcal{L}^v(t)). \tag{4.15}$$

5. Two proofs for Theorem 1.8

First proof of Theorem 1.8 – via the Robin map. First, we recall that the spectral flow of a loop, $\mathrm{Sf}_\lambda\left(H_\alpha^{\mathcal{B}}(t)\right)_{t\in\overline{\mathbb{R}}}$ does not depend on λ. Hence, for the sake of the proof we choose $\lambda \notin \mathrm{Spec}\left(H_\alpha^{\mathcal{B}}(\infty)\right)$. Consider the Robin map $\Lambda_\alpha^{\mathcal{B}}(\lambda)$ which is a $|\mathcal{B}| \times |\mathcal{B}|$ symmetric matrix (see, e.g. [19, (3.5.10)] and [35]) and therefore has exactly $|\mathcal{B}|$ real eigenvalues. By Lemma 3.1, there is a one-to-one correspondence between the eigenvalues of $\Lambda_\alpha^{\mathcal{B}}(\lambda)$ and the intersections of the spectral curves with a fixed horizontal line λ and these spectral curves are monotone increasing (by Lemma 2.1). Therefore $\mathrm{Sf}_\lambda\left(H_\alpha^{\mathcal{B}}(t)\right)_{t\in\overline{\mathbb{R}}} = |\mathcal{B}|$. □

Before presenting the second proof of Theorem 1.8, we revisit the formalism for describing the vertex conditions of the family $\left(H_\alpha^{\mathcal{B}}(t)\right)_{t\in\overline{\mathbb{R}}}$ in terms of unitary scattering matrices assigned to the vertices of $\mathcal{B}$ (see also [15, 19, 38, 44, 45] for a thorough introduction). Namely, we provide the explicit forms for the matrices $U^v(t)$ and $U(t)$ which were already introduced in Subsections 4.2 and 4.3. First, note that the $\delta_\alpha(t)$ condition at a vertex v of an arbitrary degree may be written in matrix form as

$$A_\alpha^v(t)\,F(v) + B_\alpha^v(t)\,F'(v) = 0, \tag{5.1}$$

where $F(v)$ and $F'(v)$ are vectors of size $\deg(v)$ of the values and *outwards pointing* derivatives of f at v (these are essentially the projections of the Dirichlet and Neumann traces γ, γ' on the space $\mathbb{C}^{\deg(v)}$). By [19, lem. 2.1.3] (and using $U^v(t) = \sigma^{(v)}(-1)$ to convert between notations), we have

$$U_\alpha^v(t) := -\left(A_\alpha^v(t) - iB_\alpha^v(t)\right)^{-1}\left(A_\alpha^v(t) + iB_\alpha^v(t)\right), \tag{5.2}$$

for the local unitary matrix, representing the δ_α vertex conditions at a vertex v (as in Corollary 4.7).

It is easy to verify that the matrices associated with the $\delta_\alpha(t)$ condition are given by

$$A_\alpha^v(t) = \begin{pmatrix} \cos(\alpha) & -\cos(\alpha) \\ \frac{1}{2}\cos(\alpha)\,t + \sin(\alpha) & \frac{1}{2}\cos(\alpha)\,t - \sin(\alpha) \end{pmatrix}, \tag{5.3}$$

$$B_\alpha^v(t) = \begin{pmatrix} \sin(\alpha) & \sin(\alpha) \\ \frac{1}{2}\sin(\alpha)\,t - \cos(\alpha) & -\frac{1}{2}\sin(\alpha)\,t - \cos(\alpha) \end{pmatrix}, \tag{5.4}$$

which gives that the corresponding scattering matrices at each $v \in \mathcal{B}$ are

$$U_\alpha^v(t) = \frac{1}{2i + t}\begin{pmatrix} -e^{2i\alpha}t & 2i \\ 2i & -e^{-2i\alpha}t \end{pmatrix}. \tag{5.5}$$

Using these unitary scattering matrices and the theory presented in Section 4, we present another proof of Theorem 1.8.

Second proof of Theorem 1.8 – via winding numbers of the scattering matrices. We aim to apply Corollary 4.7 in order to calculate the spectral flow of the loop $\left\{H_\alpha^{\mathcal{B}}(t)\right\}_{t\in\overline{\mathbb{R}}}$. For every vertex $v \notin \mathcal{B}$ the scattering matrix $U^v(t)$ at v corresponds to the Neumann-Kirchhoff conditions and does not depend on t. In particular, for each $v \notin \mathcal{B}$, the determinant $\det U^v(t)$ is constant and its winding number vanishes, meaning that these vertices do not contribute to the spectral flow. For $v \in \mathcal{B}$, the scattering matrix is given by (5.5), and its determinant is equal to $(t-2i)/(t+2i)$ (independently of α). The degree of the map

$$\det U_\alpha^v \colon \overline{\mathbb{R}} \to \mathcal{U}(\mathbb{C}), \quad t \mapsto (t-2i)/(t+2i)$$

is equal to 1. Summing over all $v \in \mathcal{B}$, we arrive at the desired equality

$$\mathrm{Sf}(H_\alpha(t))_{t\in\overline{\mathbb{R}}} = \sum_{v\in\mathcal{B}} \mathrm{wind}(U_\alpha^v) = |\mathcal{B}|. \tag{5.6}$$

□

Remark 5.1. There is an interesting variation of the second proof which is worth mentioning. Since $U_\alpha^v(t)$ depends continuously on the parameters (α, t), we get that the loop $(U_\alpha^v(t))_{t\in\overline{\mathbb{R}}}$ is homotopic to the loop $(U_0^v(t))_{t\in\overline{\mathbb{R}}}$ (with α itself taking the role of the homotopy parameter). Therefore, the winding number is independent of α (which is also apparent from the explicit form of $\det U_\alpha^v(t)$ computed above). Therefore, we could have equivalently performed the calculation above using the more familiar form (see e.g. [38, eq. 3.6]) of the scattering matrix for the delta condition ($\alpha = 0$),

$$U_0^v(t) = \begin{pmatrix} \frac{2}{2-it} - 1 & \frac{2}{2-it} \\ \frac{2}{2-it} & \frac{2}{2-it} - 1 \end{pmatrix}. \tag{5.7}$$

Similarly, instead of the homotopy invariance of the winding number, one can use the homotopy invariance of the Maslov index in order to justify taking $\alpha = 0$.

6. Proofs of propositions 1.9, 1.10 and a formula for the Nodal-Neumann count difference

Proof of Proposition 1.9. By Theorem 1.8, we know that $\mathrm{Sf}_{+\epsilon}\left(H^{\mathcal{B}}_{\pi/2}(t)\right)_{t\in\overline{\mathbb{R}}} = |\mathcal{B}|$ for $\epsilon > 0$. Recall that the spectral curves are continuous and monotone increasing (Lemma 2.1). Then by an intersection counting argument of the spectral curves, for $\epsilon > 0$ small enough we have that

$$\begin{aligned} &\mathrm{Sf}_{+\epsilon}\left(H^{\mathcal{B}}_{\pi/2}(t)\right)_{t\in[-\infty,0]} - \mathrm{Sf}_{-\epsilon}\left(H^{\mathcal{B}}_{\pi/2}(t)\right)_{t\in[-\infty,0]} \\ &= \mathrm{Mult}_{\lambda=0}\left(H_{\pi/2}(-\infty)\right) - \mathrm{Mult}_{\lambda=0}\left(H_{\pi/2}(0)\right). \end{aligned} \tag{6.1}$$

Moreover, for $t < 0$ the operator $H_{\pi/2}(t)$ is non-negative, as can be seen from the quadratic form $\mathcal{Q}^{(t)}_{\pi/2}(f,f)$, see proof of Lemma 2.2. This means that there are no spectral curves in the third quadrant, and so $\mathrm{Sf}_{-\epsilon}\left(H^{\mathcal{B}}_{\pi/2}(t)\right)_{t\in[-\infty,0]} = 0$.

Recall that $H_{\pi/2}(0)$ is the standard Neumann-Kirchhoff Laplacian (at all graph vertices). Hence, since Γ is connected, $\mathrm{Mult}_{\lambda=0}\left(H_{\pi/2}(0)\right) = 1$, and the unique eigenfunction is constant. In addition, $H_{\pi/2}(-\infty)$ is the standard Neumann-Kirchhoff Laplacian on the graph Γ_{cut}, obtained by cutting Γ at the points in $\mathcal{B}$. Therefore, $\mathrm{Mult}_{\lambda=0}\left(H_{\pi/2}(-\infty)\right)$ is equal to the number of connected components of the cut graph (this number is denoted by $|\pi_0(\Gamma_{\mathrm{cut}})|$), with piecewise constant eigenfunctions supported on each individual connected component.

Combining all of these and applying Theorem 1.8 we get

$$\begin{aligned}\text{Sf}_{+\epsilon}\left(H^{\mathcal{B}}_{\pi/2}(t)\right)_{t\in[0,\infty]} &= \text{Sf}_{+\epsilon}\left(H^{\mathcal{B}}_{\pi/2}(t)\right)_{t\in\overline{\mathbb{R}}} - \text{Sf}_{+\epsilon}\left(H^{\mathcal{B}}_{\pi/2}(t)\right)_{t\in[-\infty,0]} \\ &= |\mathcal{B}| - \left(|\pi_0(\Gamma_{\text{cut}})| - 1\right).\end{aligned} \tag{6.2}$$

The first Betti number of a graph Γ is $\beta_\Gamma = |\mathcal{E}_\Gamma| - |\mathcal{V}_\Gamma| + |\pi_0(\Gamma)|$. Applying this formula for both β_Γ and $\beta_{\Gamma_{\text{cut}}}$and using $|\mathcal{E}_{\Gamma_{\text{cut}}}| = |\mathcal{E}_\Gamma| + |\mathcal{B}|$, $|\mathcal{V}_{\Gamma_{\text{cut}}}| = |\mathcal{V}_\Gamma| + 2|\mathcal{B}|$ yields

$$|\pi_0(\Gamma_{\text{cut}})| - |\pi_0(\Gamma)| = |\mathcal{B}| + \beta_{\Gamma_{\text{cut}}} - \beta_\Gamma. \tag{6.3}$$

Collecting all of the above (and recalling that Γ is connected, $\pi_0(\Gamma) = 1$) we obtain

$$\text{Sf}_{+\epsilon}\left(H^{\mathcal{B}}_{\pi/2}(t)\right)_{t\in[0,\infty]} = \beta_\Gamma - \beta_{\Gamma_{\text{cut}}}. \tag{6.4}$$

□

Proof of Proposition 1.10. Since we assume that $\lambda > \left(\frac{\pi}{\ell_{\min}}\right)^2$, there is at least one α-Robin point at each edge, and hence these points, $\mathcal{P}_\alpha(f)$, partition Γ such that $\Gamma\backslash\mathcal{P}_\alpha(f)$ contains no cycle. This implies the following relation between the numbers of Robin domains and Robin points (see [8, eq. (1.11)])

$$\nu_\alpha(f) = |\mathcal{P}_\alpha(f)| - \beta_\Gamma + 1. \tag{6.5}$$

By Lemma 3.3,

$$\text{Mult}_{\lambda_n}(H_\alpha(-\infty)) = \nu_\alpha(f) = |\mathcal{P}_\alpha(f)| - \beta_\Gamma + 1. \tag{6.6}$$

Recall from the proof of Theorem 1.5 that $\lambda \in \text{Spec}\left(H^{\mathcal{P}(f)}_\alpha(t)\right)$ for all t, which means that λ is a constant (i.e., horizontal) spectral curve. Since λ is a simple eigenvalue of $H^{\mathcal{P}(f)}_\alpha(0)$ (Assumption 1.4), and by monotonicity of the spectral curves (Lemma 2.1) we conclude that all other $|\mathcal{P}_\alpha(f)| - \beta_\Gamma$ spectral curves that are equal to λ at $t = -\infty$ must intersect $\lambda + \epsilon$ somewhere along $(-\infty, \infty)$. This gives $|\mathcal{P}_\alpha(f)| - \beta_\Gamma$ intersections of spectral curves with $\lambda + \epsilon$.

By Theorem 1.8, there should overall be $|\mathcal{P}_\alpha(f)|$ such intersections along $[-\infty, \infty]$. This leaves us with β_Γ additional spectral curves (that were not counted before) which must cross through $\lambda + \epsilon$ as well. By assumption, these curves are not equal to λ_n at $t = \pm\infty$, and so they must also intersect λ somewhere along $(-\infty, \infty)$, which gives us exactly β_Γ intersections with λ, providing the desired result. □

As a an additional corollary, we may obtain an index formula on the difference of nodal and Neumann counts. The importance of this difference was pointed out in [4] (see more about Neumann domains in [5, 9, 10, 11, 50]).

Proposition 6.1. *Let (λ, f) be an eigenpair of H satisfying Assumption 1.4 and $\lambda > \left(\frac{\pi}{\ell_{min}}\right)^2$. Consider the family of operators $(H(t))_{t\in[-\infty,\infty]}$ given by concatenating $\left(H^{\mathcal{P}_0(f)}_0(t)\right)_{t\in[-\infty,0]}$ with $\left(H^{\mathcal{P}_{\pi/2}(f)}_{\pi/2}(-t)\right)_{t\in[0,\infty]}$. Then for $\epsilon > 0$ small enough:*

$$\text{Sf}_{\lambda+\epsilon}(H(t))_{t\in[-\infty,\infty]} = |\mathcal{P}_0(f)| - \left|\mathcal{P}_{\pi/2}(f)\right|, \tag{6.7}$$

and in particular:

$$1-\beta \le \mathrm{Sf}_{\lambda+\epsilon}\left(H\left(t\right)\right)_{t\in[-\infty,\infty]} \le \beta - 1 + \left|\partial\Gamma\right|. \tag{6.8}$$

Remark. Note that the family $\left(H\left(t\right)\right)_{t\in[-\infty,\infty]}$ is well-defined since $H_{0}^{\mathcal{P}_0}\left(0\right)=H_{\pi/2}^{\mathcal{P}_{\pi/2}}\left(0\right)$, which is the standard Neumann-Kirchhoff Laplacian. However, it is not a loop of operators since in general $H_{0}^{\mathcal{P}_0}\left(-\infty\right)$ differs than $H_{\pi/2}^{\mathcal{P}_{\pi/2}}\left(-\infty\right)$.

Proof of Proposition 6.1. The proof uses additivity of the spectral flow (alternatively, of the Maslov index or winding number) with respect to concatenation of paths (see e.g. [32, thm. 4.2.1]). Utilizing the spectral flow computations (3.13) in the proof of Theorem 1.5 (and noting that the orientation of the family $\left(H_{\pi/2}^{\mathcal{P}_{\pi/2}}\left(-t\right)\right)_{t\in[0,\infty]}$ is reversed), we get:

$$\begin{aligned}\mathrm{Sf}_{\lambda+\epsilon}\left(H\left(t\right)\right)_{t\in[-\infty,\infty]} &= \mathrm{Sf}_{\lambda+\epsilon}\left(H_{0}^{\mathcal{P}_0(f)}\left(t\right)\right)_{t\in[-\infty,0]} - \mathrm{Sf}_{\lambda+\epsilon}\left(H_{\pi/2}^{\mathcal{P}_{\pi/2}(f)}\left(t\right)\right)_{t\in[-\infty,0]}\\ &= \nu_0\left(f\right)+\left|\mathcal{P}_0\left(f\right)\right|-n-\left(\nu_{\pi/2}\left(f\right)+\left|\mathcal{P}_0\left(f\right)\right|-n\right)\\ &= \nu_0\left(f\right)-\nu_{\pi/2}\left(f\right)=\left|\mathcal{P}_0\left(f\right)\right|-\left|\mathcal{P}_{\pi/2}\left(f\right)\right|,\end{aligned}$$

where n above is the position of the eigenvalue λ in the spectrum of the standard Neumann-Kirchhoff Laplacian $H_{0}^{\mathcal{P}_0}\left(0\right)=H_{\pi/2}^{\mathcal{P}_{\pi/2}}\left(0\right)$ (see (3.13)). This proves (6.7). The bounds in (6.8) are then obtained as an immediate application from the bounds in [4, eq. (3.1)]. □

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Department of Mathematics, Technion - Israel Institute of Technology, Haifa, Israel, and, Institute of Mathematics, University of Potsdam, Potsdam, Germany
Email address: ramband@technion.ac.il

Department of Mathematics, University of Haifa, Israel, and, Department of Mathematics, Technion - Israel Institute of Technology, Haifa, Israel
Email address: marina.p@technion.ac.il

Department of Mathematics, Technion - Israel Institute of Technology, Haifa, Israel
Email address: gilad.sofer@campus.technion.ac.il