THE DRY TEN MARTINI PROBLEM FOR STURMIAN HAMILTONIANS

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Abstract. The Dry Ten Martini Problem for Sturmian Hamiltonians is solved. Concretely, we prove that all the spectral gaps are open for all the Schrödinger operators with Sturmian potentials and non-vanishing coupling constant. A key approach towards the solution is a representation of the spectrum as the boundary of an infinite tree. This tree is constructed via periodic approximations and it encodes substantial spectral characteristics.

1. Introduction and main results

For $\alpha \in [0,1]$ and $V \in \mathbb{R}$, consider the self-adjoint operator $H_{\alpha,V} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ defined by

$$(H_{\alpha,V}\psi)(n) := \psi(n+1) + \psi(n-1) + V\chi_{[1-\alpha,1)}(n\alpha \mod 1)\, \psi(n), \tag{1.1}$$

where $\chi_{[1-\alpha,1)}$ is the characteristic function of the interval $[1-\alpha,1)$ and $V \in \mathbb{R}$ is the strength of the potential, which is called the *coupling constant*. When $\alpha \notin \mathbb{Q}$, this operator $H_{\alpha,V}$ is called a *Sturmian Hamiltonian,* since the sequence $\chi_{[1-\alpha,1)}(\xi + n\alpha \mod 1)$ is called a *Sturmian sequence*. The parameter ξ may be set to zero for the purpose of the current paper, see Appendix I.

Let $H_{\alpha,V}|_{[0,n-1]}$ be the restriction of the operator to $\ell^2(\{0,\ldots,n-1\})$. Then $H_{\alpha,V}|_{[0,n-1]}$ is a hermitian $n \times n$ matrix with $\sigma\left(H_{\alpha,V}|_{[0,n-1]}\right)$ denoting its multiset of n eigenvalues (repeated according to their multiplicities). The limit

$$N_{\alpha,V}(E) := \lim_{n\to\infty} \frac{\#\left\{\lambda \in \sigma\left(H_{\alpha,V}|_{[0,n-1]}\right) \,:\, \lambda \le E\right\}}{n} \tag{1.2}$$

is known to exist for all $\alpha \in [0,1]\backslash\mathbb{Q}$, $V \in \mathbb{R}$ and $E \in \mathbb{R}$, see e.g. [Hof93, DF22]. The function $E \mapsto N_{\alpha,V}(E)$ is called the integrated density of states of $H_{\alpha,V}$, and we abbreviate it as IDS. We denote the spectrum of $H_{\alpha,V}$ by $\sigma(H_{\alpha,V})$, and mention two fundamental properties of the IDS:

(IDS1) The IDS, $N_{\alpha,V} : \mathbb{R} \to [0,1]$ is a monotone, non-decreasing and continuous function.
(IDS2) We have $E \in \mathbb{R}\backslash\sigma(H_{\alpha,V})$ if and only if there exists an $\varepsilon > 0$ such that the restriction $N_{\alpha,V}$ is constant on $(E-\varepsilon, E+\varepsilon)$.

The connected components of $\mathbb{R}\backslash\sigma(H_{\alpha,V})$ are called *spectral gaps* (or just gaps). As pointed out above, the IDS is constant on the spectral gaps. The values that the IDS attains at the gaps (so-called gap labels) are given in our first main theorem - solving the Dry Ten Martini Problem for Sturmian Hamiltonians.

Theorem 1.1 (All gaps are there)**.** *For all $\alpha \in [0,1] \setminus \mathbb{Q}$ and $V \in \mathbb{R} \setminus \{0\}$,*

$$\{N_{\alpha,V}(E) \,:\, E \in \mathbb{R}\backslash\sigma(H_{\alpha,V})\} = \{l\alpha \mod 1 \,:\, l \in \mathbb{Z}\} \cup \{1\}. \tag{1.3}$$

In the next subsection, we provide a brief historical account of the Dry Ten Martini Problem. In the subsequent subsections, we provide two additional main theorems, and immediately use them to shortly prove Theorem 1.1.

1.1. **The Dry Ten Martini Problem.** "Are all gaps there?", asked Kac in 1981 during a talk at the AMS annual meeting, and offered ten Martinis for the solution. This led Simon [Sim82] to coin the names the *Ten Martini Problem (TMP)* and the *Dry Ten Martini Problem (DTMP)* for two related questions concerning the almost Mathieu operator (AMO). The first problem, TMP, is whether the AMO has Cantor spectrum for all irrational frequencies and non-zero coupling constants. An affirmative answer for the TMP was given by Avila and Jitomirskaya [AJ09]. Further remarkable results on Cantor spectrum for generic quasiperiodic Schrödinger operators are found in [BS82, Eli92, Pui06, ABD09, GS11, GJYZ23, GJY23]. Historical overviews on this problem, the route to its resolution and further substantial results appear in [MJ17, Jit19, DF25].

The DTMP deals with the values that the IDS attains at the spectral gaps. The gap labelling theorem [Bel82, JM82, Bel92, BBG92, DF23, DFZ23] predicts the possible set of values, which the IDS may attain at the spectral gaps. The predicted gap labels for the AMO are exactly the ones as for the Sturmian Hamiltonians, see the right hand side of (1.3). The DTMP is whether or not all these values are attained, or quoting Kac, "Are all gaps there?". We do not exhaustively cover here the literature on the DTMP for the AMO. A substantial progress towards its solution was achieved in [CEY90, Pui04, AJ10, LY15]. The most up to date result appears in [AYZ23], where Avila, You and Zhou solve the DTMP for the non-critical AMO. A more thorough historical account on the DTMP for the AMO can be found there. Some very recent DTMP results for additional classes of operators other than the AMO appear in [Han18, DL24, DEF24, CL25, GWX25].

In the current work, we treat a different class of operators, the Sturmian Hamiltonians (1.1). This model was introduced and studied in [KKT83, OK85] being a guiding model for one-dimensional quasicrystals. We now describe the state of the art results for TMP and DTMP for these operators. A first mathematical study of the spectral properties of the Sturmian Hamiltonians can be found in Casdagli's paper [Cas86] that influenced many of the forthcoming works. In [Süt89, BIST89] it was shown that the spectrum of the Sturmian Hamiltonians is a Cantor set of Lebesgue measure zero, thus solving the TMP. This was substantially generalized in [Len02, DL06a, DL06b] by Damanik and Lenz for aperiodic Schrödinger operators satisfying the so-called Boshernitzan condition [Bos85]. This was also extended to Jacobi operators in [BP13]. A substantial step towards the DTMP was done by Raymond [Ray95a], who proved (1.3) for all $\alpha \notin \mathbb{Q}$ under the additional assumption that $V > 4$. This unpublished result is part of his thesis [Ray95b] and will appear in a revised version in [Ray]. The reader is also referred to [BBB$^+$] for a review of [Ray95a], which is adapted to the conventions of the current paper. Damanik and Gorodetski [DG11] showed (1.3) for the Fibonacci Hamiltonian, i.e. $\alpha = \frac{\sqrt{5}-1}{2}$, if the coupling constant V is small enough. Mei [Mei14] extended the previous result proving (1.3) for $\alpha \notin \mathbb{Q}$ with eventually periodic continued fraction expansion, also in the small coupling regime. The most recent achievement for the DTMP was done in 2016. In an extensive study of the Fibonacci Hamiltonian Damanik, Gorodetski and Yessen [DGY16] proved (1.3) for $\alpha = \frac{\sqrt{5}-1}{2}$ and all coupling constants $V > 0$. The current paper provides the complete affirmative solution of the DTMP for Sturmian Hamiltonians – Theorem 1.1.

1.2. **The spectra of the periodic (rational) approximations of $H_{\alpha,V}$.** The first step towards the proof of Theorem 1.1 is done by considering the spectra of the periodic (also known as rational) approximations of $H_{\alpha,V}$, which are introduced next.

The periodic approximations of $H_{\alpha,V}$ are defined via Diophantine approximations of $\alpha \in [0,1] \setminus \mathbb{Q}$. Each $\alpha \in [0,1] \setminus \mathbb{Q}$ is uniquely presented in terms of its continued fraction expansion,

$$\alpha = c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{\ddots}}}, \tag{1.4}$$

where $c_0 = 0$ in our case and $c_n \in \mathbb{N}$ for all $n \in \mathbb{N}$. Truncating the expansion above gives finite continued fraction expansions,

$$\alpha_k := c_0 + \cfrac{1}{c_1 + \cfrac{1}{\ddots + \cfrac{1}{c_k}}} = \frac{p_k}{q_k}, \qquad k \in \mathbb{N} \cup \{0\}, \tag{1.5}$$

where for $k \in \mathbb{N}$, $p_k, q_k \in \mathbb{N}$ are chosen to be coprime. By convention, $\alpha_0 = \frac{p_0}{q_0} = \frac{0}{1}$ (as $c_0 = 0$), and $\alpha_{-1} = \frac{p_{-1}}{q_{-1}} = \frac{1}{0} = \infty$.

This allows to approximate the spectrum of $H_{\alpha,V}$ in terms of spectra of periodic operators of the form $H_{\frac{p}{q},V}$ (where p, q are coprime). Such an operator $H_{\frac{p}{q},V}$ is q-periodic and hence its spectral properties are given by the Floquet-Bloch theory.

Proposition 1.2. *Let $V \in \mathbb{R} \setminus \{0\}$ and $\frac{p}{q} \in [0,1]$ such that p and q are coprime. Then $H_{\frac{p}{q},V}$ has absolutely continuous spectrum and the spectrum $\sigma(H_{\frac{p}{q},V})$ consists of exactly q connected components, each being a closed interval.*

These are well-known properties of the periodic Schrödinger operators $H_{\frac{p}{q},V}$ with p, q coprime, see e.g. [Tes00, Ray95a, DF25, BBB$^+$].

We slightly abbreviate the notation above by setting

$$\sigma_\infty(V) := \mathbb{R} \quad \text{and} \quad \sigma_{\frac{p}{q}}(V) := \sigma(H_{\frac{p}{q},V}). \tag{1.6}$$

The introduction of the auxiliary spectrum $\sigma_\infty(V)$ seems artificial at first sight, but its role becomes clearer in the next subsection, keeping in mind that $\alpha_{-1} = \infty$.

The following shows that indeed the spectra of the operators $H_{\alpha_k,V}$ approximate the spectrum of the Sturmian Hamiltonian $H_{\alpha,V}$.

Proposition 1.3. [Süt87, BIST89, BIT91] *For all $k \in \mathbb{N}$, and $V \in \mathbb{R}$, the following monotonicity property holds*

$$\sigma_{\alpha_{k+1}}(V) \subseteq \sigma_{\alpha_k}(V) \cup \sigma_{\alpha_{k-1}}(V).$$

In addition,

$$\lim_{k\to\infty} \big(\sigma_{\alpha_k}(V) \cup \sigma_{\alpha_{k+1}}(V)\big) = \bigcap_{k\in\mathbb{N}} \big(\sigma_{\alpha_k}(V) \cup \sigma_{\alpha_{k+1}}(V)\big) = \sigma(H_{\alpha,V}),$$

with the limit taken with respect to the Hausdorff metric[1] *on compact subsets of* $\mathbb{R}$.

These spectral approximations, $\sigma_{\alpha_k}(V)$, may be used to define an ordered (directed) tree graph, $\mathcal{T}_\alpha$, whose boundary represents the spectrum $\sigma(H_{\alpha,V})$. After introducing this tree graph and stating its properties, we are able to prove Theorem 1.1.

1.3. **The spectral α-tree .** Next, we define the ordered (directed) tree graph, $\mathcal{T}_\alpha$. Towards this, recall basic graph theory terminology. A *directed graph* G consists of a countable set $\mathcal{V}$, called *vertex set*, and a set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, called the *edge set*. There is an edge from $u \in \mathcal{V}$ to $w \in \mathcal{V}$ if $(u, w) \in E$. A *path* is a (finite or infinite) sequence $(u_0, u_1, \dots)$ of vertices such that $(u_m, u_{m+1}) \in \mathcal{E}$ for all $m \in \mathbb{N} \cup \{0\}$. For a finite path $(u_0, u_1, \dots, u_m)$, we say it is from vertex u_0 to vertex u_m and denote this by $u_0 \to u_m$. If in this case $u_0 = u_m$ and $m \geq 1$, then such a path is called a *cycle*. A directed graph without cycles is called a *tree*. A *rooted tree* is a tree which has a single vertex designated as a root. In the following, we consider an *ordered tree, which* is a directed rooted tree with a strict (i.e., irreflexive) partial order relation, $\prec$, defined on its vertex set.

[1] See e.g. Lemma 3.1 for definition of this Hausdorff metric.

Fix $\alpha \in [0,1] \setminus \mathbb{Q}$ and let $(c_k)_{k=0}^{\infty}$ be the coefficients of its continued fraction expansion, (1.4). We recursively describe in the following a specific ordered rooted tree, $\mathcal{T}_\alpha$, whose edge and vertex sets are denoted by $\mathcal{E}_\alpha$ and $\mathcal{V}_\alpha$, correspondingly. Figure 1.1 accompanies the tree description.

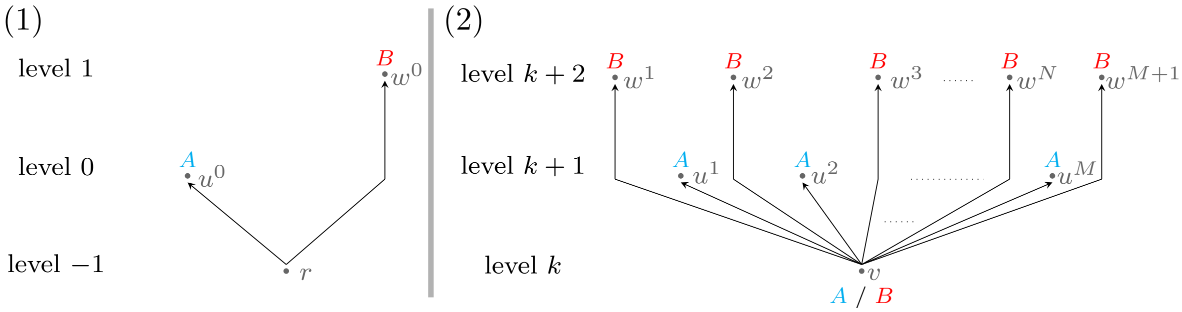

FIGURE 1.1. (1) The root of the tree graph $\mathcal{T}_\alpha$ and two adjacent vertices. (2) A vertex v in level k (for $k \geq 0$) and its outgoing edges to level $k+1$ and $k+2$.

We start by designating a single vertex to be the root, r. We say that the root belongs to level $k = -1$ of the tree. Starting from the root, all other vertices belong to ascending levels k in the tree and in addition they carry one of the two labels: A or B. There are two vertices to which the root r is connected, $(r, u^0) \in \mathcal{E}_\alpha$ and $(r, w^0) \in \mathcal{E}_\alpha$:

- We set the vertex u^0 to be in level $k = 0$ and assign u^0 with the label A. The vertex u^0 is the only vertex in level $k = 0$.
- We set the vertex w^0 to be in level $k = 1$ and assign w^0 with the label B. Note that there might be other vertices in level $k = 1$, see, e.g. Figure 1.2.
- These two vertices are ordered $u^0 \prec w^0$.

We continue defining the ordered tree $\mathcal{T}_\alpha$ recursively. For every vertex v in level k ($k \geq 0$), denote

$$M := \begin{cases} c_{k+1} - 1, & \text{if } v \text{ has the label } A, \\ c_{k+1}, & \text{if } v \text{ has the label } B, \end{cases}$$

and

- connect the vertex v to M vertices, $u^1, \ldots, u^M$, with the label A in level $k+1$, namely, $(v, u^i) \in \mathcal{E}_\alpha$ for $1 \leq i \leq M$.
- connect the vertex v to $M+1$ vertices, $w^1, \ldots, w^{M+1}$, with the label B in level $k+2$, namely, $(v, w^j) \in \mathcal{E}_\alpha$ for $1 \leq j \leq M+1$.
- These vertices are ordered $w^1 \prec u^1 \prec w^2 \prec \ldots \prec u^M \prec w^{M+1}$.

Definition 1.4. For $\alpha \in [0,1] \setminus \mathbb{Q}$, the previously described ordered tree, $\mathcal{T}_\alpha$, is called the *spectral α-tree*. The following two strict (i.e., irreflexive) partially order relations are defined on the vertex set $\mathcal{V}_\alpha$ of $\mathcal{T}_\alpha$:

- We denote $u \to w$ whenever there is a directed path connecting u to w.
- If $u_1, u_2 \in \mathcal{V}_\alpha$ satisfy $u_1 \prec u_2$, then we define $w_1 \prec w_2$ for all $w_1, w_2 \in \mathcal{V}$ satisfying $(u_1 \to w_1$ or $u_1 = w_1)$ and $(u_2 \to w_2$ or $u_2 = w_2)$.

Remark.

- We note that the relation $\prec$ is not a total order. But for any two vertices $u, w \in \mathcal{V}$ with no directed path between them, either $u \prec w$ or $w \prec u$.
- We emphasize that the level of a vertex in $\mathcal{T}_\alpha$ is not necessarily its combinatorial distance from the root. This is since the B vertices are connected by a single edge to a vertex which is two levels below.

In order to connect the spectral α-tree in Definition 1.4 to the spectral approximations, $\sigma_{\alpha_k}(V)$, we introduce the following conventions. By Proposition 1.2, for $k \geq 0$ and $V \neq 0$, the spectrum $\sigma_{\alpha_k}(V)$ consists of exactly q_k intervals (recalling that $\alpha_k = \frac{p_k}{q_k}$). This leads to the following definition.

Definition 1.5. For $\alpha \in [0,1] \setminus \mathbb{Q}$, $k \geq 0$ and α_k as in (1.5). A map $I : V \mapsto I(V),\ V > 0$, is called a *spectral band* in σ_{α_k} if there is a $0 \leq j < q_k$, such that for all $V > 0$, $I(V)$ is the j-th interval (counted from the left) of $\sigma_{\alpha_k}(V)$.

Remark. In the following, we will abuse terminology and also refer to the evaluation of that map, i.e., $I(V)$, as a spectral band. This is a common terminology in the literature. Whether a spectral band means the map itself or its evaluation will be either understood from the context or explicitly mentioned.

In addition, we note that for every spectral band the map $I : V \mapsto I(V),\ V > 0$, is actually Lipschitz continuous, see Corollary 3.2.

Next, we introduce order relations for spectral bands and use these to connect them to vertices of the ordered tree $\mathcal{T}_\alpha$. This is done within the following definition and theorem (see Figure 1.2 for a demonstration).

Definition 1.6. Let $I : V \mapsto [L(I(V)), R(I(V))]$ and $J : V \mapsto [L(J(V)), R(J(V))]$ be two spectral bands. We define the following strict (i.e., irreflexive) order relations.

(a) The spectral band I *is* strictly contained *in* J:

$$I \subseteq_{\text{str}} J \quad \Leftrightarrow \quad \forall V > 0 : \quad L(J(V)) < L(I(V)) < \mathrm{R(I(V))} < \mathrm{R(J(V))}.$$

(b) The spectral band I *is* to the left of J *(respectively* J *is* to the right of I*):*

$$I \prec J \quad \Leftrightarrow \quad \forall V > 0 : \quad L(I(V)) < L(J(V)) \text{ and } \mathrm{R(I(V))} < \mathrm{R(J(V))}.$$

Note that it is possible that I is to the left of J even if $I(V) \cap J(V) \neq \emptyset$ for some value of V. The proof of the following theorem is provided in Section 7.

Theorem 1.7. *Let $\alpha \in [0,1] \setminus \mathbb{Q}$. Let $\mathcal{T}_\alpha$ be the spectral α-tree. Then there exists a unique bijection Ψ between the vertices $\mathcal{V}_\alpha$ of $\mathcal{T}_\alpha$ and all spectral bands of $\{\sigma_{\alpha_k}\}_{k \in \mathbb{N} \cup \{-1,0\}}$ for $V > 0$, such that:*

(a) For each $k \in \mathbb{N} \cup \{-1, 0\}$, the bijection Ψ maps each vertex in level k of $\mathcal{T}_\alpha$ to a spectral band of σ_{α_k}.
(b) For every two vertices u, w, if $u \to w$ then $\Psi(w) \subseteq_{\text{str}} \Psi(u)$.
(c) If u_1, u_2 are vertices in levels k_1, k_2 (respectively) such that $|k_1 - k_2| \leq 1$, then

$$u_1 \prec u_2 \quad \Longleftrightarrow \quad \Psi(u_1) \prec \Psi(u_2).$$

A similar version of Theorem 1.7 holds for $V < 0$, but for this sake one needs to adjust the definition of the tree $\mathcal{T}_\alpha$ (see discussion in Remark 7.6). Figure 1.2 demonstrates the bijection between the graph vertices and the corresponding spectral bands, for $\mathcal{T}_\alpha$ if α has continued fraction expansion $(c_k)_{k=0}^{\infty}$ starting with $0, 1, 2, 3$.

Example 1.8. Theorem 1.7 (c) claims that Ψ preserves the order relation $\prec$ only for vertices that are in the same level or in consecutive levels. In Figure 1.2, the vertices u in level 0 and w in level 2 satisfy $u \prec w$ by the order relation defined on $\mathcal{T}_\alpha$ but $\Psi(u) \not\prec \Psi(w)$ as sketched in the figure since $(\Psi(w))(V) \subseteq_{\text{str}} (\Psi(u))(V)$ for some values of V, see Example 7.2 for more details on the computation of these spectral bands.

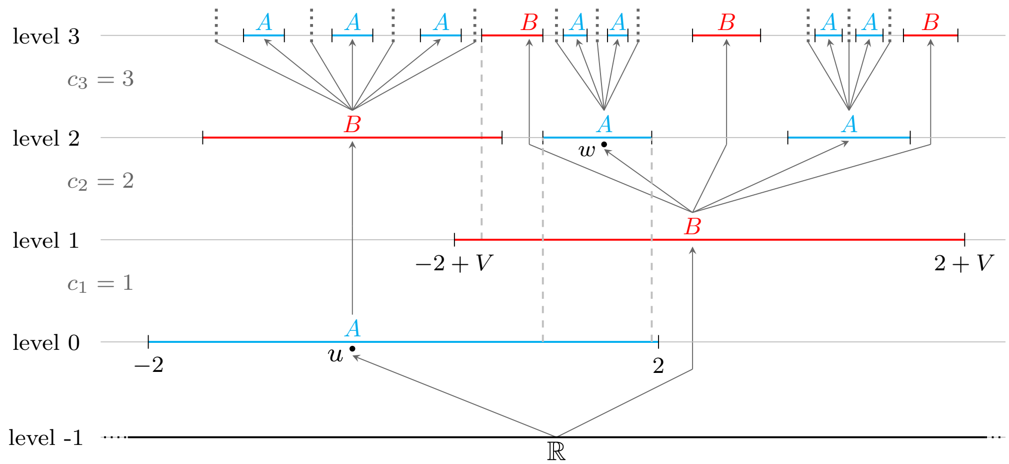

FIGURE 1.2. An example of a spectral α-tree is sketched if α has continued fraction expansion $(c_k)_{k=0}^{\infty}$ starting with $0, 1, 2, 3$, see Definition 1.4. The vertices of the graph are drawn as the spectral bands to which they are mapped by Ψ. The two vertices u, w which are marked satisfy $u \prec w$, but for their corresponding spectral bands, $\Psi(u) \not\prec \Psi(w)$.

1.4. **Connecting $\partial\mathcal{T}_\alpha$ with $\sigma(H_{\alpha,V})$ and proving Theorem 1.1 .** Theorem 1.7 connects the spectra of the approximant operators $H_{\alpha_k,V}$ with the vertices of $\mathcal{T}_\alpha$. Our next step is to connect the spectrum of $H_{\alpha,V}$ with the boundary of $\mathcal{T}_\alpha$, and use this connection to express the IDS value. This will allow us to prove Theorem 1.1.

Denote the boundary of $\mathcal{T}_\alpha$ by

$$\partial\mathcal{T}_\alpha := \{\gamma = (u_0, u_1, u_2 \dots) \,:\, u_0 \text{ is the root of } \mathcal{T}_\alpha \text{ and } (u_{m-1}, u_m) \in \mathcal{E}_\alpha \text{ for all } m \in \mathbb{N}\},$$

i.e. the set of all infinite paths which start from the root. This boundary $\partial\mathcal{T}_\alpha$ inherits a natural total order from the partial order $\prec$ on the vertex set $\mathcal{V}$. Specifically, let $\gamma_1 = (u_0, u_1, \dots)$ and $\gamma_2 = (w_0, w_1, \dots)$ be in $\partial\mathcal{T}_\alpha$. If $\gamma_1 = \gamma_2$, we set $\gamma_1 \preceq \gamma_2$ and $\gamma_2 \preceq \gamma_1$ (so that the order is reflexive). Otherwise, there exists a unique $k \geq 0$ such that $u_{k-1} = v_{k-1}$ and $u_k \neq w_k$. By construction (see Definition 1.4), either $u_m \prec w_m$ for all $m \geq k$ or $w_m \prec u_m$ for all $m \geq k$. In the former case, we set $\gamma_1 \preceq \gamma_2$ and in the latter case, we set $\gamma_2 \preceq \gamma_1$.

Given an infinite path $\gamma = (u_0, u_1, \dots) \in \partial\mathcal{T}_\alpha$ and $V > 0$, Theorem 1.7 (b) implies $\Psi(u_m) \subseteq_{\text{str}} \Psi(u_{m-1})$ for all $m \in \mathbb{N}$. Thus, for all values $V > 0$, the intersection $\cap_{m\in\mathbb{N}} (\Psi(u_m))(V)$ of nested compact intervals is non-empty and connected. By Proposition 1.3, this intersection is contained in the spectrum $\sigma(H_{\alpha,V})$. Furthermore, by [BIST89] the spectrum $\sigma(H_{\alpha,V})$ is of Lebesgue measure zero implying that the intersection $\cap_{m\in\mathbb{N}} (\Psi(u_m))(V)$ consists of a single point (see also [Ray95a] or [BBB$^+$, lem. 5.11]) where this is proven for $V > 4$, but the same proof applies for all $V > 0$). We denote this point by $E_\alpha(\gamma; V)$, i.e., $\cap_{m\in\mathbb{N}} (\Psi(u_m))(V) = \{E_\alpha(\gamma; V)\}$ and this defines the map

$$E_\alpha(\,\cdot\,; V) : \partial\mathcal{T}_\alpha \to \sigma(H_{\alpha,V}).$$

This map satisfies the following properties.

Theorem 1.9. *Let $\alpha \in [0,1] \setminus \mathbb{Q}$ and $V > 0$.*

(a) The map $E_\alpha(\,\cdot\,; V) : \partial\mathcal{T}_\alpha \to \sigma(H_{\alpha,V})$ is a bijection.
(b) The map $E_\alpha(\,\cdot\,; V) : \partial\mathcal{T}_\alpha \to \sigma(H_{\alpha,V})$ is order preserving, i.e. $\gamma_1 \preceq \gamma_2$ implies $E_\alpha(\gamma_1\,; V) \leq E_\alpha(\gamma_2\,; V)$.
(c) For all $\gamma \in \partial\mathcal{T}_\alpha$, the map $E_\alpha(\gamma;\,\cdot\,) : (0,\infty) \to \mathbb{R}$ is Lipschitz continuous.
(d) There exists a function $N_\alpha : \partial\mathcal{T}_\alpha \to [0,1]$ such that for all $V > 0$,

$$N_{\alpha,V}\left(E_\alpha(\gamma; V)\right) = N_\alpha(\gamma).$$

(e) We have $\sigma(H_{\alpha,V}) = -\sigma(H_{\alpha,-V})$*. Furthermore, for all* $\gamma \in \partial\mathcal{T}_\alpha$ *and* $V < 0$*,*

$$N_{\alpha,V}\left(-E_\alpha(\gamma;-V)\right) = 1 - N_\alpha(\gamma).$$

Remark. Note that the tree graph $\mathcal{T}_\alpha$ as well as the function N_α are V-independent. Furthermore, one can explicitly describe the function $N_\alpha : \partial\mathcal{T}_\alpha \to [0,1]$ by the local tree structure, see Remark 7.8.

The stage is now set for the proof of the main theorem.

Proof of Theorem 1.1. Let $\alpha \in [0,1] \setminus \mathbb{Q}$ and $V \in \mathbb{R} \setminus \{0\}$. The inclusion

$$\left\{N_{\alpha,V}(E) \,|\, E \in \mathbb{R}\backslash\sigma(H_{\alpha,V})\right\} \subseteq \left\{l\alpha \mod 1 \,|\, l \in \mathbb{Z}\right\} \cup \{1\}$$

is part of the gap labelling theorem [BBG92, DF23]. We need only to show the other inclusion. Clearly, the values of the IDS at the two unbounded spectral gaps are 0 and 1. More precisely, we have $N_{\alpha,V}(E) = 0$ for $E < \inf\sigma\left(H_{\alpha,V}\right)$ and $N_{\alpha,V}(E) = 1$ for $E > \sup\sigma\left(H_{\alpha,V}\right)$. Thus, the gap labels 0 ($l = 0$) and 1 are contained in $\left\{N_{\alpha,V}(E) \,|\, E \in \mathbb{R}\backslash\sigma(H_{\alpha,V})\right\}$.

Let $l \in \mathbb{Z} \setminus \{0\}$ and $V > 0$. By [Ray95a] (see also [BBB+, thm. 5.25]), there exists a $\widetilde{V} > 4$ and two different values $\widetilde{E_1}, \widetilde{E_2} \in \sigma(H_{\alpha,\widetilde{V}})$ such that

$$N_{\alpha,\widetilde{V}}(\widetilde{E_1}) = N_{\alpha,\widetilde{V}}(\widetilde{E_2}) = l\alpha \mod 1.$$

By the surjectivity of the map $E_\alpha(\,\cdot\,;\widetilde{V})$ (Theorem 1.9 (a)), we have two different infinite paths $\gamma_1, \gamma_2 \in \partial\mathcal{T}_\alpha$ such that $\widetilde{E_1} = E_\alpha(\gamma_1;\widetilde{V})$ and $\widetilde{E_2} = E_\alpha(\gamma_2;\widetilde{V})$.

We use these paths γ_1,γ_2 to designate another pair of energy values $E_1 := E_\alpha(\gamma_1;V)$ and $E_2 := E_\alpha(\gamma_2;V)$. By the injectivity of the map $E_\alpha(\,\cdot\,;V)$ (Theorem 1.9 (a)), we get that $E_1 \neq E_2$. Applying Theorem 1.9 (d) yields

$$N_{\alpha,V}(E_i) = N_\alpha(\gamma_i) = N_{\alpha,\widetilde{V}}(\widetilde{E_i}) = l\alpha \mod 1, \qquad i \in \{1,2\}\,.$$

Thus, we have identified two different spectral values $E_1, E_2 \in \sigma(H_{\alpha,V})$ such that $N_{\alpha,V}(E_1) = N_{\alpha,V}(E_2) = l\alpha \mod 1$. By the monotonicity of the IDS (see (IDS1)) we get that $N_{\alpha,V}$ is constant on the interval (E_1, E_2). By (IDS2), the interval (E_1, E_2) is a spectral gap with the required gap label $l\alpha \mod 1$. We have thus proven the equality in (1.3) for all $V > 0$.

If $V < 0$, the proof follows similarly as above with the following slight modifications. Let $l \in \mathbb{Z}$ and $V < 0$. By [Ray95a] (see also [BBB+, thm. 5.25]), there exists a $\widetilde{V} > 4$ and two different values $\widetilde{E_1}, \widetilde{E_2} \in \sigma(H_{\alpha,\widetilde{V}})$ such that

$$N_{\alpha,\widetilde{V}}(\widetilde{E_1}) = N_{\alpha,\widetilde{V}}(\widetilde{E_2}) = (-l)\alpha \mod 1.$$

Now we proceed as before, defining $\gamma_1, \gamma_2 \in \partial\mathcal{T}_\alpha$ such that $\widetilde{E_1} = E_\alpha(\gamma_1;\widetilde{V})$ and $\widetilde{E_2} = E_\alpha(\gamma_2;\widetilde{V})$. Let $E_1 := -E_\alpha(\gamma_1;-V)$ and $E_2 := -E_\alpha(\gamma_2;-V)$, which are different by Theorem 1.9 (a). Then Theorem 1.9 (d) and (e) imply $E_1, E_2 \in \sigma(H_{\alpha,V})$ and for $i \in \{1,2\}$,

$$N_{\alpha,V}(E_i) = 1 - N_\alpha(\gamma_i) = 1 - N_{\alpha,\widetilde{V}}(\widetilde{E_i}) = 1 - (-l)\alpha \mod 1 = l\alpha \mod 1.$$

Exactly as above we conclude that E_1, E_2 are the edges of a spectral gap at which the IDS attains the required gap label $l\alpha \mod 1$. □

Remark. We note that we have actually proven above that if a gap label appears (i.e., it is attained by the IDS) for some V value, then it appears for all $V \neq 0$.

1.5. **A bird's-eye view on the spectrum of Sturmian Hamiltonians.** Sturmian Hamiltonians belong to the family of dynamically-defined Schrödinger operators. In the literature, one can find various characterizations of the spectrum of Sturmian Hamiltonians, which turned out to be useful for analyzing their spectral properties. We refer the reader to [DEG15, Dam17, DF22, DF25] for a detailed elaboration and we just mention the main tools used for Sturmian dynamical systems. The spectrum can be characterized by a few alternative approaches:

- The energies for which the Lyapunov exponent vanishes (denoted by $\mathcal{Z}$) are used to prove Cantor spectrum of Lebesgue measure zero [Süt89, BIST89, Len02, DL06a, DL06b] by applying Kotani theory [Kot89].
- The energies for which the positive semiorbit under the trace map stays bounded (denoted by $\mathcal{B}$) are used to estimate the fractal dimension of the spectrum [Cas86, Ray95a, DEGT08, DGY16].
- The energies described in terms of a coding scheme (denoted here by Π). This is an approach influenced by Casdagli [Cas86] and fully developed by Raymond [Ray95a] to show that all the gaps are there for $V > 4$. This coding scheme turns also to be useful for studying the transport exponents and the fractal dimensions of the spectrum [Ray95a, KKL03, DEGT08, DG11, LQW14, DG15, CQ23]. Similar coding scheme appears also for the Period doubling sequence [LQY22].

In the current paper, we change the viewpoint from a coding scheme to the boundary of a tree, $\partial\mathcal{T}_\alpha$, and the map E_α. Adding this perspective, we may briefly summarize the different representations of the spectrum by

$$\sigma(H_{\alpha,V}) = \mathcal{B} = \mathcal{Z} = \Pi = E_\alpha\left(\partial\mathcal{T}_\alpha; V\right).$$

This last perspective is a substantial step providing an access to the solution of Dry Ten Martini Problem for Sturmian Hamiltonians. In order to get this spectral representation, the spectral bands of suitable approximations are classified in Theorem 2.15. Towards this, two additional approaches are important:

- We study the space of all finite continued fraction expansions simultaneously allowing us to use a two-level induction.
- We use a perturbation argument leading to an interlacing theorem of the spectral band edges.

These two approaches are introduced in Section 2 respectively Section 3 and further developed throughout the paper. Sections 4, 5 and 6 gradually develop the proof of Theorem 2.15(the structure of these sections is explained at the end of Section 3). Combining these tools, the main Theorems 1.7 and 1.9 are proven in Section 7.

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2. The A/B type classification theorem for the spectral bands

In the previous section we proved that all the gaps are there (Theorem 1.1) using the spectral α-tree $\mathcal{T}_\alpha$ and its properties, as given in Theorems 1.7 and 1.9. The current section develops some important notions and culminates with the statement of Theorem 2.15. This statement is the main tool, which is needed to prove Theorems 1.7 and 1.9.

2.1. **The space $\mathcal{C}$ of finite continued fraction expansions.** In the previous section we recognized the importance of the rational approximants, $H_{\alpha_k,V}$ and their spectra $\sigma_{\alpha_k}(V)$. We also observed the significant role played by the continued fraction expansion of α_k,

$$\alpha_k = c_0 + \cfrac{1}{c_1 + \cfrac{1}{\ddots + \frac{1}{c_k}}} \in [0,1]\,.$$

We write the sequence of coefficients above as a tuple $\mathbf{c} := [0, c_0, c_1, \ldots, c_k]$. Writing it in this form implies $c_{-1} = c_0 = 0$. That $c_0 = 0$ is clear from $\alpha_k \in [0,1]$, whereas the role of the additional entry $c_{-1} = 0$ will become clear later. We merely remark that $c_{-1} = 0$ corresponds to level $k = -1$ (i.e., the root vertex) of the spectral α-tree $\mathcal{T}_\alpha$. Next, we change our point of view. Rather than focusing on a single sequence of rational approximations, $\{\alpha_k\}$, of a particular $\alpha \notin \mathbb{Q}$, we need to consider the whole space of finite continued fraction expansions, as defined next.

Definition 2.1. [Space of finite continued fraction expansions]

Let $\mathbb{N}_{-1} := \mathbb{N} \cup \{-1, 0\}$ and define the space of *finite continued fraction expansions* by

$$\mathcal{C} := \{[0],\ [0,0]\} \cup \bigcup_{k\in\mathbb{N}} \{[0,0,c_1,\ldots,c_k] \,:\, c_1,\ldots,c_{k-1} \in \mathbb{N},\ c_k \in \mathbb{N}_{-1}\}\,.$$

This notation uses the convention that the two first entries of all $\mathbf{c} \in \mathcal{C}$, satisfy $c_{-1} = c_0 = 0$. Additionally, denote

$$[\mathbf{c}, m] := [0, 0, c_1, \ldots, c_k, m], \qquad m \in \mathbb{N}_{-1}.$$

This notation is used only when $[\mathbf{c}, m] \in \mathcal{C}$. To assure this, we assume when using the notation $[\mathbf{c}, m]$ that either $\mathbf{c} = [0,0]$ or $\mathbf{c} = [0,0,c_1,\ldots,c_k] \in \mathcal{C}$ with $c_k \notin \{-1, 0\}$.

The connection between the finite continued fraction expansions and the rational numbers is as follows.

Definition 2.2. The *evaluation map* $\varphi : \mathcal{C} \to \mathbb{R} \cup \{\infty\}$ is defined for all $\mathbf{c} \in \mathcal{C} \setminus \{[0], [0,0,-1]\}$ by

$$\varphi([0,c_0,c_1,\ldots,c_k]) := \begin{cases} \varphi([0,c_0,c_1,\ldots,c_{k-2}]), & k \in \mathbb{N} \text{ and } c_k = 0, \\ \varphi([0,c_0,c_1,\ldots,c_{k-2},c_{k-1}-1]), & k \in \mathbb{N}\setminus\{1\} \text{ and } c_k = -1, \\ c_0 + \cfrac{1}{c_1 + \cfrac{1}{\ddots + \frac{1}{c_k}}}, & \text{otherwise.} \end{cases} \tag{2.1}$$

In addition to that we set $\varphi([0]) := \infty$ and $\varphi([0,0,-1]) := -1$.

Remark.

(a) By the definition of the evaluation map φ we have

$$\varphi([0,c_0,\ldots,c_{k-1},0]) = \varphi([0,c_0,\ldots,c_{k-2}]).$$

This identity may be intuitively understood if one allows c_k to take real values and then consider the limit $c_k \to 0$, see further discussion in [BBB⁺, sec. 2.1].

(b) Observe that Image(φ) $\subseteq$ ($\mathbb{Q} \cap [0,1]) \cup \{-1, \infty\}$. The values -1 and ∞ deserve a special treatment and this is done in the forthcoming statements, see also [BBB$^+$, sec. 2.1] Currently, just note that $\varphi(\mathbf{c}) = -1 \Leftrightarrow \mathbf{c} = [0,0,-1]$, and

$$\varphi(\mathbf{c}) = \infty \quad \Leftrightarrow \quad \mathbf{c} \in \{[0],\ [0,0,0],\ [0,0,1,-1]\}.$$

In Section 1, we fixed some $\alpha \in [0,1] \setminus \mathbb{Q}$ and considered all its finite continued fraction expansions, giving rise to $\alpha_k = \frac{p_k}{q_k}$. From this section and later on we consider the space of all rational numbers represented by their finite continued fraction expansions, $\mathbf{c} \in \mathcal{C}$. It should be noted that the evaluation map φ is surjective but not injective.

2.2. **The spectra** $\{\sigma_{\mathbf{c}}\}_{\mathbf{c} \in \mathcal{C}}$ **.** We present the well-known formalism for transfer matrices, which describes the rational approximants spectra $\sigma_{\alpha_k}(V)$. For $V \in \mathbb{R}$, define

$$M_{[0]}(E,V) := \begin{pmatrix} 1 & -V \\ 0 & 1 \end{pmatrix}, \qquad M_{[0,0]}(E,V) := \begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix}$$

and recursively define the transfer matrices for $\mathbf{c} = [0,0,c_1,\ldots,c_k] \in \mathcal{C}$ (where $k \in \mathbb{N}$) by

$$M_{\mathbf{c}}(E,V) := M_{[0,0,c_1\ldots,c_{k-2}]}(E,V) M_{[0,0,c_1,\ldots,c_{k-1}]}(E,V)^{c_k}.$$

Consequently, denote the traces of the transfer matrices by

$$t_{\mathbf{c}}(E,V) := tr(M_{\mathbf{c}}(E,V)). \tag{2.2}$$

Example 2.3. We explicitly write here the expressions of a few traces, which will turn to be useful in the sequel. We have

$$t_{[0,0,1,-1]}(E,V) = t_{[0]}(E,V) = t_{[0,0,0]}(E,V) = 2,$$

and

$$\begin{aligned} t_{[0,0]}(E,V) &= E, & t_{[0,0,-1]}(E,V) &= E + V, \\ t_{[0,0,1]}(E,V) &= E - V, & t_{[0,0,2]}(E,V) &= E^2 - EV - 2, \\ t_{[0,0,1,2]}(E,V) &= E^3 - 2E^2V + E(V^2 - 3) + 2V, & t_{[0,0,3]}(E,V) &= E^3 - E^2V - 3E + V. \end{aligned}$$

Representing the spectra via the traces of these transfer matrices is a classical approach [Cas86, Süt87, BIST89, Ray95a]. Our description only slightly deviates from the conventional one, by referring to all the elements of $\mathcal{C}$ (within the literature above we take a route which is the closest to [Ray95a]). This approach is expressed in the next definition and proposition.

Definition 2.4. For all $V \in \mathbb{R}$, and $\mathbf{c} \in \mathcal{C}$ denote

$$\sigma_{\mathbf{c}}(V) := \{E \in \mathbb{R} : |t_{\mathbf{c}}(E,V)| \leq 2\}.$$

Using this notation, we bring the following well-known fact, see e.g. [BBB$^+$, prop. 3.5, lem. 3.6].

Lemma 2.5. . *For all* $\boldsymbol{c}, \widetilde{\boldsymbol{c}} \in \mathcal{C}$ *with* $\varphi(\widetilde{\boldsymbol{c}}) = \varphi(\boldsymbol{c})$, *we have*

$$\sigma_{\widetilde{\boldsymbol{c}}}(V) = \sigma_{\boldsymbol{c}}(V) \quad \text{and} \quad t_{\widetilde{\boldsymbol{c}}}(E,V) = t_{\boldsymbol{c}}(E,V), \quad \text{for all } E, V \in \mathbb{R}.$$

Furthermore, if $\boldsymbol{c} \in \mathcal{C}$ *with* $\varphi(\boldsymbol{c}) \notin \{-1, \infty\}$, *then the spectrum* $\sigma_{\varphi(\boldsymbol{c})}(V)$ *of the operator* $H_{\varphi(\boldsymbol{c}),V}$ *satisfies*

$$\sigma_{\boldsymbol{c}}(V) = \sigma_{\varphi(\boldsymbol{c})}(V).$$

The different spectra $\sigma_{\mathbf{c}}$ for the traces given in Example 2.3 are plotted in Figure 2.1.

2.3. **Classification of spectral bands in $\sigma_{\mathbf{c}}(V)$ to A and B types.** We classify the spectral bands in $\sigma_{\mathbf{c}}(V)$ into types. Towards this, we introduce a partial order relations on the spectral bands of the spectra $\{\sigma_{\mathbf{c}}(V)\}_{\mathbf{c}\in\mathcal{C}}$, similarly to the ones which were already introduced in Definition 1.6.

Definition 2.6. Let $I \subseteq \mathbb{R}$ be a closed interval. We define its left and right endpoints by

$$L(I) := \inf_{x\in I} x \quad \text{respectively} \quad R(I) := \sup_{x\in I} x.$$

For two closed intervals I and J define the following order relations.

(a) The interval I *is* contained *in* J:

$$I \subseteq J \quad \Leftrightarrow \quad L(J) \leq L(I) < \mathrm{R(I)} \leq \mathrm{R(J)}.$$

(b) The interval I *is* strictly contained *in* J:

$$I \subseteq_{\mathrm{str}} J \quad \Leftrightarrow \quad L(J) < L(I) < \mathrm{R(I)} < \mathrm{R(J)}.$$

(c) The interval I *is* to the left of J *(respectively* J *is* to the right of I*):*

$$I \prec J \quad \Leftrightarrow \quad L(I) < L(J) \text{ and } \mathrm{R(I)} < \mathrm{R(J)}.$$

(d) The interval I *is* strictly to the left of J *(respectively* J *is* strictly to the right of I*):*

$$I \prec_{\mathrm{str}} J \quad \Leftrightarrow \quad R(I) < L(J).$$

Remark. For closed intervals I, J we have that $I \subseteq_{\mathrm{str}} J$ implies $I \subseteq J$ and $I \prec_{\mathrm{str}} J$ implies $I \prec J$. Note also that I is *strictly* to the left of J if and only if $I \prec J$ and $I \cap J = \emptyset$. In the special case of the closed interval $J = (-\infty, +\infty)$, we have $L(J) = -\infty$ and $R(J) = \infty$ and hence $I \subseteq_{\mathrm{str}} J$, for any compact interval I.

One might compare Definition 2.6 to Definition 1.6, where similar notation and naming were used. Note that in Definition 1.6 the spectral bands are considered as maps, whereas in Definition 2.6 they are considered as intervals, i.e., for a fixed value of V. A certain relation ($\subseteq_{\mathrm{str}}$, or $\prec$) holds between two spectral bands in the sense of Definition 1.6 if and only if it holds for all $V > 0$ in the sense of Definition 2.6. Although we use the same symbols in both definitions, it will be either clear from the context or explicitly mentioned whether it is associated to a fixed interval (as in Definition 2.6) or to a map (as in Definition 1.6).

The order relations for intervals are used for the following classification of spectral bands.

Definition 2.7.

Let $V \in \mathbb{R}$ and $\mathbf{c} \in \mathcal{C}$ be such that $\varphi(\mathbf{c}) \notin \{-1, \infty\}$ and $[\mathbf{c}, 0], [\mathbf{c}, -1] \in \mathcal{C}$. A spectral band $I(V)$ of $\sigma_{\mathbf{c}}(V)$ is called

- *backward type A*
  if there exists a spectral band $J(V)$ in $\sigma_{[\mathbf{c},0]}(V)$ such that $I(V) \subseteq_{\mathrm{str}} J(V)$.
- *weak backward type A*
  if there exists a spectral band $J(V)$ in $\sigma_{[\mathbf{c},0]}(V)$ such that $I(V) \subseteq J(V)$.
- *backward type B*
  if there exists a spectral band $J(V)$ in $\sigma_{[\mathbf{c},-1]}(V)$ such that $I(V) \subseteq_{\mathrm{str}} J(V)$.
- *weak backward type B*
  if there exists a spectral band $J(V)$ in $\sigma_{[\mathbf{c},-1]}(V)$ such that $I(V) \subseteq J(V)$.

Remark 2.8. The notations A and B are deliberately used above, since they carry a similar meaning as the labels A and B introduced for the spectral α-tree, $\mathcal{T}_\alpha$, in Definition 1.4. The bijection in Theorem 1.7 allows to assign the labels A or B to any spectral band according to its vertex in $\mathcal{T}_\alpha$. More precisely, we see later in Proposition 7.1 and Proposition 7.19 that

- A spectral band $I(V)$ of $\sigma_{\alpha_k}(V)$ is of backward type A for all $V > 0$ if and only $\Psi^{-1}(I)$ is labeled A.
- A spectral band $I(V)$ of $\sigma_{\alpha_k}(V)$ is of backward type B for all $V > 0$ if and only $\Psi^{-1}(I)$ is labeled B.

We note that according to the definition above, whether a spectral band is a (weak) backward type A or B (or not at all) depends on the value of V. In other words, the definition above considers the spectral bands as intervals and not as maps. We will eventually state in Theorem 2.15 that each spectral band has exactly one type (either backward type A or B) and that this property is fixed for all $V > 0$ (and fixed for all $V < 0$). There is no merit in considering the backward type properties for $V = 0$, as in this case all spectra of all operators $H_{\alpha,V}$ are equal to $[-2, 2]$. Finally, we note that we adopted this notation of A and B for spectral bands from [KKL03], where it appeared for the specific case $\alpha = \frac{\sqrt{5}-1}{2}$ (for visual reasons we do not use the II, III notation as in [Ray95a]).

The following example continues Example 2.3 and demonstrates the backward types of some spectral bands (see also Figure 2.1).

Example 2.9. Using Example 2.3, we get for all $V \in \mathbb{R}$ that

$$\sigma_{[0]}(V) = \sigma_{[0,0,0]}(V) = \sigma_{[0,0,1,-1]}(V) = \mathbb{R} \quad \text{and} \quad \sigma_{[0,0,-1]}(V) = [-2 - V, 2 - V] = \sigma\left(H_{1,-V}\right).$$

As a byproduct of this calculation we get that Lemma 2.5 may be actually extended to include also $\varphi(\mathbf{c}) = \infty$, since we defined $\sigma_\infty(V) = \mathbb{R}$ in (1.6). Now, consider $\sigma_{[0,0]}(V) = [-2, 2]$. Given the spectra above, one may verify that for all $V \neq 0$ the spectral band $[-2, 2]$ is of backward type A but not of weak backward type B.

As another demonstration, calculate the spectra

$$\sigma_{[0,0,1]}(V) = [-2 + V, 2 + V],\ \sigma_{[0,0,1,-1]}(V) = \mathbb{R} \text{ and } \sigma_{[0,0,1,0]}(V) = \sigma_{[0,0]}(V) = [-2, 2],$$

and observe that for all $V \neq 0$ the spectral band $I_{[0,0,1]}(V) = [-2 + V, 2 + V]$ of $\sigma_{[0,0,1]}(V)$ is of backward type B but not of weak backward type A.

We have seen in Lemma 2.5 that $\sigma_{\widetilde{\mathbf{c}}}(V) = \sigma_{\mathbf{c}}(V)$ for $\mathbf{c}, \widetilde{\mathbf{c}} \in \mathcal{C}$ with $\varphi(\mathbf{c}) = \varphi(\widetilde{\mathbf{c}})$. Nevertheless, we emphasize that the backward type of a spectral band $I(V)$ of $\sigma_{\mathbf{c}}(V)$ depends on $\mathbf{c} \in \mathcal{C}$ and not on its evaluation $\varphi(\mathbf{c}) \in [0, 1]$. This is demonstrated in the next proposition, which is also useful later in the paper.

Lemma 2.10. *Let $V \in \mathbb{R}$ and $\mathbf{c} \in \mathcal{C}$ with $\varphi(\mathbf{c}) \notin \{-1, \infty\}$ and $[\mathbf{c}, m] \in \mathcal{C}$ for all $m \in \mathbb{N}_{-1}$. For $m \geq 2$, we have $\sigma_{[\mathbf{c},m]}(V) = \sigma_{[\mathbf{c},m-1,1]}(V)$. Moreover, if $I(V)$ is a spectral band in $\sigma_{[\mathbf{c},m]}(V) = \sigma_{[\mathbf{c},m-1,1]}(V)$, then both of the following hold*

- *$I(V)$ is of (weak) backward type A in $\sigma_{[\mathbf{c},m]}(V)$ if and only if $I(V)$ is of (weak) backward type B in $\sigma_{[\mathbf{c},m-1,1]}(V)$.*
- *$I(V)$ is of (weak) backward type B in $\sigma_{[\mathbf{c},m]}(V)$ if and only if $I(V)$ is of (weak) backward type A in $\sigma_{[\mathbf{c},m-1,1]}(V)$.*

Proof. If $m \geq 2$, then $\varphi([\mathbf{c}, m]) = \varphi([\mathbf{c}, m - 1, 1])$ follows by the definition of the evaluation map (this is actually a well-known duality for finite continued fraction expansions [Khi64, Ch. I.4]). Now, $\sigma_{[\mathbf{c},m]}(V) = \sigma_{[\mathbf{c},m-1,1]}(V)$ follows from Lemma 2.5. We suppress the V dependence in the rest of the proof, as it does not affect the arguments.

Let I be a spectral band in $\sigma_{[\mathbf{c},m]} = \sigma_{[\mathbf{c},m-1,1]}$. By definition, I is of backward type A in $\sigma_{[\mathbf{c},m]}$ if and only if it is strictly contained in a spectral band of $\sigma_{[\mathbf{c},m,0]} = \sigma_{\mathbf{c}}$ (where we used $\varphi([\mathbf{c}, m, 0]) = \varphi(\mathbf{c})$ and Lemma 2.5). On the other hand, I is of backward type B in $\sigma_{[\mathbf{c},m-1,1]}$ if and only if it is strictly contained in a spectral band of $\sigma_{[\mathbf{c},m-1,1,-1]} = \sigma_{\mathbf{c}}$ (where we used

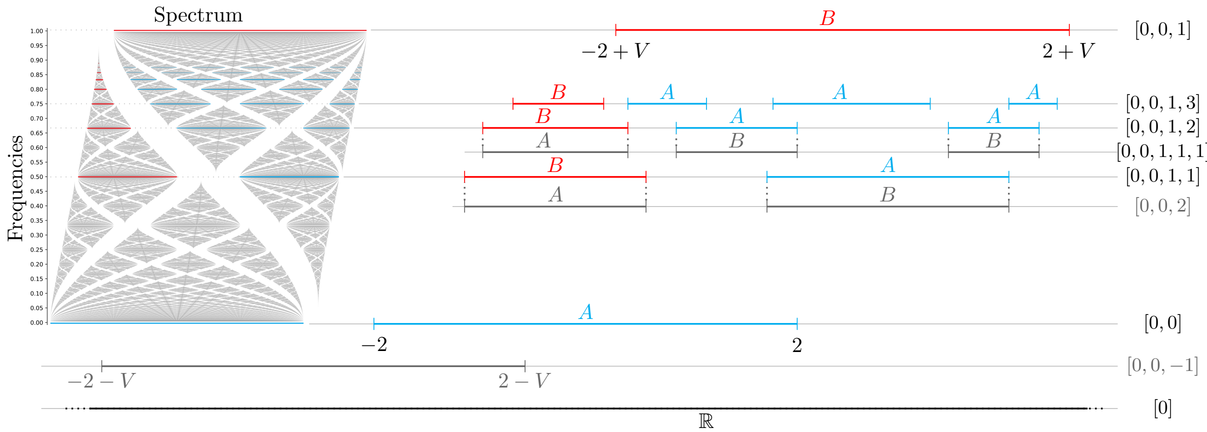

FIGURE 2.1. A plot of various spectra for different $\mathbf{c} \in \mathcal{C}$. The spectral bands are colored according to their backward types (A in blue and B in red). The embedding of these spectral bands within the Kohmoto butterfly is highlighted. The reader is referred to Example 2.9, Example 2.11 and Example 2.13 for a detailed description.

$\varphi([\mathbf{c}, m-1, 1, -1]) = \varphi([\mathbf{c}, m-1, 0]) = \varphi(\mathbf{c})$ and Lemma 2.5). This proves the first equivalence and the second one follows similarly. □

The last proposition demonstrates why it is advantageous to use the space $\mathcal{C}$ and not just rational numbers. Here it is observed from the classification of backward types, but this will also be evident from other perspectives to be seen in the sequel.

Example 2.11. To demonstrate Lemma 2.10, see Figure 2.1 where the spectra for $\varphi([0, 0, 2]) = \varphi([0, 0, 1, 1])$ and $\varphi([0, 0, 1, 2]) = \varphi([0, 0, 1, 1, 1])$ are sketched. Depending on the representation of the finite continued fraction expansion their backward type changes (see color marking vs. gray marking). The colored bands are plotted at a height which corresponds to their $\varphi(\mathbf{c})$ value. Note that $0, 1 \in [0, 1]$ admit a unique representation as a continued fraction expansion and henceforth their backward type is fixed, see Example 2.3. The spectral bands are colored in blue if they are of backward type A and are colored in red if they are of backward type B.

Next, we introduce the concept of forward types A and B. We will eventually prove that the backward and forward types properties are equivalent, see Proposition 7.19.

Definition 2.12. Let $V \in \mathbb{R}$. Let $\mathbf{c} \in \mathcal{C}$ and $m \in \mathbb{N}$ be such that $\varphi(\mathbf{c}) \notin \{-1, \infty\}$ and $[\mathbf{c}, m] \in \mathcal{C}$. A spectral band $I_{\mathbf{c}}(V)$ of $\sigma_{\mathbf{c}}(V)$ is called of *m-forward type* A with $M = m - 1$ (respectively *m-forward type* B with $M = m$) if the following holds.

(A) There exist M spectral bands in $\sigma_{[\mathbf{c},m]}(V)$ (denoted $I^1_{[\mathbf{c},m]}(V), \ldots, I^M_{[\mathbf{c},m]}(V)$) such that

- (A1) $I^i_{[\mathbf{c},m]}(V) \subseteq_{\mathrm{str}} I_{\mathbf{c}}(V)$ for all $1 \leq i \leq M$. In particular, these bands are of backward type A.
- (A2) $I^i_{[\mathbf{c},m]}(V)$ is not of weak backward type B for all $1 \leq i \leq M$.

(B) For each $n \in \mathbb{N}$, there exist $M+1$ spectral bands in $\sigma_{[\mathbf{c},m,n]}(V)$ (denoted $I^1_{[\mathbf{c},m,n]}(V), \ldots, I^{M+1}_{[\mathbf{c},m,n]}(V)$) such that

- (B1) $I^j_{[\mathbf{c},m,n]}(V) \subseteq_{\mathrm{str}} I^j_{[\mathbf{c},m,n-1]}(V)$ for all $1 \leq j \leq M+1$, where $I^j_{[\mathbf{c},m,0]}(V) := I_{\mathbf{c}}(V)$. In particular, these bands are of backward type B.

(B2) $I^j_{[\mathbf{c},m,n]}$ is not of weak backward type A for all $1 \le j \le M+1$.

(I) For each $n \in \mathbb{N}$, we have

$$I^1_{[\mathbf{c},m,n]} \prec I^1_{[\mathbf{c},m]} \prec I^2_{[\mathbf{c},m,n]} \prec I^2_{[\mathbf{c},m]} \dots \prec I^M_{[\mathbf{c},m]} \prec I^{M+1}_{[\mathbf{c},m,n]}.$$

Example 2.13. In Example 2.9 we saw that $I_{[0,0]} = [-2,2]$ is of backward type A and $I_{[0,0,1]} = [-2+V, 2+V]$ is of backward type B. We demonstrate here their forward type as well, by means of Figure 2.1. For $I_{[0,0]} = [-2,2]$, we take $\mathbf{c} = [0,0]$, $m = 1$ and $M = m - 1$. We notice that $I_{[0,0]}$ contains $M = 0$ spectral bands in $\sigma_{[0,0,1]}$ (which fits Definition 2.12, (A)). In addition, taking $n = 1,2,3$ we see (in Figure 1.2) that $I_{[0,0]}$ contains exactly $M+1 = 1$ band from each of $\sigma_{[0,0,1,1]}$, $\sigma_{[0,0,1,2]}$ and $\sigma_{[0,0,1,3]}$ (Definition 2.12, (B)). These three spectral bands are a nested sequence and in particular are of backward type B (Definition 2.12, (B1)). Property (B1) is given the name *tower* property, since these spectral bands are stacked on each other (as visually seen in the Kohmoto butterfly within Figure 2.1). To conclude, we have demonstrated that $I_{[0,0]} = [-2,2]$ is of 1-forward type A (to verify this one has to check all values $n \in \mathbb{N}$), see Lemma 5.3.

For $I_{[0,0,1]} = [-2+V, 2+V]$ we take $\mathbf{c} = [0,0,1]$, $m = 1$ and $M = m$. Then there is $M = 1$ spectral band in $\sigma_{[0,0,1,1]}$ which is contained in $I_{[0,0,1]}$ and denoted by $I^1_{[0,0,1,1]}$ (property (A1)). For $n = 1$, there are $M + 1 = 2$ spectral bands in $\sigma_{[0,0,1,1,1]}$ which are contained in $I_{[0,0,1]}$ and denoted by $I^1_{[0,0,1,1,1]}$,$I^2_{[0,0,1,1,1]}$ (property (B1)). We see in Figure 2.1 that these three bands interlace, i.e., $I^1_{[0,0,1,1,1]} \prec I^1_{[0,0,1,1]} \prec I^2_{[0,0,1,1,1]}$ (property (I)). To conclude, we have demonstrated that $I_{[0,0,1]}$ is of 1-forward type B (to verify this one has to check all values $n \in \mathbb{N}$), see Lemma 5.4.

Finally, the notions above are combined to define the type A and B spectral bands and state their dichotomy, which is another main theorem of the current paper.

Definition 2.14. Let $V \in \mathbb{R}$. Let $\mathbf{c} \in \mathcal{C}$ and $m \in \mathbb{N}$ such that $\varphi(\mathbf{c}) \notin \{-1, \infty\}$ and $[\mathbf{c}, m] \in \mathcal{C}$. A spectral band $I_{\mathbf{c}}(V)$ of $\sigma_{\mathbf{c}}(V)$ is called

- *m-type A* if $I_{\mathbf{c}}(V)$ is of backward type A and of m-forward type A.
- *m-type B* if $I_{\mathbf{c}}(V)$ is of backward type B and of m-forward type B.
- *type A* if $I_{\mathbf{c}}(V)$ is of m-type A for all $m \in \mathbb{N}$.
- *type B* if $I_{\mathbf{c}}(V)$ is of m-type B for all $m \in \mathbb{N}$.

There are equivalent formulations to the definition of A, B types (used in [BBL23]). Their equivalence is stated and proven in Proposition 7.19.

Theorem 2.15. *For all $V > 0$ and $\mathbf{c} \in \mathcal{C}$ with $\varphi(\mathbf{c}) \notin \{-1, \infty\}$, every spectral band in $\sigma_{\mathbf{c}}(V)$ is either of type A or B and its type is independent of the value of $V > 0$.*

Remark. We point out that a similar statement holds for $V < 0$; each spectral band in $\sigma_{\mathbf{c}}(V)$ is either of type A or B, see Corollary 7.7.

Theorem 2.15 is proven in Sections 3, 4, 5 and 6. Then this theorem is used in Section 7 as a main tool for proving Theorems 1.7 and Theorem 1.9.

The first step towards the proof of Theorem 2.15 is the following substantial result of Raymond which appeared nearly three decades ago.

Theorem 2.16. [Ray95a] *For all $V > 4$ and $\mathbf{c} \in \mathcal{C}$ with $\varphi(\mathbf{c}) \notin \{-1, \infty\}$, every spectral band in $\sigma_{\mathbf{c}}(V)$ is either of type A or B and its type is independent of the value of $V > 4$.*

Moreover, for a spectral band $I_{\mathbf{c}}(V)$ and $m, n \in \mathbb{N}$, the spectral bands $I^i_{[\mathbf{c},m]}(V)$ and $I^j_{[\mathbf{c},m,n]}(V)$ introduced in the forward property (A) and (B) are unique for $V > 4$, i.e. $I_{\mathbf{c}}(V)$ does not contain any other spectral band of $\sigma_{[\mathbf{c},m]}(V)$ respectively $\sigma_{[\mathbf{c},m,n]}(V)$.

We took liberty with phrasing Theorem 2.16 differently than it originally appeared in [Ray95a] (it actually did not appear there as a single theorem). In particular, the notation used in [Ray95a] is different than ours; we had to adapt the notation for the sake of our proofs. We have done such an adaptation already in [BBB$^+$, thm. 4.22], as a preliminary step towards the current paper.

Furthermore, we point out that for $V > 4$, the interlacing property (I) in Definition 2.12 may be replaced by

$$I^1_{[\mathbf{c},m,n]}(V) \prec_{\text{str}} I^1_{[\mathbf{c},m]}(V) \prec_{\text{str}} I^2_{[\mathbf{c},m,n]}(V) \prec I^2_{[\mathbf{c},m]}(V) \ldots \prec_{\text{str}} I^M_{[\mathbf{c},m]}(V) \prec_{\text{str}} I^{M+1}_{[\mathbf{c},m,n]}(V),$$

for all $n \in \mathbb{N}$. This yields a stricter definition of types A and B and Theorem 2.16 actually holds also for this stricter version (see further details in [BBB$^+$]). However, when $V < 4$ the spectral bands in this interlacing property overlap and are ordered merely by $\prec$ and not by $\prec_{\text{str}}$. Hence, our Theorem 2.15 (which is valid for all $V > 0$) does not apply to this stricter definition of types A and B. This explains why we had to use the milder version. The existence of such overlaps forms one of the major difficulties one needs to overcome in order to get from Theorem 2.16 to Theorem 2.15.

3. Basic spectral analysis - preliminary tools for the proofs

In the previous section, we have already introduced one classical tool to study the spectral approximations - the transfer matrices and their traces. Two additional tools are developed in this section for the spectral band edges: a uniform Lipschitz continuity and an interlacing theorem.

3.1. **Lipschitz continuity of spectral band edges.** As introduced in Definition 1.5, we view spectral bands as maps and next we show that these maps are Lipschitz continuous.

Lemma 3.1. *Let $\alpha \in [0,1]$. For all $V, V' \in \mathbb{R}$,*

$$d_H\big(\sigma(H_{\alpha,V}), \sigma(H_{\alpha,V'})\big) \leq |V - V'|,$$

where d_H denotes the Hausdorff metric on the compact subsets of $\mathbb{R}$ induced by the Euclidean distance, $d_H(X,Y) := \max\big\{\sup_{x\in X} d(x,Y),\ \sup_{y\in Y} d(y,X)\big\}$.

Proof. The statement follows immediately by standard arguments using the operator norm estimate $\|H_{\alpha,V} - H_{\alpha,V'}\| \leq |V - V'|$. □

Taking $\mathbf{c} \in \mathcal{C}$ with $\{-1, \infty\} \not\ni \varphi(\mathbf{c}) = \frac{p}{q}$ (coprime p, q), we know that for all $V \neq 0$, $\sigma_{\mathbf{c}}(V)$ consists of exactly q non-touching intervals (see Propositions 1.2 and 2.5). As a direct consequence from Lemma 3.1, we get the following for each of these spectral bands.

Corollary 3.2. *Let $\mathbf{c} \in \mathcal{C}$ such that $\varphi(\mathbf{c}) \notin \{-1, \infty\}$. If $I_{\mathbf{c}}$ is a spectral band of $\sigma_{\mathbf{c}}$, then for all $V, V' > 0$*

$$\max\big\{|L(I_{\mathbf{c}}(V)) - L\big(I_{\mathbf{c}}(V')\big)|, |R(I_{\mathbf{c}}(V)) - R\big(I_{\mathbf{c}}(V')\big)|\big\} \leq |V - V'|.$$

Remark. Thanks to Corollary 3.2 we may view each spectral band $I_{\mathbf{c}}$ in $\sigma_{\mathbf{c}}$ as a continuous map $V \mapsto I_{\mathbf{c}}(V)$. Combined with Raymond's Theorem 2.16 this allows to introduce the following notions.

Definition 3.3. Let $m, n \in \mathbb{N}$, $\mathbf{c} \in \mathcal{C}$ be such that $\varphi(\mathbf{c}) \notin \{-1, \infty\}$ and $[\mathbf{c}, m] \in \mathcal{C}$. For a spectral band $I_{\mathbf{c}}$ in $\sigma_{\mathbf{c}}$ define the associated unique value

$$M := \begin{cases} m-1, & I_{\mathbf{c}}(V) \text{ is of backward type } A \text{ for all } V > 4, \\ m, & I_{\mathbf{c}}(V) \text{ is of backward type } B \text{ for all } V > 4, \end{cases}$$

and the unique spectral bands $\left\{I^i_{[\mathbf{c},m]}\right\}_{i=1}^{M}$ of $\sigma_{[\mathbf{c},m]}$ and the unique spectral bands $\left\{I^j_{[\mathbf{c},m,n]}\right\}_{j=1}^{M+1}$ of $\sigma_{[\mathbf{c},m,n]}$ satisfying (A), (B) and (I) for all $V>4$.

Note that M actually depends both on $I_{\mathbf{c}}$ and m, but we omit this dependence from the notation.

Remark. The existence and uniqueness of the spectral bands $\left\{I^i_{[\mathbf{c},m]}\right\}_{i=1}^{M}$ and $\left\{I^j_{[\mathbf{c},m,n]}\right\}_{j=1}^{M+1}$ are justified by Theorem 2.16 proven by Raymond [Ray95a], see also [BBB$^+$, thm. 4.22]. Due to Corollary 3.2, we may consider the continuous maps $V\mapsto I^i_{[\mathbf{c},m]}(V)$ and $V\mapsto I^j_{[\mathbf{c},m,n]}(V)$ on $V\in(0,\infty)$. However, it is not a-priori clear whether these spectral bands still satisfy the properties (A), (B) and (I) in Definition 2.12 for $0<V\leq 4$. It turns to be so by Theorem 2.15, but we cannot use it prior to proving this theorem.

A word of caution is needed regarding the notation in Definition 3.3. In order to know to which spectral band the notation $I^i_{[\mathbf{c},m]}$ refers to within $\sigma_{[\mathbf{c},m]}$, one needs to know which spectral band $I_{\mathbf{c}}$ was designated. For different choices of spectral bands $I_{\mathbf{c}}$ within $\sigma_{\mathbf{c}}$, the spectral bands $I^i_{[\mathbf{c},m]}$ and $I^j_{[\mathbf{c},m,n]}$ will also differ. This should not lead to confusion, since in the beginning of each proof or discussion, the spectral band $I_{\mathbf{c}}$ will be explicitly chosen.

3.2. **Interlacing theorem for spectral band edges.** Given an $n\times n$ matrix H and $\theta\in[0,2\pi]$, define the $n\times n$ matrix

$$H(\theta):=H+e^{-i\theta}\mathbb{I}_{1,n}+e^{i\theta}\mathbb{I}_{n,1},$$

where $\mathbb{I}_{i,j}$ denotes the $n\times n$ matrix that has only zeros except at the (i,j)-th entry where it is equal to one.

For $\alpha\in[0,1]$, we use in the following the notation $\omega_\alpha(n):=\chi_{[1-\alpha,1)}(n\alpha \mod 1)$ for the potential. Let $V\in\mathbb{R}$, $\mathbf{c}\in\mathcal{C}$ be such that $\{-1,\infty\}\not\ni\varphi(\mathbf{c})=\frac{p}{q}$ with p,q coprime. Recall the self-adjoint operator $H_{\varphi(\mathbf{c}),V}:\ell^2(\mathbb{Z})\to\ell^2(\mathbb{Z})$ introduced in Equation (1.1). The spectral analysis of $H_{\varphi(\mathbf{c}),V}$ is done via the following related hermitian $q\times q$ matrix

$$H_{\mathbf{c},V}:=H_{\varphi(\mathbf{c}),V}|_{[0,q-1]}=\begin{pmatrix} V\omega_{\varphi(\mathbf{c})}(0) & 1 & 0 & \dots & & & 0\\ 1 & V\omega_{\varphi(\mathbf{c})}(1) & 1 & \dots & & & \\ 0 & 1 & \ddots & & & & \\ \vdots & \ddots & & \ddots & & & 0\\ 0 & & & & & & 1\\ 0 & 0 & \cdots & & 0 & 1 & V\omega_{\varphi(\mathbf{c})}(q-1)\end{pmatrix}.$$

Note the ambiguity in the notation between the operator $H_{\varphi(\mathbf{c}),V}$ on $\ell^2(\mathbb{Z})$ and the $q\times q$ matrix $H_{\mathbf{c},V}$.

Standard Floquet-Bloch theory, [Hoc75], gives

$$\sigma_{\mathbf{c}}(V)=\sigma\left(H_{\varphi(\mathbf{c}),V}\right)=\bigcup_{\theta\in[0,\pi]}\sigma\left(H_{\mathbf{c},V}(\theta)\right). \tag{3.1}$$

We have already mentioned (Proposition 1.2) that $\sigma_{\mathbf{c}}(V)$ consists of exactly q intervals (spectral bands). By standard arguments, the endpoints of these intervals are given by the eigenvalues of $H_{\mathbf{c},V}(0)$ and $H_{\mathbf{c},V}(\pi)$. Hence, the values $\theta\in\{0,\pi\}$ play a significance role in (3.1). In addition, denoting by $\chi_{H_{\mathbf{c},V}(\theta)}$ the characteristic polynomial of the matrix $H_{\mathbf{c},V}(\theta)$, we have (see e.g., [Hoc75, eq. (23)], [Sim11, thm. 5.4.1,(iii)] or [BBB$^+$, Lemma II.2]) that

$$\chi_{H_{\mathbf{c},V}(\theta)}(E)=t_{\mathbf{c}}(E,V)-2\cos(\theta), \tag{3.2}$$

where $t_{\mathbf{c}}(E;V)$ are the traces discussed in the previous section.

The spectral decomposition (3.1) may be also written in terms of the following $nq \times nq$-matrix

$$H^{\times n}_{\mathbf{c},V} := H_{\varphi(\mathbf{c}),V}|_{[0,nq-1]} = \begin{pmatrix} H_{\mathbf{c},V} & \mathbb{I}_{q,1} & 0 & \dots & & 0 \\ \mathbb{I}_{1,q} & H_{\mathbf{c},V} & \mathbb{I}_{q,1} & \dots & & \\ 0 & \mathbb{I}_{1,q} & \ddots & & & \\ \vdots & \ddots & & \ddots & & 0 \\ 0 & & & & & \mathbb{I}_{q,1} \\ 0 & 0 & \cdots & 0 & \mathbb{I}_{1,q} & H_{\mathbf{c},V} \end{pmatrix}.$$

Note that the diagonal of $H^{\times n}_{\mathbf{c},V}$ is an n-times repetition of the diagonal of $H_{\mathbf{c},V}$, which is the minimal period of the potential sequence, $V\omega_{\varphi(\mathbf{c})}(0), \dots, V\omega_{\varphi(\mathbf{c})}(q-1)$. The same spectral decomposition holds for the larger matrix $H^{\times n}_{\mathbf{c},V}$ and the same spectrum is obtained,

$$\sigma_{\mathbf{c}}(V) = \sigma\left(H_{\varphi(\mathbf{c}),V}\right) = \bigcup_{\theta\in[0,\pi]} \sigma\left(H^{\times n}_{\mathbf{c},V}(\theta)\right). \tag{3.3}$$

Now, the eigenvalues of $H^{\times n}_{\mathbf{c},V}(0)$ and $H^{\times n}_{\mathbf{c},V}(\pi)$ appear both as the endpoints of the q spectral bands, but also as interior points within these intervals (a detailed description appears in the proof of Lemma 4.5). On a first sight it might seem unnecessary to consider the larger matrix $H^{\times n}_{\mathbf{c},V}$, but its role is briefly revealed here and then it is extensively used in the next section. By the discussion above, the matrices $H_{\mathbf{c},V}$, $H^{\times n}_{[\mathbf{c},m],V}$ and $H_{[\mathbf{c},m,n],V}$ are useful to describe the spectra $\sigma_{\mathbf{c}}$, $\sigma_{[\mathbf{c},m]}$ and $\sigma_{[\mathbf{c},m,n]}$. By Lemma I.1 the main diagonal of $H_{[\mathbf{c},m,n],V}$ consists of a concatenation of the diagonal of $H_{\mathbf{c},V}$ with the diagonal of $H^{\times n}_{[\mathbf{c},m]}$ (the order of this concatenation depends on the parity of the length of $\mathbf{c}$). Hence, either $H_{[\mathbf{c},m,n],V} = H^{\times n}_{[\mathbf{c},m],V} \oplus H_{\mathbf{c},V}$ or $H_{[\mathbf{c},m,n],V} = H_{\mathbf{c},V} \oplus H^{\times n}_{[\mathbf{c},m],V}$ (depending on that same parity). Furthermore, there exist $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]} \in \{0,\pi\}$ such that the matrices

$$H_{[\mathbf{c},m,n],V}\left(\theta_{[\mathbf{c},m,n]}\right) \text{ and } H^{\times n}_{[\mathbf{c},m],V}\left(\theta_{[\mathbf{c},m]}\right) \oplus H_{\mathbf{c},V}(\theta_{\mathbf{c}})$$

differ by a traceless rank two matrix. This is verified from basic computations in Appendix III. This observation allows us to apply a perturbation theorem in order to get a useful eigenvalue interlacing theorem. Denoting the eigenvalues (counted with multiplicities) of a hermitian $q \times q$ matrix H by

$$\lambda_0(H) \le \lambda_1(H) \le \ldots \le \lambda_{q-1}(H), \tag{3.4}$$

we get the following.

Theorem 3.4 (Interlacing theorem)**.** *Let $V > 0$. Let $m, n \in \mathbb{N}$ and $\boldsymbol{c} \in \mathcal{C}$ be such that $\varphi(\boldsymbol{c}) \notin \{-1, \infty\}$ and $[\boldsymbol{c}, m] \in \mathcal{C}$. Let $\theta_{\boldsymbol{c}}, \theta_{[\boldsymbol{c},m]}, \theta_{[\boldsymbol{c},m,n]} \in \{0,\pi\}$ and denote*

$$Y = H_{[\boldsymbol{c},m,n],V}\left(\theta_{[\boldsymbol{c},m,n]}\right) \quad \textit{and} \quad X = H^{\times n}_{[\boldsymbol{c},m],V}\left(\theta_{[\boldsymbol{c},m]}\right) \oplus H_{\boldsymbol{c},V}(\theta_{\boldsymbol{c}}).$$

If $\theta_{\boldsymbol{c}} + \theta_{[\boldsymbol{c},m]} + \theta_{[\boldsymbol{c},m,n]} \in \{0, 2\pi\}$, then

$$\lambda_{j-1}(Y) \le \lambda_j(X) \le \lambda_{j+1}(Y).$$

Furthermore, if $\lambda_j(X)$ is a simple eigenvalue of X, then both inequalities are strict.

Theorem 3.4 is proven in the Appendix III. Note that even though the eigenvalues depend on the parameter $V > 0$, the inequalities of the eigenvalues hold independently of the value $V > 0$ attains[2]. The condition in the previous theorem naturally leads to the following useful definition.

Definition 3.5 (Admissibility)**.** Let $m, n \in \mathbb{N}$ and $\mathbf{c} \in \mathcal{C}$ be such that $\varphi(\mathbf{c}) \notin \{-1, \infty\}$ and $[\mathbf{c}, m] \in \mathcal{C}$.

[2]Note that simplicity may depend on $V > 0$.

(a) The values $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]} \in \{0,\pi\}$ are called *admissible* if $\theta_{\mathbf{c}} + \theta_{[\mathbf{c},m]} + \theta_{[\mathbf{c},m,n]} \in \{0, 2\pi\}$.

(b) If $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]} \in \{0,\pi\}$ are admissible and

$$\lambda_{\mathbf{c}} \in H_{\mathbf{c},V}(\theta_{\mathbf{c}}),\ \lambda_{[\mathbf{c},m]} \in H^{\times n}_{[\mathbf{c},m],V}(\theta_{[\mathbf{c},m]}),\ \lambda_{[\mathbf{c},m,n]} \in H_{[\mathbf{c},m,n],V}(\theta_{[\mathbf{c},m,n]}),$$

then we call $\lambda_{\mathbf{c}}, \lambda_{[\mathbf{c},m]}, \lambda_{[\mathbf{c},m,n]}$ admissible.

Remark 3.6. We emphasize here that $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}$ are admissible if the triple has an even number of π's. In particular,

$$\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]} \text{ are not admissible} \quad \Leftrightarrow \quad \theta_{\mathbf{c}}, \pi - \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]} \text{ are admissible}$$

With these basic tools at hand, we can start proving Theorem 2.15. This proof spreads over the next three sections. First, Section 4 aims at proving Proposition 4.20, which states that if $I_{\mathbf{c}}$ is of backward type A or B, then it is of forward type A or B. This proposition is used to prove Theorem 2.15 by means of induction. The induction base is shown in Section 5, whereas the induction steps consist of 'vertical' and 'horizontal' steps over the space $\mathcal{C}$, and they are done in Section 6. Since Section 4 and Section 5 contain large parts of somewhat tedious computations, we offer the reader the possibility to skip these sections, read only the statement of Proposition 4.20 and then continue to Section 6. Independently (or in a second reading), it might be insightful to see how the basic ingredients and tools presented here are used in Section 4 to prove Proposition 4.20.

4. Admissibility, index relations and the forward type properties

This section is devoted to various technical tools used in the induction base (Section 5) and to prove that backward type implies forward type - Proposition 4.20.

Let us provide a short overview of this technical section. In Subsection 4.1, we introduce an eigenvalue counting function, which later plays a crucial role in application of the interlacing theorem (Theorem 3.4). Since eigenvalue admissibility is a necessary condition in the interlacing theorem, we give a useful characterization of it in Subsection 4.2. With this at hand, in Subsection 4.3 we provide Lemma 4.7 which is a manifestation of the interlacing theorem (Theorem 3.4). In effect, it is this lemma which is going to be directly applied, rather than Theorem 3.4. In Subsection 4.4, we develop index relations which are needed whenever we apply Lemma 4.7. In Subsection 4.5 we bring some useful trace estimate and in Subsection 4.6, we consider the products of such traces appearing in the Fricke-Vogt invariant and relate them to admissibility. This connection is crucial in order to prove that all spectral gaps are open in Section 7. Then, Subsection 4.7 applies the various index relations, eigenvalue estimates and trace estimates to prove that the spectral bands $I^i_{[\mathbf{c},m]}$ and $I^j_{[\mathbf{c},m,n]}$ maintain certain properties from Definition 2.12 if V decreases to zero. This is finally used in Subsection 4.8 to prove that if the spectral bands are of backward type A, respectively B, then they are also of forward type A, respectively B. This is the main tool to inductively prove Theorem 2.15 for all $V > 0$ in Section 6.

Throughout this section we use the notational conventions of Definition 3.3 without pointing them out all the time.

4.1. **Counting spectral bands and eigenvalues.** In this subsection we consider two types of counting functions: for the spectral bands in $\sigma_{\mathbf{c}}$ and for the eigenvalues of the matrices $H_{\mathbf{c},V}(\theta)$, $H^{\times n}_{\mathbf{c},V}(\theta)$ and relate both types of functions.

First, we recall that $\sigma_{\mathbf{c}}(V)$ consists of exactly q intervals (Proposition 1.2 and Lemma 2.5) and that we consider each spectral band as a Lipschitz continuous map, $V \mapsto I_{\mathbf{c}}(V)$, for $V > 0$ (Definition 1.5 and Lemma 3.1). This, together with Definition 1.6, justifies the following.

Definition 4.1. [Index of a spectral band] Let $I_{\mathbf{c}}$ be a spectral band of $\sigma_{\mathbf{c}}$. The *index* of $I_{\mathbf{c}}$ (in $\sigma_{\mathbf{c}}$) is defined by

$$\operatorname{ind}(I_{\mathbf{c}}) := \left|\{I \text{ is a spectral band of } \sigma_{\mathbf{c}} \,:\, I \prec I_{\mathbf{c}}\}\right|.$$

Remark 4.2. Note that the index counting starts from zero, namely $0 \le \operatorname{ind}(I_{\mathbf{c}}) \le q-1$ where $\varphi(\mathbf{c}) = \frac{p}{q}$ with p, q coprime. Moreover, we emphasize that $\operatorname{ind}(I_{\mathbf{c}})$ is independent of $V > 0$, allowing us to assume $V > 4$ in some instances and use Theorem 2.16.

In order to apply the interlacing theorem (Theorem 3.4), we need to count eigenvalues. Let $\{\lambda_i(H)\}_{i=0}^{n-1}$ be the eigenvalues (increasingly arranged and counted with multiplicity) of an $n \times n$ matrix H, as in (3.4).

Definition 4.3. [Counting function] For an $n\times n$ hermitian matrix H, the *eigenvalue counting function* is defined by

$$N(\lambda; H) := \left|\{0 \le i \le n-1 \,:\, \lambda_i(H) < \lambda\}\right|.$$

Remark. Note that $N(\lambda;\ H)$ may attain the value zero and also that $N(\lambda_i(H);\ H) = i$ for each $0 \le i \le n-1$ where $\lambda_i(H)$ is simple.

We will be in particular interested in evaluating the counting function for an eigenvalue which is also an edge of a certain spectral band. The index of that spectral band is then related to the counting of its edge point, as follows.

Lemma 4.4. *Let $V > 0$, $\boldsymbol{c} \in \mathcal{C}$ and $\{-1, \infty\} \not\ni \varphi(\boldsymbol{c}) = \frac{p}{q}$ with p, q coprime. Let $I_{\boldsymbol{c}}$ be a spectral band of $\sigma_{\boldsymbol{c}}$ and $\theta \in \{0, \pi\}$.*

(a) We have

$$ind(I_{\boldsymbol{c}}) - q \equiv \frac{1}{\pi}\theta \mod 2 \quad \Leftrightarrow \quad L(I_{\boldsymbol{c}}(V)) \in \sigma\left(H_{\boldsymbol{c},V}(\theta)\right)$$

and

$$ind(I_{\boldsymbol{c}}) + 1 - q \equiv \frac{1}{\pi}\theta \mod 2 \quad \Leftrightarrow \quad R(I_{\boldsymbol{c}}(V)) \in \sigma\left(H_{\boldsymbol{c},V}(\theta)\right).$$

(b) If $L(I_{\boldsymbol{c}}(V)) \in \sigma\left(H_{\boldsymbol{c},V}(\theta)\right)$, then

$$ind(I_{\boldsymbol{c}}) = N\left(L(I_{\boldsymbol{c}}(V)); H_{\boldsymbol{c},V}(\theta)\right).$$

(c) If $R(I_{\boldsymbol{c}}(V)) \in \sigma\left(H_{\boldsymbol{c},V}(\theta)\right)$, then

$$ind(I_{\boldsymbol{c}}) = N\left(R(I_{\boldsymbol{c}}(V)); H_{\boldsymbol{c},V}(\theta)\right).$$

Proof. This follows from the next lemma and the fact that $H_{\mathbf{c},V}(\theta) = H^{\times n}_{\mathbf{c},V}(\theta)$ if $n = 1$. ☐

Lemma 4.4 can be generalized as follows.

Lemma 4.5. *Let $V > 0$, $\boldsymbol{c} \in \mathcal{C}$ and $\{-1, \infty\} \not\ni \varphi(\boldsymbol{c}) = \frac{p}{q}$ with p, q coprime. Let $I_{\boldsymbol{c}}$ be a spectral band of $\sigma_{\boldsymbol{c}}$ and $\theta \in \{0, \pi\}$. Then the following holds for $n \in \mathbb{N}$.*

(a) If $n \in \mathbb{N}$ is even, then $L(I_{\boldsymbol{c}}(V)) \in \sigma\left(H^{\times n}_{\boldsymbol{c},V}(0)\right)$ and $R(I_{\boldsymbol{c}}(V)) \in \sigma\left(H^{\times n}_{\boldsymbol{c},V}(0)\right)$.

(b) If $n \in \mathbb{N}$ is odd, then

$$ind(I_{\boldsymbol{c}}) - q \equiv \frac{1}{\pi}\theta \mod 2 \quad \Leftrightarrow \quad L(I_{\boldsymbol{c}}(V)) \in \sigma\left(H^{\times n}_{\boldsymbol{c},V}(\theta)\right)$$

and

$$ind(I_{\boldsymbol{c}}) + 1 - q \equiv \frac{1}{\pi}\theta \mod 2 \quad \Leftrightarrow \quad R(I_{\boldsymbol{c}}(V)) \in \sigma\left(H^{\times n}_{\boldsymbol{c},V}(\theta)\right).$$

(c) *If* $L(I_{\boldsymbol{c}}(V)) \in \sigma\left(H^{\times n}_{\boldsymbol{c},V}(\theta)\right)$, *then*

$$n \cdot ind(I_{\boldsymbol{c}}) = N\left(L(I_{\boldsymbol{c}}(V)); H^{\times n}_{\boldsymbol{c},V}(\theta)\right)$$

and there exists $\lambda \in \sigma\left(H^{\times n}_{\boldsymbol{c},V}(\pi-\theta)\right)$ *such that*

$$L(I_{\boldsymbol{c}}(V)) < \lambda \leq R(I_{\boldsymbol{c}}(V)) \quad \text{and} \quad N\left(\lambda; H^{\times n}_{\boldsymbol{c},V}(\pi-\theta)\right) = N\left(L(I_{\boldsymbol{c}}(V)); H^{\times n}_{\boldsymbol{c},V}(\theta)\right).$$

(d) *If* $R(I_{\boldsymbol{c}}(V)) \in \sigma\left(H^{\times n}_{\boldsymbol{c},V}(\theta)\right)$, *then*

$$n \cdot (ind(I_{\boldsymbol{c}})+1) - 1 = N\left(R(I_{\boldsymbol{c}}(V)); H^{\times n}_{\boldsymbol{c},V}(\theta)\right)$$

and for $n \geq 2$, *there exists* $\lambda \in \sigma\left(H^{\times n}_{\boldsymbol{c},V}(\pi-\theta)\right)$ *such that*

$$L(I_{\boldsymbol{c}}(V)) \leq \lambda < R(I_{\boldsymbol{c}}(V)) \quad \text{and} \quad N\left(\lambda; H^{\times n}_{\boldsymbol{c},V}(\pi-\theta)\right) = N\left(R(I_{\boldsymbol{c}}(V)); H^{\times n}_{\boldsymbol{c},V}(\theta)\right) - 1.$$

(e) *We have* $\left|\left\{\lambda \in \sigma\left(H^{\times n}_{\boldsymbol{c},V}(\theta)\right) \cap I_{\boldsymbol{c}}(V)\right\}\right| = n$ *and if* $\lambda \in \sigma\left(H^{\times n}_{\boldsymbol{c},V}(\theta)\right) \cap \{L(I_{\boldsymbol{c}}(V)), R(I_{\boldsymbol{c}}(V))\}$, *then* λ *is a simple eigenvalue of* $H^{\times n}_{\boldsymbol{c},V}(\theta)$.

Proof. Recall from (3.3) that the spectrum $\sigma_{\mathbf{c}}(V)$ is given as the union of the eigenvalues of $H^{\times n}_{\mathbf{c},V}(\theta)$ over all $\theta \in [0,\pi]$. Denote by $\lambda^{(\theta)}_j := \lambda_j\left(H^{\times n}_{\mathbf{c},V}(\theta)\right)$ for $0 \leq j \leq nq-1$ the eigenvalues of $H^{\times n}_{\mathbf{c},V}(\theta)$ in increasing order counting multiplicities, see (3.4). These eigenvalues for $\theta \in \{0,\pi\}$ are arranged as follows,

$$\ldots \leq \lambda^{(\pi)}_{nq-4} \leq \lambda^{(\pi)}_{nq-3} < \lambda^{(0)}_{nq-3} \leq \lambda^{(0)}_{nq-2} < \lambda^{(\pi)}_{nq-2} \leq \lambda^{(\pi)}_{nq-1} < \lambda^{(0)}_{nq-1}, \tag{4.1}$$

noting that the strict inequalities above appear whenever we compare eigenvalues with different θ values (see e.g. [Hoc75, Eq. (25)]). We use these eigenvalues to recursively define the following intervals

$$\ldots,\ J_l := [\lambda^{(\theta_l)}_l, \lambda^{(\pi-\theta_l)}_l],\ \ldots,\ J_{nq-2} := [\lambda^{(0)}_{nq-2}, \lambda^{(\pi)}_{nq-2}],\ J_{nq-1} := [\lambda^{(\pi)}_{nq-1}, \lambda^{(0)}_{nq-1}],$$

for appropriately chosen $\theta_l \in \{0,\pi\}$. We note that these intervals are ordered, i.e. $J_l \prec J_{l+1}$ for all $0 \leq l \leq nq-2$.

We now make a connection between these intervals, and the spectral bands $I_{\mathbf{c}}$ of $\sigma_{\mathbf{c}}$. By Proposition 1.2 and Lemma 2.5, $\sigma_{\mathbf{c}}(V)$ consists of exactly q disjoint intervals - called spectral bands. For each such spectral band $I_{\mathbf{c}}$ of $\sigma_{\mathbf{c}}$, set $j = \text{ind}(I_{\mathbf{c}})$ and $I_j := I_{\mathbf{c}}(V)$ for the given $V > 0$.

We show a few auxiliary claims, and then use them to prove the statements in the lemma.

(1) For all $0 \leq l \leq nq-1$, the endpoints $L(J_l)$ and $R(J_l)$ correspond to eigenvalues with different θ values. Moreover, $R(J_l)$ and $L(J_{l+1})$ correspond to the same value of $\theta \in \{0,\pi\}$ for all $0 \leq l \leq nq-2$.
(2) The equalities

$$\sigma_{\mathbf{c}}(V) = \bigcup_{j=0}^{q-1} I_j = \bigcup_{l=0}^{nq-1} J_l \quad \text{and} \quad I_{q-1-j} = \bigcup_{l=0}^{n-1} J_{nq-1-nj-l} \quad \text{for all } 0 \leq j \leq q-1$$

hold.
(3) For $\theta \in \{0,\pi\}$, each I_j contains exactly n eigenvalues of $\sigma\left(H^{\times n}_{\mathbf{c},V}(\theta)\right)$.
(4) We have $R(I_{q-1}) = \lambda^{(0)}_{nq-1}$.

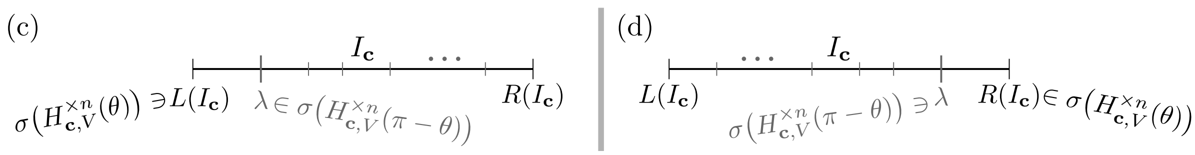

FIGURE 4.1. A sketch for the proof of (c) and (d) in Lemma 4.5.

Claim (1) is immediate from the definition of the intervals J_l. The first equality in (2) follows from (3.1) and (3.3). Thus, each I_j is the union of some of the consecutive intervals J_l. By [Hoc84, Theorem 1] each n consecutive J_l bands touch (so that their union is a single connected component) and this inductively implies the second equality in (2). This also implies (3). To deduce (4) we combine $I_{q-1} = \bigcup_{l=0}^{n-1} J_{nq-1-l}$ (which follows from (2)) with $J_{nq-1} := [\lambda^{(\pi)}_{nq-1}, \lambda^{(0)}_{nq-1}]$.

We now use the claims above to prove the different statements of the lemma.

(a): The claims (1) and (2) for even $n \in \mathbb{N}$ imply that the left and right spectral edges of $I_{\mathbf{c}}(V)$ correspond to the same value $\theta \in \{0, \pi\}$. Combining this with claim (4) implies that all spectral edges of $I_{\mathbf{c}}$ correspond to the value $\theta = 0$.

(b): The claims (1) and (2) for odd $n \in \mathbb{N}$ imply that the left and right spectral edge of $I_{\mathbf{c}}(V)$ correspond to a different value of $\theta \in \{0, \pi\}$. Hence, the value of $\theta \in \{0, \pi\}$ which corresponds to $L(I_j)$ alternates with j (and it also alternates for $R(I_j)$). Combining this with claim (4) yields the statement in (b).

(c) and (d): The first equality in (c) and (d) follows from claim (3). Note that for (d) the quantity $N\left(R(I_{\mathbf{c}}(V)); H^{\times n}_{\mathbf{c},V}(\theta)\right)$ counts $n-1$ eigenvalues in the spectral band $I_{\mathbf{c}}$(V) and n eigenvalues for each spectral band $I(V) \prec I_{\mathbf{c}}(V)$ (which are $\operatorname{ind}(I_{\mathbf{c}})$ many).

We turn to prove the second claim in (c). It follows from claim (2) that there exists $\theta \in \{0, \pi\}$ such that $L(I_{\mathbf{c}}(V)) \in \sigma\left(H^{\times n}_{\mathbf{c},V}(\theta)\right)$. Using the notation for the eigenvalues of $H^{\times n}_{\mathbf{c},V}(\theta)$ introduced in the beginning of the proof, we can write $\lambda^{(\theta)}_l := L(I_{\mathbf{c}}(V))$, for some $0 \le l \le nq-1$. Now, we define

$$\lambda := \lambda^{(\pi-\theta)}_l = \min\left\{\tilde{\lambda} \,:\, \tilde{\lambda} \in \sigma\left(H^{\times n}_{\mathbf{c},V}(\pi-\theta)\right) \quad \text{and} \quad \tilde{\lambda} > \lambda^{(\theta)}_l := L(I_{\mathbf{c}}(V))\right\}$$

(as sketched in Figure 4.1,(c)) and show that this is the desired $\lambda \in \sigma\left(H^{\times n}_{\mathbf{c},V}(\pi-\theta)\right)$ in the statement of (c). By the construction in the beginning of the proof we get that $L(I_{\mathbf{c}}(V)) < \lambda$ and $J_l = [L(I_{\mathbf{c}}(V)), \lambda]$. Furthermore, J_l is the left-most sub-interval within $I_{\mathbf{c}}(V)$, as in the decomposition of claim (2). Hence, $L(I_{\mathbf{c}}(V)) < \lambda \le R(I_{\mathbf{c}}(V))$, as stated in (c). To complete the proof of (c) we just note that $N\left(L(I_{\mathbf{c}}(V)); H^{\times n}_{\mathbf{c},V}(\theta)\right) = l$, just by the choice of $0 \le l \le nq-1$ and similarly $N\left(\lambda; H^{\times n}_{\mathbf{c},V}(\pi-\theta)\right) = l$. Hence, $N\left(\lambda; H^{\times n}_{\mathbf{c},V}(\pi-\theta)\right) = N\left(L(I_{\mathbf{c}}(V)); H^{\times n}_{\mathbf{c},V}(\theta)\right)$.

It is left to prove the second claim in (d). This follows similarly as in (c). First, there exists $\theta \in \{0, \pi\}$ such that $R(I_{\mathbf{c}}(V)) \in \sigma\left(H^{\times n}_{\mathbf{c},V}(\theta)\right)$; we write $\lambda^{(\theta)}_l := R(I_{\mathbf{c}}(V))$, for some $0 \le l \le nq-1$; we define

$$\lambda := \lambda^{(\pi-\theta)}_l = \max\left\{\tilde{\lambda} \,:\, \tilde{\lambda} \in \sigma\left(H^{\times n}_{\mathbf{c},V}(\pi-\theta)\right) \quad \text{and} \quad \tilde{\lambda} < \lambda^{(\theta)}_l = R(I_{\mathbf{c}}(V))\right\}$$

(as sketched in Figure 4.1). Then $L(I_{\mathbf{c}}(V)) \le \lambda < R(I_{\mathbf{c}}(V))$ holds. If $n \ge 2$, then λ is in the interior of $I_{\mathbf{c}}(V)$ and so the eigenvalue $\lambda \in \sigma\left(H^{\times n}_{\mathbf{c},V}(\pi-\theta)\right)$ has multiplicity two by (4.1) and claim (2). Thus, $N\left(\lambda; H^{\times n}_{\mathbf{c},V}(\theta)\right)$ counts $n-2$ eigenvalues in $I_{\mathbf{c}}(V)$ while $N\left(R(I_{\mathbf{c}}(V)); H^{\times n}_{\mathbf{c},V}(\theta)\right)$ counts $n-1$ eigenvalues in $I_{\mathbf{c}}(V)$. Hence, $N\left(\lambda; H^{\times n}_{\mathbf{c},V}(\pi-\theta)\right) = N\left(R(I_{\mathbf{c}}(V)); H^{\times n}_{\mathbf{c},V}(\theta)\right) - 1$ follows proving (d).

(e) This is an immediate consequence of claim (2) and (4.1). □

4.2. **A characterization of admissibility.** We recall the definition of admissibility (Definition 3.5) for a triple of eigenvalues. We now use the lemmas of the previous subsection in order to provide an equivalent condition for admissibility. Since the definition of admissibility is independent of $V > 0$ (as is also mentioned within Definition 3.5), we omit the V-dependence from the notation in this subsection. For example, we write $I_{\mathbf{c}}, \lambda_{\mathbf{c}}$ and $H^{\times n}_{\mathbf{c}}$ instead of writing $I_{\mathbf{c}}(V), \lambda_{\mathbf{c}}(V)$ and $H^{\times n}_{\mathbf{c},V}$.

Lemma 4.6. *Let $m, n \in \mathbb{N}$, and $\boldsymbol{c} \in \mathcal{C}$ be such that $\varphi(\boldsymbol{c}) \notin \{-1, \infty\}$ and $[\boldsymbol{c}, m] \in \mathcal{C}$. For each $\widetilde{\mathbf{c}} \in \{\boldsymbol{c},\ [\boldsymbol{c}, m],\ [\boldsymbol{c}, m, n]\}$, let $I_{\widetilde{\mathbf{c}}}$ be a spectral band in $\sigma_{\widetilde{\mathbf{c}}}$ and $\lambda_{\widetilde{\mathbf{c}}} \in \{L(I_{\widetilde{\mathbf{c}}}), R(I_{\widetilde{\mathbf{c}}})\}$ and denote*

$$\delta_R(\lambda_{\widetilde{\mathbf{c}}}) := \begin{cases} 0, & \lambda_{\widetilde{\mathbf{c}}} = L\big(I_{\widetilde{\mathbf{c}}}\big), \\ 1, & \lambda_{\widetilde{\mathbf{c}}} = R\big(I_{\widetilde{\mathbf{c}}}\big). \end{cases}$$

Then $\lambda_{\boldsymbol{c}}, \lambda_{[\boldsymbol{c},m]}, \lambda_{[\boldsymbol{c},m,n]}$ are admissible if and only if

$$ind\big(I_{\boldsymbol{c}}\big) + n\cdot ind\big(I_{[\boldsymbol{c},m]}\big) + ind\big(I_{[\boldsymbol{c},m,n]}\big) \equiv \delta_R(\lambda_{\boldsymbol{c}}) + n\cdot\delta_R(\lambda_{[\boldsymbol{c},m]}) + \delta_R(\lambda_{[\boldsymbol{c},m,n]}) \mod 2.$$

Proof. Let $\widetilde{\mathbf{c}} \in \mathcal{C}$ be such that $\{-1, \infty\} \not\ni \varphi(\widetilde{\mathbf{c}}) = \frac{p_{\widetilde{\mathbf{c}}}}{q_{\widetilde{\mathbf{c}}}}$ with $p_{\widetilde{\mathbf{c}}}, q_{\widetilde{\mathbf{c}}}$ coprime. Let $I_{\widetilde{\mathbf{c}}}$ be a spectral band of $\sigma_{\widetilde{\mathbf{c}}}$ and $\lambda_{\widetilde{\mathbf{c}}}$ an edge (left or right) of $I_{\widetilde{\mathbf{c}}}$. In particular, by Lemma 4.4 (a) $\lambda_{\widetilde{\mathbf{c}}}$ is an eigenvalue in $H_{\widetilde{\mathbf{c}}}(\theta_{\widetilde{\mathbf{c}}})$ for some $\theta_{\widetilde{\mathbf{c}}} \in \{0, \pi\}$ and

$$\mathrm{ind}(I_{\widetilde{\mathbf{c}}}) + \delta_R(\lambda_{\widetilde{\mathbf{c}}}) - q_{\widetilde{\mathbf{c}}} \equiv \frac{1}{\pi}\theta_{\widetilde{\mathbf{c}}} \mod 2. \tag{4.2}$$

We will apply (4.2) in the following for both $\widetilde{\mathbf{c}} = \mathbf{c}$ and $\widetilde{\mathbf{c}} = [\mathbf{c}, m, n]$. However, recall from the admissibility definition (Definition 3.5) that we need to consider $\lambda_{[\mathbf{c},m]}$ as an eigenvalue of the matrix $H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})$ (rather than the matrix $H_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})$). Therefore, we need to develop an alternative identity to (4.2). This is done with the aid of Lemma 4.5 (a) and (b) from which we conclude that

$$n\cdot\left(\mathrm{ind}(I_{[\mathbf{c},m]}) + \delta_R(\lambda_{[\mathbf{c},m]}) - q_{[\mathbf{c},m]}\right) \equiv \frac{1}{\pi}\theta_{[\mathbf{c},m]} \mod 2, \tag{4.3}$$

for both even and odd values of $n \in \mathbb{N}$.

To conclude the proof we sum Equation (4.2) for $\widetilde{\mathbf{c}} = \mathbf{c}$ and for $\widetilde{\mathbf{c}} = [\mathbf{c}, m, n]$ and we add to it Equation (4.3). This yields

$$\begin{aligned} &(\mathrm{ind}(I_{\mathbf{c}}) + \delta_R(\lambda_{\mathbf{c}})) + \left(\mathrm{ind}(I_{[\mathbf{c},m,n]}) + \delta_R(\lambda_{[\mathbf{c},m,n]})\right) \\ +n\cdot&\left(\mathrm{ind}(I_{[\mathbf{c},m]}) + \delta_R(\lambda_{[\mathbf{c},m]})\right) - \left(q_{\mathbf{c}} + n\cdot q_{[\mathbf{c},m]} + q_{[\mathbf{c},m,n]}\right) \equiv \frac{1}{\pi}\left(\theta_{\mathbf{c}} + \theta_{[\mathbf{c},m]} + \theta_{[\mathbf{c},m,n]}\right) \mod 2. \end{aligned}$$

By definition, admissibility of $\lambda_{\mathbf{c}}, \lambda_{[\mathbf{c},m]}, \lambda_{[\mathbf{c},m,n]}$ is equivalent to admissibility of the values $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]} \in \{0, \pi\}$, which is equivalent to $\frac{1}{\pi}\left(\theta_{\mathbf{c}} + \theta_{[\mathbf{c},m]} + \theta_{[\mathbf{c},m,n]}\right) \equiv 0 \mod 2$. To end the proof, we just substitute above the equality $q_{\mathbf{c}} + n\cdot q_{[\mathbf{c},m]} = q_{[\mathbf{c},m,n]}$, which is standard in the theory of finite continued fraction expansions (see Lemma I.1, (b)). □

4.3. **Eigenvalue inequalities resulting from interlacing theorem .** Combining the interlacing theorem (Theorem 3.4) with Lemma 4.5 gives the following useful lemma, which is applied many times in the following subsections.

Lemma 4.7. *Let* $V > 0$*,* $m, n \in \mathbb{N}$*,* $\boldsymbol{c} \in \mathcal{C}$ *be such that* $\varphi(\boldsymbol{c}) \notin \{-1, \infty\}$ *and* $[\boldsymbol{c}, m] \in \mathcal{C}$*. Let* $\theta_{\boldsymbol{c}}, \theta_{[\boldsymbol{c},m]}, \theta_{[\boldsymbol{c},m,n]} \in \{0, \pi\}$ *and*

$$\lambda_{\mathbf{o}} \in \sigma\left(H^{\times n}_{[\boldsymbol{c},m],V}(\theta_{[\boldsymbol{c},m]}) \oplus H_{\boldsymbol{c},V}(\theta_{\boldsymbol{c}})\right) \quad \text{and} \quad \mu_{\mathbf{o}} \in \sigma\left(H_{[\boldsymbol{c},m,n],V}(\theta_{[\boldsymbol{c},m,n]})\right).$$

Define

$$N_{\boldsymbol{c}} := N\left(\lambda_{\mathbf{o}};\ H_{\boldsymbol{c},V}(\theta_{\boldsymbol{c}})\right), \quad N_{[\boldsymbol{c},m]} := N\left(\lambda_{\mathbf{o}};\ H^{\times n}_{[\boldsymbol{c},m],V}(\theta_{[\boldsymbol{c},m]})\right)$$

and

$$N_{[\boldsymbol{c},m,n]} := N\left(\mu_{\mathbf{o}};\ H_{[\boldsymbol{c},m,n],V}(\theta_{[\boldsymbol{c},m,n]})\right).$$

(a) *Let* $\mathcal{M}_{\lambda_{\mathbf{o}}}$ *be the multiplicity of the eigenvalue* $\lambda_{\mathbf{o}}$ *of* $H^{\times n}_{[\boldsymbol{c},m],V}(\theta_{[\boldsymbol{c},m]}) \oplus H_{\boldsymbol{c},V}(\theta_{\boldsymbol{c}})$*. If* $\theta_{\boldsymbol{c}}, \theta_{[\boldsymbol{c},m]}, \theta_{[\boldsymbol{c},m,n]}$ *are admissible, then the following implications hold:*

$$N_{\boldsymbol{c}} + N_{[\boldsymbol{c},m]} < N_{[\boldsymbol{c},m,n]} \quad \Rightarrow \quad \lambda_{\mathbf{o}} \le \mu_{\mathbf{o}}, \tag{4.4}$$

$$N_{\boldsymbol{c}} + N_{[\boldsymbol{c},m]} + \mathcal{M}_{\lambda_{\mathbf{o}}} - 1 > N_{[\boldsymbol{c},m,n]} \quad \Rightarrow \quad \lambda_{\mathbf{o}} \ge \mu_{\mathbf{o}}. \tag{4.5}$$

If, additionally, $\lambda_{\mathbf{o}}$ *is a simple eigenvalue of* $H^{\times n}_{[\boldsymbol{c},m],V}(\theta_{[\boldsymbol{c},m]}) \oplus H_{\boldsymbol{c},V}(\theta_{\boldsymbol{c}})$ *(i.e.,* $\mathcal{M}_{\lambda_{\mathbf{o}}} = 1$*), then the two inequalities on the right hand sides of (4.4) and (4.5) are strict.*

(b) *If*
- $\theta_{\boldsymbol{c}}, \theta_{[\boldsymbol{c},m]}, \theta_{[\boldsymbol{c},m,n]}$ *are not admissible and*
- $I_{[\boldsymbol{c},m]}$ *is a spectral band in* $\sigma_{[\boldsymbol{c},m]}$ *satisfying* $\sigma\left(H_{\boldsymbol{c},V}(\theta_{\boldsymbol{c}})\right) \cap I_{[\boldsymbol{c},m]}(V) = \emptyset$*,*

then the following implications hold:

$$\lambda_{\mathbf{o}} = L(I_{[\boldsymbol{c},m]}(V)), \quad N_{\boldsymbol{c}} + N_{[\boldsymbol{c},m]} < N_{[\boldsymbol{c},m,n]} \quad \Rightarrow \quad \lambda_{\mathbf{o}} < \mu_{\mathbf{o}} \tag{4.6}$$

and for $n \ge 2$*,*

$$\lambda_{\mathbf{o}} = R(I_{[\boldsymbol{c},m]}(V)), \quad N_{\boldsymbol{c}} + N_{[\boldsymbol{c},m]} - 1 > N_{[\boldsymbol{c},m,n]} \quad \Rightarrow \quad \lambda_{\mathbf{o}} > \mu_{\mathbf{o}}. \tag{4.7}$$

Remark. We emphasize that $\lambda_{\mathbf{o}}$ and $\mu_{\mathbf{o}}$ do depend on V, but the implications of the lemma do not.

Proof. We start by noting the following rather trivial counting relation

$$N\left(\lambda_{\mathbf{o}};\ H^{\times n}_{[\mathbf{c},m],V}(\theta_{[\mathbf{c},m]}) \oplus H_{\mathbf{c},V}(\theta_{\mathbf{c}})\right) = N_{\mathbf{c}} + N_{[\mathbf{c},m]}. \tag{4.8}$$

(a) Suppose that $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}$ are admissible. Both of the required implications (4.4) and (4.5) follow from Theorem 3.4, when keeping in mind the counting relation (4.8). Explicitly, denoting the eigenvalues of $H^{\times n}_{[\mathbf{c},m],V}(\theta_{[\mathbf{c},m]}) \oplus H_{\mathbf{c},V}(\theta_{\mathbf{c}})$ by $\lambda_0 \le \lambda_1 \le \lambda_2 \le \ldots$ in increasing order, we get that $\lambda_{\mathbf{o}} = \lambda_{N_{\mathbf{c}}+N_{[\mathbf{c},m]}} = \ldots = \lambda_{N_{\mathbf{c}}+N_{[\mathbf{c},m]}+M_{\lambda_{\mathbf{o}}}-1}$. Similarly $\mu_{\mathbf{o}} = \mu_{N_{[\mathbf{c},m,n]}}$, if the eigenvalues of $H_{[\mathbf{c},m,n],V}(\theta_{[\mathbf{c},m,n]})$ are denoted by $\mu_0 \le \mu_1 \le \mu_2 \le \ldots$ in increasing order. Hence,

- (4.4) follows by applying Theorem 3.4 for $\lambda_{\mathbf{o}} = \lambda_{N_{\mathbf{c}}+N_{[\mathbf{c},m]}}$, $\mu_{\mathbf{o}} = \mu_{N_{[\mathbf{c},m,n]}}$, and
- (4.5) follows by applying Theorem 3.4 for $\lambda_{\mathbf{o}} = \lambda_{N_{\mathbf{c}}+N_{[\mathbf{c},m]}+\mathcal{M}_{\lambda_{\mathbf{o}}}-1}$, $\mu_{\mathbf{o}} = \mu_{N_{[\mathbf{c},m,n]}}$.

If, additionally, $\lambda_{\mathbf{o}}$ is a simple eigenvalue of $H^{\times n}_{[\mathbf{c},m],V}(\theta_{[\mathbf{c},m]}) \oplus H_{\mathbf{c},V}(\theta_{\mathbf{c}})$, then $\mathcal{M}_{\lambda_{\mathbf{o}}} = 1$ and the relevant statement within Theorem 3.4 yields the corresponding strict inequalities.

(b) Suppose that $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}$ are not admissible and let $I_{[\mathbf{c},m]}$ be a spectral band in $\sigma_{[\mathbf{c},m]}$ satisfying $\sigma\left(H_{\mathbf{c},V}(\theta_{\mathbf{c}})\right) \cap I_{[\mathbf{c},m]}(V) = \emptyset$.

In the first case (Equation (4.6)), we assume $\lambda_{\mathbf{o}} = L(I_{[\mathbf{c},m]}(V))$ and $N_{\mathbf{c}} + N_{[\mathbf{c},m]} < N_{[\mathbf{c},m,n]}$. We aim to apply Theorem 3.4 directly but $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}$ are not admissible. Thus, we change one of these values to attain an admissible triple. More precisely, $\theta_{\mathbf{c}}, \pi - \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}$

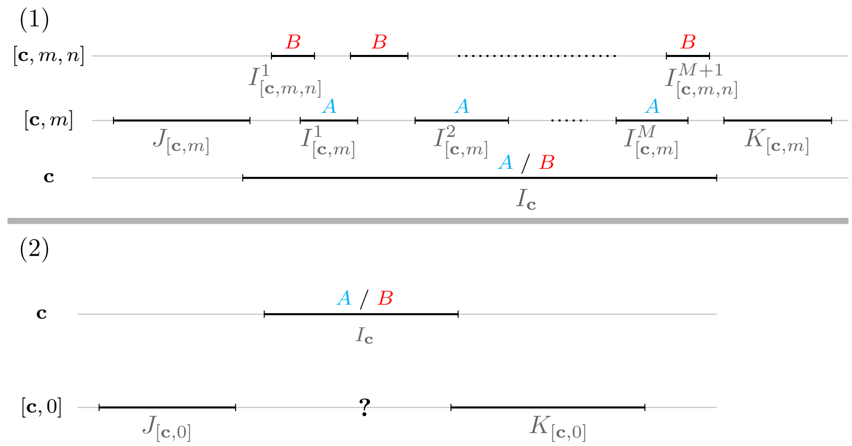

FIGURE 4.2. A sketch for the statement of Lemma 4.9. Note that if $I_\mathbf{c}$ is of backward type A for $V > 4$, then there is a spectral band between $J_{[\mathbf{c},m]}$ and $K_{[\mathbf{c},m]}$. Otherwise, there is no spectral band between them, namely $\operatorname{ind}(K_{[\mathbf{c},m]}) = \operatorname{ind}(J_{[\mathbf{c},m]}) + 1$. This is indicated by the question mark.

are admissible, see Remark 3.6. By Lemma 4.5 (c), there exists a $\lambda \in H^{\times n}_{[\mathbf{c},m],V}(\pi - \theta_{[\mathbf{c},m]})$ such that

$$\lambda_\mathbf{o} = L(I_{[\mathbf{c},m]}(V)) < \lambda \le R(I_{[\mathbf{c},m]}(V)) \quad \text{and} \quad N\left(\lambda; H^{\times n}_{[\mathbf{c},m],V}(\pi-\theta)\right) = N\left(\lambda_\mathbf{o}; H^{\times n}_{[\mathbf{c},m],V}(\theta)\right).$$

Thus, $\lambda \in I_{[\mathbf{c},m]}(V)$ and $\sigma\left(H_{\mathbf{c},V}(\theta_\mathbf{c})\right) \cap I_{[\mathbf{c},m]}(V) = \emptyset$ lead to $N\left(\lambda;\ H_{\mathbf{c},V}(\theta_\mathbf{c})\right) = N\left(\lambda_\mathbf{o};\ H_{\mathbf{c},V}(\theta_\mathbf{c})\right)$. Therefore, (4.8) implies

$$N\left(\lambda;\ H^{\times n}_{[\mathbf{c},m],V}(\pi - \theta_{[\mathbf{c},m]}) \oplus H_{\mathbf{c},V}(\theta_\mathbf{c})\right) = N_\mathbf{c} + N_{[\mathbf{c},m]}.$$

Since $N_\mathbf{c} + N_{[\mathbf{c},m]} < N_{[\mathbf{c},m,n]}$ and $\theta_\mathbf{c}, \pi - \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}$ are admissible, Theorem 3.4 yields $\lambda \le \mu_\mathbf{o}$. Using $\lambda_\mathbf{o} < \lambda$, we conclude $\lambda_\mathbf{o} < \mu_\mathbf{o}$, as claimed.

The second case (Equation (4.7)) follows similar arguments, using Lemma 4.5 (d). □

4.4. **Index identities of the spectral bands.** In order to apply Lemma 4.7, we need to be able to compare the spectral positions of $\lambda_\mathbf{o}$ and $\mu_\mathbf{o}$ ($N_\mathbf{c}$, $N_{[\mathbf{c},m]}$ and $N_{[\mathbf{c},m,n]}$) which appear in Lemma 4.7. Towards this we develop in Lemma 4.9 some connections between indices of spectral bands. For the upcoming statements and proofs, we introduce the following notations, see Figure 4.2 for a sketch.

Definition 4.8. Let $m \in \mathbb{N}_0$ and $\mathbf{c}, [\mathbf{c}, m] \in \mathcal{C}$ be such that $\varphi(\mathbf{c}) \notin \{-1, \infty\}$ and $\varphi([\mathbf{c}, m]) \notin \{-1, \infty\}$. For a spectral band $I_\mathbf{c}$ in $\sigma_\mathbf{c}$, define the *associated spectral bands* $J_{[\mathbf{c},m]}$, $K_{[\mathbf{c},m]}$ in $\sigma_{[\mathbf{c},m]}$ to be the unique spectral bands (if they exist) such that for $V > 4$,

- $J_{[\mathbf{c},m]}(V)$ is the right-most band of $\sigma_{[\mathbf{c},m]}(V)$ for which $J_{[\mathbf{c},m]}(V) \prec I_\mathbf{c}(V)$, and
- $K_{[\mathbf{c},m]}(V)$ is the left-most band of $\sigma_{[\mathbf{c},m]}(V)$ for which $I_\mathbf{c}(V) \prec K_{[\mathbf{c},m]}(V)$.

Remark. Note that it might be that some of the bands $J_{[\mathbf{c},m]}$ and $K_{[\mathbf{c},m]}$ do not exist. In such a case, this in an empty convention. Further note that $\varphi([\mathbf{c}, m]) \in \{-1, \infty\}$ for $m \in \mathbb{N}_0$ can only happen if $\mathbf{c} = [0, 0]$ and $m = 0$ in which case such spectral bands $J_{[\mathbf{c},m]}$ and $K_{[\mathbf{c},m]}$ do not exist.

The reason for including $V > 4$ in the definition above is explained in the beginning of the proof of Lemma 4.9.

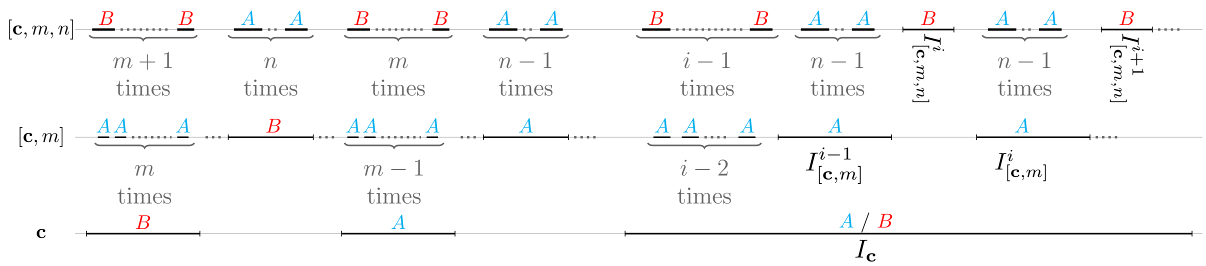

FIGURE 4.3. A sketch for the proof of (4.9) and (4.10) in Lemma 4.9.

Lemma 4.9. *Let $m, n \in \mathbb{N}$, and $\boldsymbol{c} \in \mathcal{C}$ be such that $\varphi(\boldsymbol{c}) \notin \{-1, \infty\}$ and $[\boldsymbol{c}, m] \in \mathcal{C}$. Consider a spectral band $I_{\boldsymbol{c}}$ in $\sigma_{\boldsymbol{c}}$ with associated spectral bands $\{I^i_{[\boldsymbol{c},m]}\}_{i=1}^M$ and $\{I^j_{[\boldsymbol{c},m,n]}\}_{j=1}^{M+1}$ introduced in Definition 3.3. If $M \geq 1$, then for all $1 \leq i \leq M$*

$$ind(I^i_{[\boldsymbol{c},m,n]}) = n \cdot ind(I^i_{[\boldsymbol{c},m]}) + ind(I_{\boldsymbol{c}}), \tag{4.9}$$

and

$$ind(I^{i+1}_{[\boldsymbol{c},m,n]}) = n \cdot (ind(I^i_{[\boldsymbol{c},m]}) + 1) + ind(I_{\boldsymbol{c}}). \tag{4.10}$$

Whenever the spectral bands $J_{[\boldsymbol{c},m]}$ or $K_{[\boldsymbol{c},m]}$ associated with $I_{\boldsymbol{c}}$ exist, then the following hold. If $M \geq 0$, then

$$ind(I^1_{[\boldsymbol{c},m,n]}) = n \cdot (ind(J_{[\boldsymbol{c},m]}) + 1) + ind(I_{\boldsymbol{c}}) \tag{4.11}$$

and

$$ind\left(I^{M+1}_{[\boldsymbol{c},m,n]}\right) = n \cdot ind(K_{[\boldsymbol{c},m]}) + ind(I_{\boldsymbol{c}}) \tag{4.12}$$

If $I_{\boldsymbol{c}}(V)$ is of type B for $V > 4$, then

$$ind(I^1_{[\boldsymbol{c},1]}) = ind(I_{\boldsymbol{c}}) + ind(J_{[\boldsymbol{c},0]}) + 1 = ind(I_{\boldsymbol{c}}) + ind(K_{[\boldsymbol{c},0]}). \tag{4.13}$$

Proof. We start by noting that the index of a spectral band is independent of $V > 0$ (Remark 4.2) allowing us to restrict to the case $V > 4$ where all spectral bands are either of type A or of type B by Theorem 2.16. Therefore, within this proof we allow ourselves to assume $V > 4$ and abuse notation, writing just I (meaning interval and not a map) instead of writing $I(V)$ for some $V > 4$. Namely, when writing within this proof sentences such as "I is a spectral band of type A (or B) and belongs to $\sigma_{\mathbf{c}}$", we actually mean that for some value of $V > 4$, $I(V)$ is of type A (or B) and belongs to $\sigma_{\mathbf{c}}(V)$.

We introduce the following extra notations for the band indices:

$$\begin{aligned} \mathrm{ind}_A(I_{\mathbf{c}}) &:= \left|\{I \text{ is of type } A \text{ and it belongs to } \sigma_{\mathbf{c}} : I \prec I_{\mathbf{c}}\}\right|, \\ \mathrm{ind}_B(I_{\mathbf{c}}) &:= \left|\{I \text{ is of type } B \text{ and it belongs to } \sigma_{\mathbf{c}} : I \prec I_{\mathbf{c}}\}\right|. \end{aligned}$$

Clearly, $\mathrm{ind}(I_{\mathbf{c}}) = \mathrm{ind}_A(I_{\mathbf{c}}) + \mathrm{ind}_B(I_{\mathbf{c}})$ for all $I_{\mathbf{c}}$, see Definition 4.1.

We first assume that $M \geq 1$. Start by examining $I^i_{[\mathbf{c},m,n]}$ and evaluating $\mathrm{ind}_B(I^i_{[\mathbf{c},m,n]})$ and $\mathrm{ind}_A(I^i_{[\mathbf{c},m,n]})$. The spectral band $I^i_{[\mathbf{c},m,n]}$ is of type B and belongs to $\sigma_{[\mathbf{c},m,n]}$. We know that $I^i_{[\mathbf{c},m,n]}$ is included in $I_{\mathbf{c}}$ of $\sigma_{\mathbf{c}}$. There are additional $i-1$ spectral bands of type B in $\sigma_{[\mathbf{c},m,n]}$, which are to the left of $I^i_{[\mathbf{c},m,n]}$ and included in $I_{\mathbf{c}}$. All other spectral bands of type B to the left of $I^i_{[\mathbf{c},m,n]}$ come in groups of either m or $m+1$ and each such group is included in some spectral band I in $\sigma_{\mathbf{c}}$ that is to the left of $I_{\mathbf{c}}$, see Figure 4.3. The group is of size m if I is of type A and it is of size $m+1$ if I is of type B. This discussion may be summarized in the following identity

$$\mathrm{ind}_B(I^i_{[\mathbf{c},m,n]}) = m \cdot \mathrm{ind}_A(I_{\mathbf{c}}) + (m+1) \cdot \mathrm{ind}_B(I_{\mathbf{c}}) + i - 1. \tag{4.14}$$

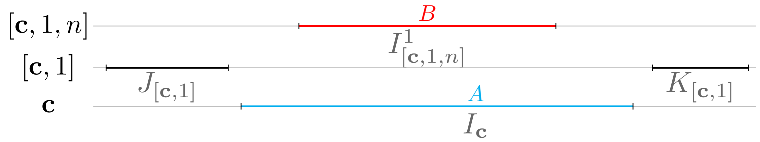

FIGURE 4.4. A sketch for the proof of (4.11) in Lemma 4.9. We have $\operatorname{ind}(K_{[\mathbf{c},m]}) = \operatorname{ind}(J_{[\mathbf{c},m]}) + 1$.

We now evaluate $\operatorname{ind}_A(I^i_{[\mathbf{c},m,n]})$. We note that all the spectral bands of type A to the left of $I^i_{[\mathbf{c},m,n]}$ come in groups of either $n-1$ or n and each such group is included in some spectral band I in $\sigma_{[\mathbf{c},m]}$ that is to the left of $I^i_{[\mathbf{c},m]}$, see Figure 4.3. The group is of size $n-1$ if I is of type A and it is of size n if I is of type B. This discussion may be summarized in the following identity

$$\operatorname{ind}_A(I^i_{[\mathbf{c},m,n]}) = (n-1)\cdot \operatorname{ind}_A(I^i_{[\mathbf{c},m]}) + n \cdot \operatorname{ind}_B(I^i_{[\mathbf{c},m]}). \tag{4.15}$$

We now evaluate $\operatorname{ind}_A(I^i_{[\mathbf{c},m]})$. We note that there are $i-1$ spectral bands of type A to the left of $I^i_{[\mathbf{c},m]}$ which are included in $I_{\mathbf{c}}$. Every other spectral band in $\sigma_{[\mathbf{c},m]}$ of type A to the left of $I^i_{[\mathbf{c},m]}$ is included in a spectral band of $\sigma_{\mathbf{c}}$ to the left of $I_{\mathbf{c}}$. Specifically, they come in groups of either $m-1$ or m and each group is included in a spectral band I in $\sigma_{\mathbf{c}}$ to the left of $I_{\mathbf{c}}$. The group is of size $m-1$ if I is of type A and it is of size m if I is of type B, see Figure 4.3. This discussion may be summarized in the following identity

$$\operatorname{ind}_A(I^i_{[\mathbf{c},m]}) = (m-1)\cdot \operatorname{ind}_A(I_{\mathbf{c}}) + m\cdot \operatorname{ind}_B(I_{\mathbf{c}}) + i - 1. \tag{4.16}$$

Combining the Equations (4.14) and (4.15) together with the identity $\operatorname{ind}(I) = \operatorname{ind}_A(I) + \operatorname{ind}_B(I)$, which holds for all I, gives

$$\begin{aligned}\operatorname{ind}(I^i_{[\mathbf{c},m,n]}) &= \operatorname{ind}_A(I^i_{[\mathbf{c},m,n]}) + \operatorname{ind}_B(I^i_{[\mathbf{c},m,n]})\\ &= \Big(n\cdot \operatorname{ind}(I^i_{[\mathbf{c},m]}) - \operatorname{ind}_A(I^i_{[\mathbf{c},m]})\Big) + (m\cdot \operatorname{ind}_A(I_{\mathbf{c}}) + (m+1)\cdot \operatorname{ind}_B(I_{\mathbf{c}}) + i - 1)\\ &= n\cdot \operatorname{ind}(I^i_{[\mathbf{c},m]}) + ind(I_{\mathbf{c}}),\end{aligned}$$

where in the last line we used (4.16). This proves Equation (4.9).

To prove (4.10), we observe that between $I^i_{[\mathbf{c},m,n]}$ and $I^{i+1}_{[\mathbf{c},m,n]}$ there are $n-1$ spectral bands of type A (the bands which are contained in $I^i_{[\mathbf{c},m]}$) and no spectral bands of type B, see Figure 4.3. We therefore get

$$\operatorname{ind}(I^{i+1}_{[\mathbf{c},m,n]}) = \Big(\operatorname{ind}(I^i_{[\mathbf{c},m,n]}) + 1\Big) + (n-1) = n\cdot(\operatorname{ind}(I^i_{[\mathbf{c},m]}) + 1) + ind(I_{\mathbf{c}}),$$

which proves Equation (4.10).

For $M \geq 1$, Equation (4.11) follows from Equation (4.9) for $i = 1$ and $\operatorname{ind}(I^1_{[\mathbf{c},m]}) = \operatorname{ind}(J_{[\mathbf{c},m]}) + 1$. Similarly, Equation (4.12) follows for $M \geq 1$ from Equation (4.10) for $i = M$ and $\operatorname{ind}(K_{[\mathbf{c},m]}) = \operatorname{ind}(I^M_{[\mathbf{c},m]}) + 1$.

For $M = 0$, (4.11) and (4.12) follow similar arguments as (4.9) and (4.10) using $\operatorname{ind}(K_{[\mathbf{c},m]}) = \operatorname{ind}(J_{[\mathbf{c},m]}) + 1$ if $M = 0$.

To prove (4.13) for the index of $I^1_{[\mathbf{c},1]}$ we note the following. There is a bijection between bands of type A in $\sigma_{[\mathbf{c},1]}$ and bands of type B in $\sigma_{\mathbf{c}}$: a spectral band $I_{\mathbf{c}}$ in $\sigma_{\mathbf{c}}$ of type A does not contain any spectral band in $\sigma_{[\mathbf{c},1]}$ but if $I_{\mathbf{c}}$ in $\sigma_{\mathbf{c}}$ is of type B, then it contains exactly (using uniqueness of these bands for $V > 4$, see Theorem 2.16) one band in $\sigma_{[\mathbf{c},1]}$ of type A. Thus,

$$\operatorname{ind}_A(I^1_{[\mathbf{c},1]}) = \operatorname{ind}_B(I_{\mathbf{c}})$$

follows. We denote by c_k the last digit in $\mathbf{c}$, namely, $\mathbf{c} := [0, 0, c_1, \ldots, c_k]$. Similar counting arguments as for (4.14) lead to

$$\operatorname{ind}_B(I^1_{[\mathbf{c},1]}) = c_k \cdot \operatorname{ind}_A(J_{[\mathbf{c},0]}) + (c_k + 1) \cdot \operatorname{ind}_B(J_{[\mathbf{c},0]}) + c_k + \begin{cases} 0 & J_{[\mathbf{c},0]} \text{ is of type } A, \\ 1 & J_{[\mathbf{c},0]} \text{ is of type } B, \end{cases} \tag{4.17}$$

Moreover, similar counting arguments as in (4.16) imply

$$\begin{aligned} \operatorname{ind}_A(I_\mathbf{c}) &= (c_k - 1) \cdot \operatorname{ind}_A(J_{[\mathbf{c},0]}) + c_k \cdot \operatorname{ind}_B(J_{[\mathbf{c},0]}) + c_k - 1 + \begin{cases} 0 & J_{[\mathbf{c},0]} \text{ is of type } A, \\ 1 & J_{[\mathbf{c},0]} \text{ is of type } B, \end{cases} \\ &= \operatorname{ind}_B(I^1_{[\mathbf{c},1]}) - \operatorname{ind}(J_{[\mathbf{c},0]}) - 1, \end{aligned}$$

where in the last line we used (4.17). Hence, we arrive at

$$\begin{aligned} \operatorname{ind}(I^1_{[\mathbf{c},1]}) &= \operatorname{ind}_B(I^1_{[\mathbf{c},1]}) + \operatorname{ind}_A(I^1_{[\mathbf{c},1]}) \\ &= \big(\operatorname{ind}_A(I_\mathbf{c}) + \operatorname{ind}(J_{[\mathbf{c},0]}) + 1\big) + \operatorname{ind}_B(I_\mathbf{c}) \\ &= \operatorname{ind}(I_\mathbf{c}) + \operatorname{ind}(J_{[\mathbf{c},0]}) + 1 \\ &= \operatorname{ind}(I_\mathbf{c}) + \operatorname{ind}(K_{[\mathbf{c},0]}) \end{aligned}$$

using $\operatorname{ind}(K_{[\mathbf{c},0]}) = \operatorname{ind}(J_{[\mathbf{c},0]}) + 1$, which holds by definition. Thus, the desired Equality (4.13) is proven. □

4.5. **Spectral band edges and trace estimates.** The interlacing theorem is not enough for getting strict inequalities whenever $\lambda_\mathbf{o}$ is not a simple eigenvalue (see exact statement in Lemma 4.7). In order to overcome this, we need to develop some properties of the traces of transfer matrices, which are provided in the next two propositions.

Lemma 4.10. *Let $V \in \mathbb{R} \setminus \{0\}$, $\boldsymbol{c} \in \mathcal{C}$ with $\varphi(\boldsymbol{c}) \notin \{-1, \infty\}$. Then the following statements hold.*

- *(a) For $E \in \mathbb{R}$, we have $|t_\boldsymbol{c}(E, V)| = 2$, if and only if $E \in \{L(I_\boldsymbol{c}(V)), R(I_\boldsymbol{c}(V))\}$* for some spectral band *$I_\boldsymbol{c}(V)$ in $\sigma_\boldsymbol{c}(V)$.*
- *(b) If a spectral band $I_\boldsymbol{c}$ in $\sigma_\boldsymbol{c}$ is*
  - *of backward type A, then $|t_{[\boldsymbol{c},0]}(E, V)| \leq 2$ for all $E \in I_\boldsymbol{c}$. The estimate is strict if $\varphi(\boldsymbol{c}) \in (0, 1)$.*
  - *of backward type B, then $|t_{[\boldsymbol{c},-1]}(E, V)| \leq 2$ for all $E \in I_\boldsymbol{c}$. The estimate is strict if $\varphi(\boldsymbol{c}) \in (0, 1)$.*
- *(c) For $m \geq 0$, we have $t_{[\boldsymbol{c},m+1]} = t_\boldsymbol{c} t_{[\boldsymbol{c},m]} - t_{[\boldsymbol{c},m-1]}$.*

Proof. (a) This is an immediate consequence of Equation (3.2) and Lemma 4.4 (a).
(b) This follows from Definition 2.7 and Equation (3.2).
(c) This well-known identity is proven in [Ray95a]. The reader is also referred to Appendix II.2 for related results and more references, see also [BBB$^+$, lem. 3.8]. □

Remark. The first statement (a) of the lemma says that the traces attain the values ± 2 exactly at the spectral band edges. This does not hold for $\mathbf{c} = [0]$ where $t_\mathbf{c}(E, V) = 2$ and $\sigma_\mathbf{c}(V) = \mathbb{R}$.

The next statement is based on well-known techniques of transfer matrix traces and its proof is included in the Appendix II.2.

Lemma 4.11. *Let $V \in \mathbb{R}$, $m \in \mathbb{N}$, $\boldsymbol{c} \in \mathcal{C}$ be such that $\varphi(\boldsymbol{c}) \notin \{-1, \infty\}$ and $[\boldsymbol{c}, m] \in \mathcal{C}$. Let $I(V)$ be a spectral band in $\sigma_\boldsymbol{c}(V)$ of backward type A or backward type B. Then for $E \in \{L(I(V)), R(I(V))\}$ and $n \in \mathbb{N}$, the following holds.*

- *(a) $|t_{[\boldsymbol{c},m]}(E, V)| \geq 2 \quad \Rightarrow \quad |t_{[\boldsymbol{c},m,n]}(E, V)| \geq 2$.*
- *(b) $|t_{[\boldsymbol{c},m]}(E, V)| > 2 \quad \Rightarrow \quad |t_{[\boldsymbol{c},m,n]}(E, V)| > 2$.*
- *(c) $\varphi(\boldsymbol{c}) \in (0, 1)$ and $|t_{[\boldsymbol{c},m]}(E, V)| \geq 2 \quad \Rightarrow \quad |t_{[\boldsymbol{c},m,n]}(E, V)| > 2$.*

4.6. **Admissibility and triple trace products.** Section 4 is mostly dedicated for developing tools towards the proof of Proposition 4.20. This subsection has a different role - it provides two lemmas which will be used only in the proof of Lemma 7.11. The reason for including Lemma 4.12 and Lemma 4.13 here is that their proofs are rather short and mainly based on the concept of admissibility and the notations introduced so far in Section 4. In this subsection we abbreviate the notation for the trace functions $t_{\mathbf{c}}(E,V)$ and write $t_{\mathbf{c}}(E(V))$, whenever $E:(0,\infty)\to\mathbb{R}$ is taken to be a V-dependent map.

Lemma 4.12. *Let $V\in\mathbb{R}$, $m\in\mathbb{N}$, $\boldsymbol{c}\in\mathcal{C}$ be such that $\varphi(\boldsymbol{c})\notin\{-1,\infty\}$ and $[\boldsymbol{c},m]\in\mathcal{C}$. Consider a spectral band $I_{\boldsymbol{c}}$ in $\sigma_{\boldsymbol{c}}$ with the associated spectral band $I^1_{[\boldsymbol{c},m,1]}$ in $\sigma_{[\boldsymbol{c},m,1]}$ introduced in Definition 3.3. Moreover, let $J_{[\boldsymbol{c},m]}$ in $\sigma_{[\boldsymbol{c},m]}$ be the associated spectral band of $I_{\boldsymbol{c}}$ defined in Definition 4.8. Then*

$$\operatorname{sign}\left(t_{\boldsymbol{c}}\left(L(I_{\boldsymbol{c}}(V))\right)\cdot t_{[\boldsymbol{c},m]}\left(R(J_{[\boldsymbol{c},m]}(V))\right)\cdot t_{[\boldsymbol{c},m,1]}\left(L(I^1_{[\boldsymbol{c},m,1]}(V))\right)\right)=+1.$$

Proof. Let $\theta_{\mathbf{c}},\theta_{[\mathbf{c},m]},\theta_{[\mathbf{c},m,1]}\in\{0,\pi\}$ be such that

$$L(I_{\mathbf{c}}(V))\in\sigma\left(H_{\mathbf{c},V}(\theta_{\mathbf{c}})\right),\quad R(J_{[\mathbf{c},m]}(V))\in\sigma\left(H^{\times 1}_{[\mathbf{c},m],V}(\theta_{[\mathbf{c},m]})\right)$$
$$\text{and}\quad L(I^1_{[\mathbf{c},m,1]}(V))\in\sigma\left(H_{[\mathbf{c},m,1],V}(\theta_{[\mathbf{c},m,1]})\right).$$

Then these three eigenvalues, respectively $\theta_{\mathbf{c}},\theta_{[\mathbf{c},m]},\theta_{[\mathbf{c},m,1]}$, are admissible by inserting the index relation (4.11) into the characterization of admissibility from Lemma 4.6 for $n=1$. Due to Equation (3.2), we conclude

$$t_{\mathbf{c}}\left(L(I_{\mathbf{c}}(V))\right)=2\cos(\theta_{\mathbf{c}})\quad\text{and}\quad t_{[\mathbf{c},m,1]}\left(L(I^1_{[\mathbf{c},m,1]}(V))\right)=2\cos(\theta_{[\mathbf{c},m,1]}).$$

Note that $H^{\times 1}_{[\mathbf{c},m],V}(\theta_{[\mathbf{c},m]})=H_{[\mathbf{c},m],V}(\theta_{[\mathbf{c},m]})$ holds by definition of the matrix and so $\theta_{[\mathbf{c},m]}$ satisfies (again by (3.2))

$$t_{[\mathbf{c},m]}\left(R(J_{[\mathbf{c},m]}(V))\right)=2\cos(\theta_{[\mathbf{c},m]}).$$

Thus, the statement follows from the fact that $\theta_{\mathbf{c}},\theta_{[\mathbf{c},m]},\theta_{[\mathbf{c},m,1]}$ are admissible (even number of π's, see Remark 3.6). □

Lemma 4.13. *Let $V\in\mathbb{R}$, $m,n\in\mathbb{N}$, $\boldsymbol{c}\in\mathcal{C}$ be such that $\varphi(\boldsymbol{c})\notin\{-1,\infty\}$ and $[\boldsymbol{c},m]\in\mathcal{C}$. Consider a spectral band $I_{\boldsymbol{c}}$ in $\sigma_{\boldsymbol{c}}$ with the associated spectral bands $I^1_{[\boldsymbol{c},m,n]}$ in $\sigma_{[\boldsymbol{c},m,n]}$ and $I^1_{[\boldsymbol{c},m,n+1]}$ in $\sigma_{[\boldsymbol{c},m,n+1]}$ introduced in Definition 3.3. Moreover, let $J_{[\boldsymbol{c},m]}$ in $\sigma_{[\boldsymbol{c},m]}$ be the associated spectral band of $I_{\boldsymbol{c}}$ defined in Definition 4.8. Then*

$$\operatorname{sign}\left(t_{[\boldsymbol{c},m]}\left(R(J_{[\boldsymbol{c},m]}(V))\right)\cdot t_{[\boldsymbol{c},m,n]}\left(L(I^1_{[\boldsymbol{c},m,n]}(V))\right)\cdot t_{[\boldsymbol{c},m,n+1]}\left(L(I^1_{[\boldsymbol{c},m,n+1]}(V))\right)\right)=+1.$$

Proof. Define $\widetilde{\mathbf{c}}=[\mathbf{c},m]$ and $\widetilde{m}=n$. Thus, $[\mathbf{c},m,n]=[\widetilde{\mathbf{c}},\widetilde{m}]$ and $[\widetilde{\mathbf{c}},\widetilde{m},1]=[\mathbf{c},m,n,1]$. Then the evaluation map satisfies $\varphi([\widetilde{\mathbf{c}},\widetilde{m},1])=\varphi([\mathbf{c},m,n+1])$. Thus, Lemma 2.5 implies

$$t_{[\widetilde{\mathbf{c}},\widetilde{m},1]}\left(L(I^1_{[\mathbf{c},m,n+1]}(V))\right)=t_{[\mathbf{c},m,n+1]}\left(L(I^1_{[\mathbf{c},m,n+1]}(V))\right).$$

Let $\theta_{\widetilde{\mathbf{c}}},\theta_{[\widetilde{\mathbf{c}},\widetilde{m}]},\theta_{[\widetilde{\mathbf{c}},\widetilde{m},1]}\in\{0,\pi\}$ be such that

$$R(J_{[\mathbf{c},m]}(V))\in\sigma\left(H_{\widetilde{\mathbf{c}},V}(\theta_{\widetilde{\mathbf{c}}})\right),\quad L(I^1_{[\mathbf{c},m,n]}(V))\in\sigma\left(H^{\times 1}_{[\widetilde{\mathbf{c}},\widetilde{m}],V}(\theta_{[\widetilde{\mathbf{c}},\widetilde{m}]})\right).$$
$$\text{and}\quad L(I^1_{[\mathbf{c},m,n+1]}(V))\in\sigma\left(H_{[\widetilde{\mathbf{c}},\widetilde{m},1],V}(\theta_{[\widetilde{\mathbf{c}},\widetilde{m},1]})\right).$$

Then these spectral edges, respectively $\theta_{\widetilde{\mathbf{c}}},\theta_{[\widetilde{\mathbf{c}},\widetilde{m}]},\theta_{[\widetilde{\mathbf{c}},\widetilde{m},1]}$, are admissible by inserting the index relation (4.11) into the characterization of admissibility from Lemma 4.6 for $\widetilde{\mathbf{c}}$, $\widetilde{m}$ and $\widetilde{n}=1$. Due to Equation (3.2), we conclude

$$t_{\widetilde{\mathbf{c}}}\left(R(J_{[\mathbf{c},m]}(V))\right)=2\cos(\theta_{\widetilde{\mathbf{c}}})\quad\text{and}\quad t_{[\widetilde{\mathbf{c}},\widetilde{m},1]}\left(L(I^1_{[\mathbf{c},m,n+1]}(V))\right)=2\cos(\theta_{[\widetilde{\mathbf{c}},\widetilde{m},1]}).$$

Note that $H^{\times 1}_{[\widetilde{\mathbf{c}},\widetilde{m}],V}(\theta_{[\widetilde{\mathbf{c}},\widetilde{m}]}) = H_{[\widetilde{\mathbf{c}},\widetilde{m}],V}(\theta_{[\widetilde{\mathbf{c}},\widetilde{m}]})$ holds by definition of the matrix and so

$$t_{[\widetilde{\mathbf{c}},\widetilde{m}]}\left(L(I^1_{[\mathbf{c},m,n]}(V))\right) = 2\cos(\theta_{[\widetilde{\mathbf{c}},\widetilde{m}]}).$$

Thus, the statement follows from the fact that $\theta_{\widetilde{\mathbf{c}}}, \theta_{[\widetilde{\mathbf{c}},\widetilde{m}]}, \theta_{[\widetilde{\mathbf{c}},\widetilde{m},1]}$ are admissible (even number of π's, see Remark 3.6). □

4.7. **Sufficient conditions for the forward type properties.** In this subsection, we prove that various properties of the forward type (A or B) as in Definition 2.12 are satisfied under some conditions. The statements which we eventually apply in the next subsection are Corollary 4.16, Lemma 4.17, Corollary 4.18 and Lemma 4.19.

We start with proving that the interlacing property (I) holds under some conditions.

Lemma 4.14. *Let $V_1 > 0$, $m, n \in \mathbb{N}$, $\mathbf{c} \in \mathcal{C}$ be such that $\varphi(\mathbf{c}) \notin \{-1, \infty\}$ and $[\mathbf{c}, m] \in \mathcal{C}$. Consider a spectral band $I_{\mathbf{c}}$ in $\sigma_{\mathbf{c}}$ with the associated spectral bands $\{I^i_{[\mathbf{c},m]}\}_{i=1}^M$ and $\{I^j_{[\mathbf{c},m,n]}\}_{j=1}^{M+1}$ introduced in Definition 3.3. If $1 \le i \le M$ and $I^i_{[\mathbf{c},m]}(V_1) \subseteq_{\mathrm{str}} I_{\mathbf{c}}(V_1)$, then*

$$I^i_{[\mathbf{c},m,n]}(V_1) \prec I^i_{[\mathbf{c},m]}(V_1) \prec I^{i+1}_{[\mathbf{c},m,n]}(V_1).$$

Proof. Let $1 \le i \le M$ and $V_1 > 0$. We need to show the following inequalities

(a) $L\left(I^i_{[\mathbf{c},m,n]}(V_1)\right) < L\left(I^i_{[\mathbf{c},m]}(V_1)\right)$,
(b) $R\left(I^i_{[\mathbf{c},m]}(V_1)\right) < R\left(I^{i+1}_{[\mathbf{c},m,n]}(V_1)\right)$,
(c) $R\left(I^i_{[\mathbf{c},m,n]}(V_1)\right) < R\left(I^i_{[\mathbf{c},m]}(V_1)\right)$,
(d) $L\left(I^i_{[\mathbf{c},m]}(V_1)\right) < L\left(I^{i+1}_{[\mathbf{c},m,n]}(V_1)\right)$.

We proceed proving the inequalities above one at a time via an appropriate application of Lemma 4.7. Although the inequalities above and the assumption $I^i_{[\mathbf{c},m]}(V_1) \subseteq_{\mathrm{str}} I_{\mathbf{c}}(V_1)$ depend on the fixed $V_1 > 0$, we will abbreviate notation, for the sake of easier reading, and omit the V_1 dependence in most parts of this proof.

(a) We aim to apply Lemma 4.7 for $\lambda_{\mathbf{o}} = L(I^i_{[\mathbf{c},m]})$, $\mu_{\mathbf{o}} = L(I^i_{[\mathbf{c},m,n]})$. Let $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]} \in \{0, \pi\}$ be such that

$$L(I_{\mathbf{c}}) \in \sigma\left(H_{\mathbf{c}}(\theta_{\mathbf{c}})\right), \quad L(I^i_{[\mathbf{c},m]}) \in \sigma\left(H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) \quad \text{and} \quad L(I^i_{[\mathbf{c},m,n]}) \in \sigma\left(H_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m,n]})\right).$$

These spectral edges, respectively $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}$, are admissible, as can be verified by using the index relation (4.9) of Lemma 4.9 in the characterization of admissibility from Lemma 4.6. Furthermore, Lemma 4.5 (c) applied to $[\mathbf{c}, m] \in \mathcal{C}$ implies

$$N_{[\mathbf{c},m]} := N\left(L(I^i_{[\mathbf{c},m]});\ H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) = n \cdot \mathrm{ind}(I^i_{[\mathbf{c},m]}).$$

Apply Lemma 4.4 (b) to $[\mathbf{c}, m, n] \in \mathcal{C}$ and use again the index relation (4.9) of Lemma 4.9 to conclude

$$N_{[\mathbf{c},m,n]} := N\left(L(I^i_{[\mathbf{c},m,n]});\ H_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m,n]})\right) = \mathrm{ind}(I^i_{[\mathbf{c},m,n]}) = \mathrm{ind}(I_{\mathbf{c}}) + n \cdot \mathrm{ind}(I^i_{[\mathbf{c},m]}).$$

Since $I^i_{[\mathbf{c},m]}(V_1) \subseteq_{\mathrm{str}} I_{\mathbf{c}}(V_1)$, we infer $L(I_{\mathbf{c}}(V_1)) < L(I^i_{[\mathbf{c},m]}(V_1))$ and $\sigma\left(H_{\mathbf{c},V_1}(\theta_{\mathbf{c}})\right) \cap I^i_{[\mathbf{c},m]}(V_1) = \emptyset$. Hence, $L(I_{\mathbf{c}}) \in \sigma\left(H_{\mathbf{c}}(\theta_{\mathbf{c}})\right)$ and Lemma 4.4 (b) applied to $\mathbf{c} \in \mathcal{C}$ lead to

$$N_{\mathbf{c}} := N\left(L(I^i_{[\mathbf{c},m]});\ H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) = N(L(I_{\mathbf{c}});\ H_{\mathbf{c}}(\theta_{\mathbf{c}})) + 1 = \mathrm{ind}(I_{\mathbf{c}}) + 1.$$

Summing up, we obtained $N_{\mathbf{c}} + N_{[\mathbf{c},m]} > N_{[\mathbf{c},m,n]}$. Moreover, $\sigma\left(H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) \cap I^i_{[\mathbf{c},m]} = \emptyset$ and Lemma 4.5 (e) that $\lambda_{\mathbf{o}} = L(I^i_{[\mathbf{c},m]})$ is a simple eigenvalue of $H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]}) \oplus H_{\mathbf{c}}(\theta_{\mathbf{c}})$. Using admissibility, Lemma 4.7 (a) yields the required inequality $\lambda_{\mathbf{o}} = L(I^i_{[\mathbf{c},m]}) > L(I^i_{[\mathbf{c},m,n]}) = \mu_{\mathbf{o}}$.

(b) We aim to apply Lemma 4.7 for $\lambda_{\mathbf{o}} = R(I^i_{[\mathbf{c},m]})$, $\mu_{\mathbf{o}} = R(I^{i+1}_{[\mathbf{c},m,n]})$. Let $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]} \in \{0,\pi\}$ be such that

$$R(I_{\mathbf{c}}) \in \sigma\left(H_{\mathbf{c}}(\theta_{\mathbf{c}})\right), \; R(I^i_{[\mathbf{c},m]}) \in \sigma\left(H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) \text{ and } R(I^{i+1}_{[\mathbf{c},m,n]}) \in \sigma\left(H_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m,n]})\right).$$

Then these spectral edges, respectively $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}$, are admissible by inserting the index relation (4.10) of Lemma 4.9 into the characterization of admissibility from Lemma 4.6. Furthermore, Lemma 4.5 (d) applied to $[\mathbf{c}, m] \in \mathcal{C}$ implies

$$N_{[\mathbf{c},m]} := N\left(R(I^i_{[\mathbf{c},m]});\; H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) = n \cdot \left(\operatorname{ind}(I^i_{[\mathbf{c},m]}) + 1\right) - 1.$$

Apply Lemma 4.4 (c) to $[\mathbf{c}, m, n] \in \mathcal{C}$ and use again the index relation (4.10) of Lemma 4.9 to conclude

$$\begin{aligned} N_{[\mathbf{c},m,n]} := N\left(R(I^{i+1}_{[\mathbf{c},m,n]});\; H_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m,n]})\right) &= \operatorname{ind}(I^{i+1}_{[\mathbf{c},m,n]}) \\ &= n \cdot \left(\operatorname{ind}(I^i_{[\mathbf{c},m]}) + 1\right) + \operatorname{ind}(I_{\mathbf{c}}). \end{aligned}$$

Since $I^i_{[\mathbf{c},m]}(V_1) \subseteq_{\text{str}} I_{\mathbf{c}}(V_1)$, we infer $R(I^i_{[\mathbf{c},m]}(V_1)) < R(I_{\mathbf{c}}(V_1))$ and $\sigma\left(H_{\mathbf{c},V_1}(\theta_{\mathbf{c}})\right) \cap I^i_{[\mathbf{c},m]}(V_1) = \emptyset$. Hence, Lemma 4.4 (c) applied to $\mathbf{c} \in \mathcal{C}$ leads to

$$N_{\mathbf{c}} := N\left(R(I^i_{[\mathbf{c},m]});\; H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) = N(R(I_{\mathbf{c}});\; H_{\mathbf{c}}(\theta_{\mathbf{c}})) = \operatorname{ind}(I_{\mathbf{c}}).$$

Summing up, we obtained $N_{\mathbf{c}} + N_{[\mathbf{c},m]} < N_{[\mathbf{c},m,n]}$. Moreover, $\sigma\left(H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) \cap I^i_{[\mathbf{c},m]} = \emptyset$ and Lemma 4.5 (e) that $\lambda_{\mathbf{o}} = R(I^i_{[\mathbf{c},m]})$ is a simple eigenvalue of $H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]}) \oplus H_{\mathbf{c}}(\theta_{\mathbf{c}})$. Using admissibility, Lemma 4.7 (a) yields the required inequality $\lambda_{\mathbf{o}} = R(I^i_{[\mathbf{c},m]}) < R(I^{i+1}_{[\mathbf{c},m,n]}) = \mu_{\mathbf{o}}$.

(c) We aim to apply Lemma 4.7 for $\lambda_{\mathbf{o}} = R(I^i_{[\mathbf{c},m]})$, $\mu_{\mathbf{o}} = R(I^i_{[\mathbf{c},m,n]})$. Let $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]} \in \{0,\pi\}$ be such that

$$L(I_{\mathbf{c}}) \in \sigma\left(H_{\mathbf{c}}(\theta_{\mathbf{c}})\right), \; R(I^i_{[\mathbf{c},m]}) \in \sigma\left(H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) \text{ and } R(I^i_{[\mathbf{c},m,n]}) \in \sigma\left(H_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m,n]})\right).$$

Lemma 4.5 (d) applied to $[\mathbf{c}, m] \in \mathcal{C}$ implies

$$N_{[\mathbf{c},m]} := N\left(R(I^i_{[\mathbf{c},m]});\; H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) = n \cdot \left(\operatorname{ind}(I^i_{[\mathbf{c},m]}) + 1\right) - 1.$$

Apply Lemma 4.4 (c) to $[\mathbf{c}, m, n] \in \mathcal{C}$ and use the index relation (4.9) of Lemma 4.9 to conclude

$$\begin{aligned} N_{[\mathbf{c},m,n]} := N\left(R(I^i_{[\mathbf{c},m,n]});\; H_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m,n]})\right) &= \operatorname{ind}(I^i_{[\mathbf{c},m,n]}) \\ &= n \cdot \operatorname{ind}(I^i_{[c,m]}) + \operatorname{ind}(I_c). \end{aligned}$$

Since $I^i_{[\mathbf{c},m]}(V_1) \subseteq_{\text{str}} I_{\mathbf{c}}(V_1)$, we infer $L(I_{\mathbf{c}}(V_1)) < R(I^i_{[\mathbf{c},m]}(V_1)) < R(I_{\mathbf{c}}(V_1))$ and $\sigma\left(H_{\mathbf{c},V_1}(\theta_{\mathbf{c}})\right) \cap I^i_{[\mathbf{c},m]}(V_1) = \emptyset$. Hence, $L(I_{\mathbf{c}}) \in \sigma\left(H_{\mathbf{c}}(\theta_{\mathbf{c}})\right)$ and Lemma 4.4 (b) applied to $\mathbf{c} \in \mathcal{C}$ lead to

$$N_{\mathbf{c}} := N\left(R(I^i_{[\mathbf{c},m]});\; H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) = N\left(L(I_{\mathbf{c}});\; H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) + 1 = \operatorname{ind}(I_{\mathbf{c}}) + 1.$$

Thus, $N_{\mathbf{c}} + N_{[\mathbf{c},m]} = N_{[\mathbf{c},m,n]} + n > N_{[\mathbf{c},m,n]}$ follows. Moreover, $\sigma\left(H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) \cap I^i_{[\mathbf{c},m]} = \emptyset$ and Lemma 4.5 (e) that $\lambda_{\mathbf{o}} = R(I^i_{[\mathbf{c},m]})$ is a simple eigenvalue of $H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]}) \oplus H_{\mathbf{c}}(\theta_{\mathbf{c}})$. Observe that $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}$ are admissible, if and only if n is odd by inserting the index relation (4.9) of Lemma 4.9 into the characterization of admissibility from Lemma 4.6. Thus, if n is odd, the previous considerations with Lemma 4.7 (a) yield $\lambda_{\mathbf{o}} = R(I^i_{[\mathbf{c},m]}(V)) > R(I^i_{[\mathbf{c},m,n]}) = \mu_{\mathbf{o}}$.

If n is even, then $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}$ are not admissible. Moreover, $N_{\mathbf{c}} + N_{[\mathbf{c},m]} - 1 > N_{[\mathbf{c},m,n]}$ follows since $n \geq 2$ if n is even. Thus, Lemma 4.7 (b) with $\sigma\left(H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) \cap I^i_{[\mathbf{c},m]} = \emptyset$implies $\lambda_{\mathbf{o}} = R(I^i_{[\mathbf{c},m]}) > R(I^i_{[\mathbf{c},m,n]}) = \mu_{\mathbf{o}}$.

(d) We aim to apply Lemma 4.7 for $\lambda_{\mathbf{o}} = L(I^i_{[\mathbf{c},m]})$, $\mu_{\mathbf{o}} = L(I^{i+1}_{[\mathbf{c},m,n]})$. Let $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]} \in \{0,\pi\}$ be such that

$$R(I_{\mathbf{c}}) \in \sigma\left(H_{\mathbf{c}}(\theta_{\mathbf{c}})\right), \quad L(I^i_{[\mathbf{c},m]}) \in \sigma\left(H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) \quad \text{and} \quad L(I^{i+1}_{[\mathbf{c},m,n]}) \in \sigma\left(H_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m,n]})\right).$$

Lemma 4.5 (c) applied to $[\mathbf{c},m] \in \mathcal{C}$ implies

$$N_{[\mathbf{c},m]} := N\left(L(I^i_{[\mathbf{c},m]});\ H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) = n \cdot \operatorname{ind}(I^i_{[\mathbf{c},m]}).$$

Apply Lemma 4.4 (b) to $[\mathbf{c},m,n] \in \mathcal{C}$ and the index relation (4.10) of Lemma 4.9 to conclude

$$N_{[\mathbf{c},m,n]} := N\left(L(I^{i+1}_{[\mathbf{c},m,n]});\ H_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m,n]})\right) = \operatorname{ind}(I^{i+1}_{[\mathbf{c},m,n]}) = n \cdot \left(\operatorname{ind}(I^i_{[c,m]}) + 1\right) + \operatorname{ind}(I_c).$$

Since $I^i_{[\mathbf{c},m]}(V_1) \subseteq_{\text{str}} I_{\mathbf{c}}(V_1)$, we infer $L(I_{\mathbf{c}}(V_1)) < L(I^i_{[\mathbf{c},m]}(V_1)) < R(I_{\mathbf{c}}(V_1))$ and $\sigma\left(H_{\mathbf{c},V_1}(\theta_{\mathbf{c}})\right) \cap I^i_{[\mathbf{c},m]}(V_1) = \emptyset$. Hence, Lemma 4.4 (c) applied to $\mathbf{c} \in \mathcal{C}$ leads to

$$N_{\mathbf{c}} := N\left(L(I^i_{[\mathbf{c},m]});\ H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) = N(R(I_{\mathbf{c}});\ H_{\mathbf{c}}(\theta_{\mathbf{c}})) = \operatorname{ind}(I_{\mathbf{c}}).$$

Thus, $N_{\mathbf{c}} + N_{[\mathbf{c},m]} < N_{[\mathbf{c},m,n]}$ follows. Moreover, $\sigma\left(H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) \cap I^i_{[\mathbf{c},m]} = \emptyset$ and Lemma 4.5 (e) that $\lambda_{\mathbf{o}} = L(I^i_{[\mathbf{c},m]})$ is a simple eigenvalue of $H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]}) \oplus H_{\mathbf{c}}(\theta_{\mathbf{c}})$. Observe that $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}$ are admissible, if and only if $n \in \mathbb{N}$ is odd by inserting the index relation (4.10) of Lemma 4.9 into the characterization of admissibility from Lemma 4.6. Thus, if $n \in \mathbb{N}$ is odd, the previous considerations with Lemma 4.7 (b) yield the required inequality $\lambda_{\mathbf{o}} = L(I^i_{[\mathbf{c},m]}) < L(I^{i+1}_{[\mathbf{c},m,n]}) = \mu_{\mathbf{o}}$.

If $n \in \mathbb{N}$ is even, then $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}$ are not admissible. Thus, Lemma 4.7 (b) with $\sigma\left(H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) \cap I^i_{[\mathbf{c},m]} = \emptyset$ and $N_{\mathbf{c}} + N_{[\mathbf{c},m]} < N_{[\mathbf{c},m,n]}$ imply $\lambda_{\mathbf{o}} = L(I^i_{[\mathbf{c},m]}) < L(I^{i+1}_{[\mathbf{c},m,n]}) = \mu_{\mathbf{o}}$. □

The next lemma is tailored towards proving property (B2).

Lemma 4.15. *Let $V_1 > 0$, $m,n \in \mathbb{N}$ and $\boldsymbol{c} \in \mathcal{C}$ be such that $\varphi(\boldsymbol{c}) \notin \{-1,\infty\}$ and $[\boldsymbol{c},m] \in \mathcal{C}$. Consider a spectral band $I_{\boldsymbol{c}}$ in $\sigma_{\boldsymbol{c}}$ with the associated spectral bands $\{I^i_{[\boldsymbol{c},m]}\}_{i=1}^M$ and $\{I^j_{[\boldsymbol{c},m,n]}\}_{j=1}^{M+1}$ introduced in Definition 3.3. Let $J_{[\boldsymbol{c},m]}$ and $K_{[\boldsymbol{c},m]}$ be the spectral bands associated with $I_{\boldsymbol{c}}$ as defined in Definition 4.8. If*

$$I^1_{[\boldsymbol{c},m,n]}(V_1) \subseteq_{\text{str}} I_{\boldsymbol{c}}(V_1) \quad \text{and} \quad I^{M+1}_{[\boldsymbol{c},m,n]}(V_1) \subseteq_{\text{str}} I_{\boldsymbol{c}}(V_1),$$

then

$$R(J_{[\boldsymbol{c},m]}(V_1)) < R(I^1_{[\boldsymbol{c},m,n]}(V_1)) \quad \text{and} \quad L(I^{M+1}_{[\boldsymbol{c},m,n]}(V_1)) < L(K_{[\boldsymbol{c},m]}(V_1)).$$

Remark. It might be that either $J_{[\mathbf{c},m]}$ or $K_{[\mathbf{c},m]}$ as defined in Definition 4.8 do not exist. In such a case, part of the statement is empty.

Combining Lemma 4.14 and Lemma 4.15, we get the following corollary which shows that properties (A1), (B2) and (I) hold under some conditions.

Corollary 4.16. *Let $V_1 > 0$, $m,n \in \mathbb{N}$ and $\boldsymbol{c} \in \mathcal{C}$ be such that $\varphi(\boldsymbol{c}) \notin \{-1,\infty\}$ and $[\boldsymbol{c},m] \in \mathcal{C}$. Consider a spectral band $I_{\boldsymbol{c}}$ in $\sigma_{\boldsymbol{c}}$ with associated spectral bands $\{I^i_{[\boldsymbol{c},m]}\}_{i=1}^M$ and $\{I^j_{[\boldsymbol{c},m,n]}\}_{j=1}^{M+1}$ introduced in Definition 3.3. If*

$$I^1_{[\boldsymbol{c},m]}(V_1),\ I^M_{[\boldsymbol{c},m]}(V_1) \subseteq_{\text{str}} I_{\boldsymbol{c}}(V_1) \qquad \text{and} \qquad I^1_{[\boldsymbol{c},m,n]}(V_1),\ I^{M+1}_{[\boldsymbol{c},m,n]}(V_1) \subseteq_{\text{str}} I_{\boldsymbol{c}}(V_1),$$

then $I_{\boldsymbol{c}}(V_1)$ satisfies the properties (A1), (B2) and (I).

Proof of Corollary 4.16. First, we note that the condition in the corollary is equivalent to $I^j_{[\mathbf{c},m,n]}(V_1) \subseteq_{\text{str}} I_{\mathbf{c}}(V_1)$ for all $1 \leq j \leq M+1$ and $I^i_{[\mathbf{c},m]}(V_1) \subseteq_{\text{str}} I_{\mathbf{c}}(V_1)$ for all $1 \leq i \leq M$, since

$$I^i_{[\mathbf{c},m]} \prec I^{i+1}_{[\mathbf{c},m]} \qquad \text{and} \qquad I^j_{[\mathbf{c},m,n]} \prec I^{j+1}_{[\mathbf{c},m,n]}.$$

The assumption that $I^i_{[\mathbf{c},m]}(V_1) \subseteq_{\rm str} I_{\mathbf{c}}(V_1)$ for all $1 \le i \le M$ is exactly property (A1) of $I_{\mathbf{c}}(V_1)$. Moreover, this assumption allows us to apply Lemma 4.14 and obtain

$$I^i_{[\mathbf{c},m,n]}(V_1) \prec I^i_{[\mathbf{c},m]}(V_1) \prec I^{i+1}_{[\mathbf{c},m,n]}(V_1) \qquad \text{for all } 1 \le i \le M.$$

Thus, $I_{\mathbf{c}}(V_1)$ satisfies property (I). Furthermore, these relations imply that each of the bands $\left\{I^j_{[\mathbf{c},m,n]}(V_1)\right\}_{j=1}^{M+1}$ is not contained in any of the bands $\left\{I^i_{[\mathbf{c},m]}(V_1)\right\}_{i=1}^{M}$, which is useful towards proving property (B2). Recall (Definition 4.8) the notation of the spectral bands $J_{[\mathbf{c},m]}$ and $K_{[\mathbf{c},m]}$ associated with $I_{\mathbf{c}}$. In order to prove (B2), it is enough to prove that $I^1_{[\mathbf{c},m,n]}(V_1)$ is not contained $J_{[\mathbf{c},m]}(V_1)$ and $I^{M+1}_{[\mathbf{c},m,n]}(V_1)$ is not contained in $K_{[\mathbf{c},m]}(V_1)$. This follows from Lemma 4.15. □

Proof of Lemma 4.15. (a) We prove that $R(J_{[\mathbf{c},m]}(V_1)) < R(I^1_{[\mathbf{c},m,n]}(V_1))$. First we note that this inequality immediately holds if $R(J_{[\mathbf{c},m]}(V_1)) \le L(I_{\mathbf{c}}(V_1))$, because $I^1_{[\mathbf{c},m,n]}(V_1) \subseteq_{\rm str} I_{\mathbf{c}}(V_1)$ by assumption. Therefore, we assume from now on that $R(J_{[\mathbf{c},m]}(V_1)) > L(I_{\mathbf{c}}(V_1))$. Although the assumptions and the conclusions of the lemma depend on the fixed $V_1 > 0$, we will abbreviate notation, for the sake of easier reading, and omit the V_1 dependence in most parts of this proof. Let $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]} \in \{0,\pi\}$ be such that

$$R(I_{\mathbf{c}}) \in \sigma\left(H_{\mathbf{c}}(\theta_{\mathbf{c}})\right), \quad R(J_{[\mathbf{c},m]}) \in \sigma\left(H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) \text{ and } R(I^1_{[\mathbf{c},m,n]}) \in \sigma\left(H_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m,n]})\right).$$

Then these spectral edges, respectively $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}$, are admissible by inserting the index relation (4.11) of Lemma 4.9 into the characterization of admissibility from Lemma 4.6. Furthermore, Lemma 4.4 (c) for $[\mathbf{c},m,n] \in \mathcal{C}$ and $\mathbf{c} \in \mathcal{C}$, the index relation (4.11) of Lemma 4.9 and Lemma 4.5 (d) for the spectral band $J_{[\mathbf{c},m]}$ imply

$$\begin{aligned} N\left(R(I^1_{[\mathbf{c},m,n]});\ H_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m,n]})\right) &= \mathrm{ind}(I^1_{[\mathbf{c},m,n]}) \\ &= \mathrm{ind}(I_{\mathbf{c}}) + n\cdot(\mathrm{ind}(J_{[\mathbf{c},m]})+1) \\ &= N\left(R(I_{\mathbf{c}});\ H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) + N\left(R(J_{[\mathbf{c},m]});\ H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) + 1. \end{aligned} \tag{4.18}$$

In order to proceed, we first show that $R(J_{[\mathbf{c},m]}) < R(I_{\mathbf{c}})$. Assume by contradiction this is not the case, namely $R(J_{[\mathbf{c},m]}) \ge R(I_c)$. We aim to apply Lemma 4.7 for $\lambda_{\mathbf{o}} = R(I_{\mathbf{c}})$ and $\mu_{\mathbf{o}} = R(I^1_{[\mathbf{c},m,n]})$. With (4.18) and $R(J_{[\mathbf{c},m]}) \ge R(I_c)$ at hand, we conclude

$$\begin{aligned} N\left(R(I^1_{[\mathbf{c},m,n]});\ H_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m,n]})\right) &= N\left(R(I_{\mathbf{c}});\ H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) + N\left(R(J_{[\mathbf{c},m]});\ H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) + 1 \\ &\ge N\left(R(I_{\mathbf{c}});\ H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) + N\left(R(I_{\mathbf{c}});\ H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) + 1. \end{aligned}$$

Using the notation from Lemma 4.7, the latter reads $N_{[\mathbf{c},m,n]} \ge N_{\mathbf{c}} + N_{[\mathbf{c},m]} + 1$. Thus, Lemma 4.7 yields $\mu_{\mathbf{o}} \ge \lambda_{\mathbf{o}}$ as $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}$ are admissible. On the other hand, $I^1_{[\mathbf{c},m,n]} \subseteq_{\rm str} I_{\mathbf{c}}$ implies $R(I^1_{[\mathbf{c},m,n]}) = \mu_{\mathbf{o}} < \lambda_{\mathbf{o}} = R(I_{\mathbf{c}})$, a contradiction. Hence, $R(J_{[\mathbf{c},m]}) < R(I_{\mathbf{c}})$ follows as claimed.

With this at hand, we continue applying once again Lemma 4.7, but this time for $\lambda_{\mathbf{o}} = R(J_{[\mathbf{c},m]})$ and $\mu_{\mathbf{o}} = R(I^1_{[\mathbf{c},m,n]})$. Using (4.18) and $R(J_{[\mathbf{c},m]}) < R(I_{\mathbf{c}})$, we infer

$$\begin{aligned} N\left(R(I^1_{[\mathbf{c},m,n]});\ H_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m,n]})\right) &= N\left(R(I_{\mathbf{c}});\ H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) + N\left(R(J_{[\mathbf{c},m]});\ H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) + 1 \\ &\ge N\left(R(J_{[\mathbf{c},m]});\ H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) + N\left(R(J_{[\mathbf{c},m]});\ H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) + 1. \end{aligned}$$

Using the notation from Lemma 4.7, the latter reads $N_{[\mathbf{c},m,n]} \ge N_{\mathbf{c}} + N_{[\mathbf{c},m]} + 1$. Recall that we showed in the beginning of the proof, $L(I_{\mathbf{c}}(V_1)) < R(J_{[\mathbf{c},m]}(V_1))$. This, together with $R(J_{[\mathbf{c},m]}(V_1)) < R(I_{\mathbf{c}}(V_1))$, implies that $R(J_{[\mathbf{c},m]}(V_1))$ is not an eigenvalue of $H_{\mathbf{c},V_1}(\theta_{\mathbf{c}})$. Hence,

$\lambda_{\mathbf{o}} = R(J_{[\mathbf{c},m]}(V_1))$ is a simple eigenvalue of $H_{\mathbf{c},V_1}(\theta_{\mathbf{c}}) \oplus H^{\times n}_{[\mathbf{c},m],V_1}(\theta_{[\mathbf{c},m]})$ using Lemma 4.5 (e). Thus, Lemma 4.7 (a) applied with $N_{[\mathbf{c},m,n]} \geq N_{\mathbf{c}} + N_{[\mathbf{c},m]} + 1$ yields that $\mu_{\mathbf{o}} > \lambda_{\mathbf{o}}$, i.e., $R(I^1_{[\mathbf{c},m,n]}) > R(J_{[\mathbf{c},m]})$, as required.

(b) We prove that $L(I^{M+1}_{[\mathbf{c},m,n]}(V_1)) < L(K_{[\mathbf{c},m]}(V_1))$. First we note that this inequality immediately holds if $R(I_{\mathbf{c}}(V_1)) \leq L(K_{[\mathbf{c},m]}(V_1))$, because $I^{M+1}_{[\mathbf{c},m,n]}(V_1) \subseteq_{\text{str}} I_{\mathbf{c}}(V_1)$ by assumption. Therefore, we assume from now on that $L(K_{[\mathbf{c},m]}(V_1)) < R(I_{\mathbf{c}}(V_1))$. In order to simplify the notation, we will omit the dependence on V_1 in the following unless we want to emphasize its dependence. Let $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]} \in \{0, \pi\}$ be such that

$$L(I_{\mathbf{c}}) \in \sigma\left(H_{\mathbf{c}}(\theta_{\mathbf{c}})\right), \; L(K_{[\mathbf{c},m]}) \in \sigma\left(H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) \text{ and } L(I^{M+1}_{[\mathbf{c},m,n]}) \in \sigma\left(H_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m,n]})\right).$$

Then these spectral edges, respectively $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}$, are admissible by inserting the index relation (4.12) of Lemma 4.9 into the characterization of admissibility from Lemma 4.6. By Definition 4.8, we have $\operatorname{ind}(K_{[\mathbf{c},m]}) = \operatorname{ind}(I^M_{[\mathbf{c},m]}) + 1$. With this at hand, Lemma 4.4 (b) for $[\mathbf{c}, m, n] \in \mathcal{C}$ and $\mathbf{c} \in \mathcal{C}$, the index relation (4.12) of Lemma 4.9 and Lemma 4.5 (c) for the spectral band $K_{[\mathbf{c},m]}$ imply

$$\begin{aligned} N\left(L(I^{M+1}_{[\mathbf{c},m,n]});\; H_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m,n]})\right) &= \operatorname{ind}(I^{M+1}_{[\mathbf{c},m,n]}) \\ &= \operatorname{ind}(I_{\mathbf{c}}) + n \cdot \operatorname{ind}(K_{[\mathbf{c},m]}) \\ &= N\left(L(I_{\mathbf{c}});\; H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) + N\left(L(K_{[\mathbf{c},m]});\; H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right). \end{aligned} \tag{4.19}$$

In order to proceed, we first show that $L(K_{[\mathbf{c},m]}) > L(I_{\mathbf{c}})$. Assume by contradiction this is not the case, namely $L(K_{[\mathbf{c},m]}) \leq L(I_c)$. We aim to apply Lemma 4.7 for $\lambda_{\mathbf{o}} = L(I_{\mathbf{c}})$ and $\mu_{\mathbf{o}} = L(I^{M+1}_{[\mathbf{c},m,n]})$. If $L(K_{[\mathbf{c},m]}) = L(I_c)$, then $\lambda_{\mathbf{o}}$ has multiplicity $\mathcal{M}_{\lambda_{\mathbf{o}}} = 2$. Thus, the previous identity (4.19) leads to

$$N\left(L(I^{M+1}_{[\mathbf{c},m,n]});\; H_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m,n]})\right) < N\left(L(I_{\mathbf{c}});\; H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) + N\left(L(I_{\mathbf{c}});\; H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) + \mathcal{M}_{\lambda_{\mathbf{o}}} - 1.$$

If $L(K_{[\mathbf{c},m]}) < L(I_c)$, then $N\left(L(K_{[\mathbf{c},m]});\; H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) \leq N\left(L(I_{\mathbf{c}});\; H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) - 1$ follows and the multiplicity of $\lambda_{\mathbf{o}}$ satisfies $\mathcal{M}_{\lambda_{\mathbf{o}}} \geq 1$. Combing these with the Equation (4.19) and $L(K_{[\mathbf{c},m]}) \in \sigma\left(H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right)$, we conclude

$$\begin{aligned} N\left(L(I^{M+1}_{[\mathbf{c},m,n]});\; H_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m,n]})\right) &= N\left(L(I_{\mathbf{c}});\; H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) + N\left(L(K_{[\mathbf{c},m]});\; H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) \\ &< N\left(L(I_{\mathbf{c}});\; H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) + N\left(L(I_{\mathbf{c}});\; H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) + \mathcal{M}_{\lambda_{\mathbf{o}}} - 1. \end{aligned}$$

Using the notation from Lemma 4.7, the latter reads $N_{[\mathbf{c},m,n]} < N_{\mathbf{c}} + N_{[\mathbf{c},m]} + \mathcal{M}_{\lambda_{\mathbf{o}}} - 1$ whenever $L(K_{[\mathbf{c},m]}) \leq L(I_c)$. Thus, Lemma 4.7 (a) yields $\mu_{\mathbf{o}} \leq \lambda_{\mathbf{o}}$ as $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}$ are admissible. On the other hand, $I^{M+1}_{[\mathbf{c},m,n]} \subseteq_{\text{str}} I_{\mathbf{c}}$ implies $L(I^{M+1}_{[\mathbf{c},m,n]}) = \mu_{\mathbf{o}} > \lambda_{\mathbf{o}} = L(I_{\mathbf{c}})$, a contradiction. Hence, $L(K_{[\mathbf{c},m]}) > L(I_{\mathbf{c}})$ follows as claimed.

With this at hand, we continue applying once again Lemma 4.7, but this time for $\lambda_{\mathbf{o}} = L(K_{[\mathbf{c},m]})$ and $\mu_{\mathbf{o}} = L(I^{M+1}_{[\mathbf{c},m,n]})$. Using (4.19), the inequality $L(K_{[\mathbf{c},m]}) > L(I_{\mathbf{c}})$ and $L(I_{\mathbf{c}}) \in \sigma(H_{\mathbf{c}}(\theta_{\mathbf{c}})$, we infer

$$\begin{aligned} N\left(L(I^{M+1}_{[\mathbf{c},m,n]});\; H_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m,n]})\right) &= N\left(L(I_{\mathbf{c}});\; H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) + N\left(L(K_{[\mathbf{c},m]});\; H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) \\ &< N\left(L(K_{[\mathbf{c},m]});\; H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) + N\left(L(K_{[\mathbf{c},m]});\; H^{\times n}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right). \end{aligned}$$

Using the notation from Lemma 4.7, the latter reads $N_{[\mathbf{c},m,n]} < N_{\mathbf{c}} + N_{[\mathbf{c},m]}$. Recall that we showed in the beginning of the proof, $L(K_{[\mathbf{c},m]}(V_1)) < R(I_{\mathbf{c}}(V_1))$. This, together with $L(I_{\mathbf{c}}(V_1)) < L(K_{[\mathbf{c},m]}(V_1))$, implies that $L(K_{[\mathbf{c},m]}(V_1))$ is not an eigenvalue of $H_{\mathbf{c},V_1}(\theta_{\mathbf{c}})$. Hence,

$\lambda_{\mathbf{o}} = L(K_{[\mathbf{c},m]}(V_1))$ is a simple eigenvalue of $H_{\mathbf{c},V_1}(\theta_{\mathbf{c}}) \oplus H^{\times n}_{[\mathbf{c},m],V_1}(\theta_{[\mathbf{c},m]})$ using Lemma 4.5 (e). Thus, Lemma 4.7 applied with $N_{[\mathbf{c},m,n]} < N_{\mathbf{c}} + N_{[\mathbf{c},m]}$ yields that $\mu_{\mathbf{o}} < \lambda_{\mathbf{o}}$, i.e., $L(I^{M+1}_{[\mathbf{c},m,n]}) < L(K_{[\mathbf{c},m]})$. □

Next we show that the assumptions in the previous Corollary 4.16 are satisfied whenever the spectral band is of backward type A or B. The proofs of the previous lemmas in this subsection were mainly based on applications of Lemma 4.7. This lemma will keep being applied in the next proofs, but we will also need to make use of some trace identities, as appear in Lemma 4.10 and Lemma 4.11.

Lemma 4.17. *Let $m \in \mathbb{N}$ and $\boldsymbol{c} \in \mathcal{C}$ be such that $\varphi(\boldsymbol{c}) \in (0,1)$. Let $V_1 > 0$ and $I_{\boldsymbol{c}}$ be a spectral band in $\sigma_{\boldsymbol{c}}$ with associated spectral bands $\{I^i_{[\boldsymbol{c},m]}\}_{i=1}^M$ and $\{I^j_{[\boldsymbol{c},m,n]}\}_{j=1}^{M+1}$ introduced in Definition 3.3. If either*

- $I_{\mathbf{c}}(V)$ is of backward type A for all $V \geq V_1$ and $M := m-1$, or
- $I_{\mathbf{c}}(V)$ is of backward type B for all $V \geq V_1$ and $M := m$,

then

$$I^1_{[\mathbf{c},m,1]}(V),\ I^{M+1}_{[\mathbf{c},m,1]}(V) \subseteq_{\text{str}} I_{\mathbf{c}}(V)$$

and

$$I^1_{[\mathbf{c},m]}(V),\ I^{M}_{[\mathbf{c},m]}(V) \subseteq_{\text{str}} I_{\mathbf{c}}(V)$$

for all $V \geq V_1$.

Remark. We have to exclude the cases $\varphi(\mathbf{c}) \in \{0, \pm 1, \infty\}$ so that we can apply Lemma 4.11 (c).

Proof. The claim follows once we show that for all $V \geq V_1$,

$$\begin{aligned} L(I_{\mathbf{c}}(V)) &< \min\left\{L(I^1_{[\mathbf{c},m,1]}(V)),\ L(I^1_{[\mathbf{c},m]}(V))\right\}, \\ \max\left\{R(I^M_{[\mathbf{c},m]}(V)),\ R(I^{M+1}_{[\mathbf{c},m,1]}(V))\right\} &< R(I_{\mathbf{c}}(V)). \end{aligned} \tag{4.20}$$

Assume by contradiction that (4.20) does not hold for some $V \geq V_1$. Due to Theorem 2.16, these strict inequalities in (4.20) hold for $V > 4$. Thus, the continuity of the spectral band edges in $V > 0$ (Corollary 3.2) implies that the maximum

$$V_2 := \max\{V \geq V_1 \,:\, (4.20) \text{ does not hold}\}$$

exists and $V_2 \in [V_1, 4]$. Due to Lemma 4.14, the strict inclusions $I^1_{[\mathbf{c},m]}(V) \subseteq_{\text{str}} I_{\mathbf{c}}(V)$ and $I^M_{[\mathbf{c},m]}(V) \subseteq_{\text{str}} I_{\mathbf{c}}(V)$ for $V > V_2$ yield

$$L(I^1_{[\mathbf{c},m,1]}(V)) < L(I^1_{[\mathbf{c},m]}(V)) \qquad \text{and} \qquad R(I^M_{[\mathbf{c},m]}(V)) < R(I^{M+1}_{[\mathbf{c},m,1]}(V)) \qquad \text{for } V > V_2.$$

Let $J_{[\mathbf{c},m]}$ and $K_{[\mathbf{c},m]}$ be the spectral bands associated with $I_{\mathbf{c}}$ (Definition 4.8). Since the strict inclusions $I^1_{[\mathbf{c},m,n]}(V) \subseteq_{\text{str}} I_{\mathbf{c}}(V)$ and $I^{M+1}_{[\mathbf{c},m,n]}(V) \subseteq_{\text{str}} I_{\mathbf{c}}(V)$ hold for $V > V_2$, Lemma 4.15 asserts

$$R(J_{[\mathbf{c},m]}(V)) < R(I^1_{[\mathbf{c},m,n]}(V)) \quad \text{and} \quad L(I^{M+1}_{[\mathbf{c},m,n]}(V)) < L(K_{[\mathbf{c},m]}(V)) \qquad \text{for } V > V_2.$$

Note that we have $R(I^1_{[\mathbf{c},m,n]}(V)) \leq R(I^{M+1}_{[\mathbf{c},m,n]}(V))$ and $L(I^1_{[\mathbf{c},m,n]}(V)) \leq L(I^{M+1}_{[\mathbf{c},m,n]}(V))$ for $V > 0$. Hence, the continuity of the spectral band edges in $V > 0$ (Corollary 3.2) leads to

$$\begin{aligned} L(I^1_{[\mathbf{c},m,1]}(V_2)) &\leq \min\left\{L(I^1_{[\mathbf{c},m]}(V_2)), L(K_{[\mathbf{c},m]}(V_2))\right\}, \\ \max\left\{R(I^M_{[\mathbf{c},m]}(V_2)), R(J_{[\mathbf{c},m]}(V_2))\right\} &\leq R(I^{M+1}_{[\mathbf{c},m,1]}(V_2)). \end{aligned} \tag{4.21}$$

Note that the spectral bands $J_{[\mathbf{c},m]}$ and $K_{[\mathbf{c},m]}$ may not exist simplifying our considerations below. This in particular implies

$$V_2 = \max\left\{ V \geq V_1 \,:\, L(I_{\mathbf{c}}(V)) = L(I^1_{[\mathbf{c},m,1]}(V)) \text{ or } R(I_{\mathbf{c}}(V)) = R(I^{M+1}_{[\mathbf{c},m,1]}(V)) \right\}.$$

We continue proving that this leads to a contradiction.

Case 1: We show that $L(I_{\mathbf{c}}(V_2)) = L(I^1_{[\mathbf{c},m,1]}(V_2))$ yields a contradiction. Set $E := L(I_{\mathbf{c}}(V_2))$. Thus, $\left|t_{[\mathbf{c},m,1]}(E;\ V_2)\right| = 2$ follows from Lemma 4.10 (a). Since $I_{\mathbf{c}}(V_2)$ is of backward type A or B (using $V_2 \geq V_1$) and $\varphi(\mathbf{c}) \in (0,1)$, Lemma 4.11 (c) yields $\left|t_{[\mathbf{c},m]}(E;\ V_2)\right| < 2$. Hence, E must lie in the interior of a spectral band in $\sigma_{[\mathbf{c},m]}(V_2)$ by Lemma 4.10 (a). Thus, Equation (4.21) and $L(I_{\mathbf{c}}(V_2)) = L(I^1_{[\mathbf{c},m,1]}(V_2))$ lead to

$$E = L(I_{\mathbf{c}}(V_2)) < R(J_{[\mathbf{c},m]}(V_2)).$$

Note that if $J_{[\mathbf{c},m]}$ does not exist, then there is no spectra to the left of $L(I^1_{[\mathbf{c},m]}(V_2))$ contradicting $\left|t_{[\mathbf{c},m]}(E;\ V_2)\right| < 2$ and (4.21). Hence, we may continue assuming that $J_{[\mathbf{c},m]}$ exists. Next we aim to apply Lemma 4.7 for $\lambda_{\mathbf{o}} = L(I_{\mathbf{c}}(V_2))$ and $\mu_{\mathbf{o}} = L(I^1_{[\mathbf{c},m,1]}(V_2))$. For the sake of simplification, we drop the V_2 notation in the following. Let $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,1]} \in \{0,\pi\}$ be such that

$$L(I_{\mathbf{c}}) \in \sigma\left(H_{\mathbf{c}}(\theta_{\mathbf{c}})\right),\ R(J_{[\mathbf{c},m]}) \in \sigma\left(H^{\times 1}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) \text{ and } L(I^1_{[\mathbf{c},m,1]}) \in \sigma\left(H_{[\mathbf{c},m,1]}(\theta_{[\mathbf{c},m,1]})\right).$$

Then these spectral edges, respectively $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,1]}$, are admissible by inserting the index relation (4.11) into the characterization of admissibility from Lemma 4.6 for $n = 1$. With this at hand, and Lemma 4.4 (b) applied to $\mathbf{c}, [\mathbf{c},m,1] \in \mathcal{C}$ leads to

$$N_{\mathbf{c}} := N\left(L(I_{\mathbf{c}});\ H_{\mathbf{c}}(\theta_{\mathbf{c}})\right) = \operatorname{ind}(I_{\mathbf{c}})$$

and using (4.11)

$$N_{[\mathbf{c},m,1]} := N\left(L(I^1_{[\mathbf{c},m,1]});\ H_{[\mathbf{c},m,1]}(\theta_{[\mathbf{c},m,1]})\right) = \operatorname{ind}(I^1_{[\mathbf{c},m,1]}) = \operatorname{ind}(J_{[\mathbf{c},m]}) + 1 + \operatorname{ind}(I_{\mathbf{c}}).$$

Furthermore, $L(I_{\mathbf{c}}) < R(J_{[\mathbf{c},m]})$ and Lemma 4.5 (d) applied to $[\mathbf{c},m] \in \mathcal{C}$ for $n = 1$ lead to

$$N_{[\mathbf{c},m]} := N\left(L(I_{\mathbf{c}});\ H^{\times 1}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) \leq N\left(R(J_{[\mathbf{c},m]});\ H^{\times 1}_{[\mathbf{c},m]}(\theta_{[\mathbf{c},m]})\right) = \operatorname{ind}(J_{[\mathbf{c},m]}).$$

Thus, $N_{\mathbf{c}} + N_{[\mathbf{c},m]} < N_{[\mathbf{c},m,1]}$ follows. Since $\lambda_{\mathbf{o}} = E = L(I_{\mathbf{c}}(V_2))$ lies in the interior of a spectral band in $\sigma_{[\mathbf{c},m]}(V_2)$ and the eigenvalues of $H^{\times 1}_{[\mathbf{c},m],V_2}(\theta_{[\mathbf{c},m]})$ are contained in the spectral band edges of $\sigma_{[\mathbf{c},m]}(V_2)$ (by Lemma 4.5), we conclude that $\lambda_{\mathbf{o}}$ is not an eigenvalue of $H^{\times 1}_{[\mathbf{c},m],V_2}(\theta_{[\mathbf{c},m]})$. Thus, $\lambda_{\mathbf{o}}$ is a simple eigenvalue of $H^{\times 1}_{[\mathbf{c},m],V_2}(\theta_{[\mathbf{c},m]}) \oplus H_{\mathbf{c},V_2}(\theta_{\mathbf{c}})$ using Lemma 4.5 (e). Hence, Lemma 4.7 (a) yields

$$\lambda_{\mathbf{o}} = L(I_{\mathbf{c}}(V_2)) < L(I^1_{[\mathbf{c},m,1]}(V_2)) = \mu_{\mathbf{o}},$$

contradicting that these two values are equal by the initial assumption of the considered case.

Case 2: Similarly as in Case 1, $R(I^{M+1}_{[\mathbf{c},m,1]}(V_2)) = R(I_{\mathbf{c}}(V_2))$ yields a contradiction. □

We have seen that Corollary 4.16 is set towards proving the forward properties (A1), (B2), (I). Next, we aim to prove the forward property (B1), (also called the *tower* property).

Corollary 4.18. *Let $V_1 > 0$, $m \in \mathbb{N}$ and $\boldsymbol{c} \in \mathcal{C}$ be such that $\varphi(\boldsymbol{c}) \notin \{-1, \infty\}$ and $[\boldsymbol{c}, m] \in \mathcal{C}$. Consider a spectral band $I_{\boldsymbol{c}}$ in $\sigma_{\boldsymbol{c}}$ with associated spectral bands $\{I^i_{[\boldsymbol{c},m]}\}_{i=1}^M$ and $\{I^j_{[\boldsymbol{c},m,n]}\}_{j=1}^{M+1}$ introduced in Definition 3.3. If $1 \leq j \leq M+1$ and $I^j_{[\boldsymbol{c},m,n]}(V) \subseteq_{\text{str}} I_{\boldsymbol{c}}(V)$ holds for all $V \geq V_1$ and all $n \in \mathbb{N}$, then*

$$I^j_{[\boldsymbol{c},m,n]}(V) \subseteq_{\text{str}} I^j_{[\boldsymbol{c},m,n-1]}(V)$$

holds for all $n \in \mathbb{N}$ and $V \geq V_1$ where $I^j_{[\boldsymbol{c},m,0]}(V) = I_{\boldsymbol{c}}(V)$.

Proof. The proof is by induction over $n \in \mathbb{N}$. The induction base ($n = 1$) holds trivially since $I^j_{[\mathbf{c},m,1]}(V) \subseteq_{\mathrm{str}} I_{\mathbf{c}}(V)$ for all $V \geq V_1$ and $\sigma_{[\mathbf{c},m,n-1]}(V) = \sigma_{\mathbf{c}}(V)$ if $n = 1$ by Proposition II.2 (a).

For the induction step, suppose $I^j_{[\mathbf{c},m,n]}(V) \subseteq_{\mathrm{str}} I^j_{[\mathbf{c},m,n-1]}(V)$ holds for all $V \geq V_1$. We show that $I^j_{[\mathbf{c},m,n+1]}(V) \subseteq_{\mathrm{str}} I^j_{[\mathbf{c},m,n]}(V)$ holds for all $V \geq V_1$. Due to Proposition II.2 (a), we have $\sigma_{[\mathbf{c},m,n+1]}(V) = \sigma_{[\mathbf{c},m,n,1]}(V)$. Furthermore, $I^j_{[\mathbf{c},m,n+1]}(V) \subseteq_{\mathrm{str}} I^j_{[\mathbf{c},m,n]}(V)$ holds for $V > 4$ since $I_{\mathbf{c}}(V)$ is either of type A or B for $V > 4$ by Theorem 2.16. Thus, $I^j_{[\mathbf{c},m,n+1]}(V)$ equals to the unique spectral band $I^1_{[\mathbf{c},m,n,1]}(V)$ of type A that is strictly contained in $I^j_{[\mathbf{c},m,n]}(V)$ for $V > 4$. Hence, it suffices to prove $I^1_{[\mathbf{c},m,n,1]}(V) \subseteq_{\mathrm{str}} I^j_{[\mathbf{c},m,n]}(V)$ for all $V \geq V_1$.

Let $V \geq V_1$. By induction hypothesis, we have $I^j_{[\mathbf{c},m,n]}(V) \subseteq_{\mathrm{str}} I^j_{[\mathbf{c},m,n-1]}(V)$ for all $V \geq V_1$, namely $I^j_{[\mathbf{c},m,n]}(V)$ is of backward type B for all $V \geq V_1$. Furthermore, $\varphi([\mathbf{c},m,n]) \in (0,1)$ holds as $m,n \in \mathbb{N}$. Thus, Lemma 4.17 applied to $[\mathbf{c},m,n]$ implies $I^1_{[\mathbf{c},m,n,1]}(V) \subseteq_{\mathrm{str}} I^j_{[\mathbf{c},m,n]}(V)$ for all $V \geq V_1$. □

The next lemma is the crucial ingredient to prove the forward property (A2).

Lemma 4.19. *Let $V_1 > 0$ and $\mathbf{c} \in \mathcal{C}$ be such that $\varphi(\mathbf{c}) \in (0,1)$. Consider a spectral band $V \mapsto I_{\mathbf{c}}(V)$ in $\sigma_{\mathbf{c}}(V)$ which is of backward type B for all $V \geq V_1$ and $I^1_{[\mathbf{c},1]}$ is the associated spectral band introduced in Definition 3.3. Then for all $V \geq V_1$, $I^1_{[\mathbf{c},1]}(V) \subseteq_{\mathrm{str}} I_{\mathbf{c}}(V)$ (namely $I^1_{[\mathbf{c},1]}(V)$ is of backward type A) and $I^1_{[\mathbf{c},1]}(V)$ is not of weak backward type B.*

Proof. Since $I_{\mathbf{c}}(V)$ is of backward type B for all $V \geq V_1$, it follows that $I_{\mathbf{c}}(V)$ is of type B for all $V > 4$ by Theorem 2.16. Thus, there is a unique spectral band $I^1_{[\mathbf{c},1]}$ in $\sigma_{[\mathbf{c},1]}$ such that $I^1_{[\mathbf{c},1]}(V) \subseteq_{\mathrm{str}} I_{\mathbf{c}}(V)$ for all $V > 4$. By Theorem 2.16, the lemma holds for all $V_1 > 4$, and so we can assume in the proof that $V_1 \leq 4$.

Consider the spectral bands $J_{[\mathbf{c},0]}$ and $K_{[\mathbf{c},0]}$ associated with $I_{\mathbf{c}}$ (see Definition 4.8). Since $I_{\mathbf{c}}(V)$is of backward type B for $V > 4$, we have $\mathrm{ind}(K_{[\mathbf{c},0]}) = \mathrm{ind}(J_{[\mathbf{c},0]}) + 1$ (i.e., there is no other spectral band between those two) and

$$\forall V > 4, \qquad J_{[\mathbf{c},0]}(V) \prec I_{\mathbf{c}}(V) \prec K_{[\mathbf{c},0]}(V).$$

Lemma 4.17 implies $I^1_{[\mathbf{c},1]}(V) \subseteq_{\mathrm{str}} I_{\mathbf{c}}(V)$ for all $V \geq V_1$. It is left to prove that for $V \geq V_1$, $I^1_{[\mathbf{c},1]}(V)$ is not contained in any spectral band of $\sigma_{[\mathbf{c},1,-1]}(V) = \sigma_{[\mathbf{c},0]}(V)$ (where the last equality follows from Proposition II.2). Actually, it suffices to prove that for all $V \geq V_1$,

$$R(J_{[\mathbf{c},0]}(V)) < R(I^1_{[\mathbf{c},1]}(V)) \quad \text{and} \quad L(I^1_{[\mathbf{c},1]}(V)) < L(K_{[\mathbf{c},0]}(V)). \tag{4.22}$$

Assume by contradiction that (4.22) does not hold for some $V \geq V_1$. By Theorem 2.16, (4.22) holds for $V > 4$. Thus, the continuity of the spectral band edges in $V > 0$ (Corollary 3.2) implies that the maximum

$$V_2 := \max\left\{ V \geq V_1 \,:\, R(J_{[\mathbf{c},0]}(V)) = R(I^1_{[\mathbf{c},1]}(V)) \text{ or } L(I^1_{[\mathbf{c},1]}(V)) = L(K_{[\mathbf{c},0]}(V)) \right\}$$

exists and $V_2 \in [V_1, 4]$. We split into cases according to the nature of failure of (4.22) at $V = V_2$, and show a contradiction for each of these cases. First note that Equation (4.13) of Lemma 4.9 implies

$$\mathrm{ind}(I^1_{[\mathbf{c},1]}) = \mathrm{ind}(J_{[\mathbf{c},0]}) + 1 + \mathrm{ind}(I_{\mathbf{c}}) = \mathrm{ind}(K_{[\mathbf{c},0]}) + \mathrm{ind}(I_{\mathbf{c}}). \tag{4.23}$$

Since $\varphi(\mathbf{c}) \in (0,1)$, there is a $k \in \mathbb{N}$ such that $\mathbf{c} = [0, c_0, \ldots, c_k]$. In the following we apply Lemma 4.7 to $\widetilde{\mathbf{c}}, [\widetilde{\mathbf{c}}, m], [\widetilde{\mathbf{c}}, m, n] \in \mathcal{C}$ where $\widetilde{\mathbf{c}} = [0, c_0, \ldots, c_{k-1}]$, $m = c_k$ and $n = 1$. Note that $\varphi(\widetilde{\mathbf{c}}) = \varphi([\mathbf{c}, 0])$, $\varphi([\widetilde{\mathbf{c}}, m]) = \varphi(\mathbf{c})$ and $\varphi([\widetilde{\mathbf{c}}, m, n]) = \varphi([\mathbf{c}, 1])$. Thus, in effect it is as if we apply Lemma 4.7 to $[\mathbf{c},0],\mathbf{c},[\mathbf{c},1]\in\mathcal{C}$ (rather than to $\widetilde{\mathbf{c}},[\widetilde{\mathbf{c}},m],[\widetilde{\mathbf{c}},m,n]\in\mathcal{C}$). We use this convention until the end of the current proof.

Case 1: We show that $R(J_{[\mathbf{c},0]}(V_2)) = R(I^1_{[\mathbf{c},1]}(V_2))$ yields a contradiction. We aim to apply Lemma 4.7 for $\lambda_{\mathbf{o}} = R(J_{[\mathbf{c},0]}(V_2))$ and $\mu_{\mathbf{o}} = R(I^1_{[\mathbf{c},1]}(V_2))$. For the sake of simplification, we drop the V_2 notation in the following unless we want to emphasize its dependence. Let $\theta_{[\mathbf{c},0]},\theta_{\mathbf{c}},\theta_{[\mathbf{c},1]}\in\{0,\pi\}$ be such that

$$R(J_{[\mathbf{c},0]})\in\sigma\left(H_{[\mathbf{c},0]}(\theta_{[\mathbf{c},0]})\right),\ R(I_{\mathbf{c}})\in\sigma\left(H^{\times 1}_{\mathbf{c}}(\theta_{\mathbf{c}})\right)\text{ and }R(I^1_{[\mathbf{c},1]})\in\sigma\left(H_{[\mathbf{c},1]}(\theta_{[\mathbf{c},1]})\right).$$

Then these spectral edges, respectively $\theta_{[\mathbf{c},0]},\theta_{\mathbf{c}},\theta_{[\mathbf{c},1]}$, are admissible by inserting the index relation (4.23) into the characterization of admissibility from Lemma 4.6 for $n=1$. With this at hand, Lemma 4.4 (c) applied to $[\mathbf{c},0]\in\mathcal{C}$ and $[\mathbf{c},1]\in\mathcal{C}$ leads to

$$N_{[\mathbf{c},0]} := N\left(R(J_{[\mathbf{c},0]});\ H_{[\mathbf{c},0]}(\theta_{[\mathbf{c},0]})\right) = \operatorname{ind}(J_{[\mathbf{c},0]})$$

and

$$N_{[\mathbf{c},1]} := N\left(R(I^1_{[\mathbf{c},1]});\ H_{[\mathbf{c},1]}(\theta_{[\mathbf{c},1]})\right) = \operatorname{ind}(I^1_{[\mathbf{c},1]}).$$

Furthermore, $I^1_{[\mathbf{c},1]}(V_2)\subseteq_{\text{str}} I_{\mathbf{c}}(V_2)$ and the assumption $R(J_{[\mathbf{c},0]}(V_2)) = R(I^1_{[\mathbf{c},1]}(V_2))$ imply $R(J_{[\mathbf{c},0]}(V_2)) < R(I_{\mathbf{c}}(V_2))\in\sigma\left(H^{\times 1}_{\mathbf{c},V_2}(\theta_{\mathbf{c}})\right)$. Thus, Lemma 4.5 (d) applied to $n=1$ and $\mathbf{c}\in\mathcal{C}$ imply

$$N_{\mathbf{c}} := N\left(R(J_{[\mathbf{c},0]});\ H^{\times 1}_{\mathbf{c}}(\theta_{\mathbf{c}})\right) \le N\left(R(I_{\mathbf{c}});\ H^{\times 1}_{\mathbf{c}}(\theta_{\mathbf{c}})\right) = \operatorname{ind}(I_{\mathbf{c}}).$$

Thus, (4.23) implies $N_{[\mathbf{c},1]} > N_{\mathbf{c}} + N_{[\mathbf{c},0]}$. If we prove that $\lambda_{\mathbf{o}} = R(J_{[\mathbf{c},0]}(V_2))$ is a simple eigenvalue of $H^{\times 1}_{\mathbf{c},V_2}(\theta_{\mathbf{c}})\oplus H_{[\mathbf{c},0],V_2}(\theta_{[\mathbf{c},0]})$, then Lemma 4.7 yields $\lambda_{\mathbf{o}} = R(J_{[\mathbf{c},0]}(V_2)) < R(I^1_{[\mathbf{c},1]}(V_2)) = \mu_{\mathbf{o}}$, a contradiction.

By Lemma 4.5 (e), simplicity of the eigenvalue $\lambda_{\mathbf{o}}$ holds if it is not an eigenvalue of $H^{\times 1}_{\mathbf{c},V_2}(\theta_{\mathbf{c}}) = H_{\mathbf{c},V_2}(\theta_{\mathbf{c}})$. Using Lemma 4.5 (e), $R(I_{\mathbf{c}}(V_2))$ is the only eigenvalue of $H_{\mathbf{c},V_2}(\theta_{\mathbf{c}})$ in $I_{\mathbf{c}}(V_2)$. Thus, our working assumption, $R(J_{[\mathbf{c},0]}(V_2)) = R(I^1_{[\mathbf{c},1]}(V_2)) < R(I_{\mathbf{c}}(V_2))$ implies that $R(J_{[\mathbf{c},0]}(V_2))$ is not an eigenvalue of $H_{\mathbf{c},V_2}(\theta_{\mathbf{c}})$.

Case 2: We show that $L(I^1_{[\mathbf{c},1]}(V_2)) = L(K_{[\mathbf{c},0]}(V_2))$ yields a contradiction. We aim to apply Lemma 4.7 for $\lambda_{\mathbf{o}} = L(K_{[\mathbf{c},0]}(V_2))$ and $\mu_{\mathbf{o}} = L(I^1_{[\mathbf{c},1]}(V_2))$. For the sake of simplification, we drop the V_2 notation in the following unless we want to emphasize its dependence. Let $\theta_{[\mathbf{c},0]},\theta_{\mathbf{c}},\theta_{[\mathbf{c},1]}\in\{0,\pi\}$ be such that

$$L(K_{[\mathbf{c},0]})\in\sigma\left(H_{[\mathbf{c},0]}(\theta_{[\mathbf{c},0]})\right),\ L(I_{\mathbf{c}})\in\sigma\left(H^{\times 1}_{\mathbf{c}}(\theta_{\mathbf{c}})\right)\text{ and }L(I^1_{[\mathbf{c},1]})\in\sigma\left(H_{[\mathbf{c},1]}(\theta_{[\mathbf{c},1]})\right).$$

Then these spectral edges, respectively $\theta_{[\mathbf{c},0]},\theta_{\mathbf{c}},\theta_{[\mathbf{c},1]}$, are admissible by inserting the index relation (4.23) into the characterization of admissibility from Lemma 4.6 for $n=1$. With this at hand, Lemma 4.4 (b) applied to $[\mathbf{c},0]\in\mathcal{C}$ and $[\mathbf{c},1]\in\mathcal{C}$ leads to

$$N_{[\mathbf{c},0]} := N\left(L(K_{[\mathbf{c},0]});\ H_{[\mathbf{c},0]}(\theta_{[\mathbf{c},0]})\right) = \operatorname{ind}(K_{[\mathbf{c},0]})$$

and

$$N_{[\mathbf{c},1]} := N\left(L(I^1_{[\mathbf{c},1]});\ H_{[\mathbf{c},1]}(\theta_{[\mathbf{c},1]})\right) = \operatorname{ind}(I^1_{[\mathbf{c},1]}).$$

Furthermore, $I^1_{[\mathbf{c},1]}(V_2)\subseteq_{\text{str}} I_{\mathbf{c}}(V_2)$ and the assumption $L(I^1_{[\mathbf{c},1]}(V_2)) = L(K_{[\mathbf{c},0]}(V_2))$ imply $\sigma\left(H^{\times 1}_{\mathbf{c},V_2}(\theta_{\mathbf{c}})\right)\ni L(I_{\mathbf{c}}(V_2)) < L(K_{[\mathbf{c},0]}(V_2))$. Thus, Lemma 4.5 (c) applied to $n=1$ and $\mathbf{c}\in\mathcal{C}$ leads to

$$N_{\mathbf{c}} := N\left(L(K_{[\mathbf{c},0]});\ H^{\times 1}_{\mathbf{c}}(\theta_{\mathbf{c}})\right) \ge N\left(L(I_{\mathbf{c}});\ H^{\times 1}_{\mathbf{c}}(\theta_{\mathbf{c}})\right) + 1 = \operatorname{ind}(I_{\mathbf{c}}) + 1.$$

Thus, (4.23) implies $N_{[\mathbf{c},1]} < N_{\mathbf{c}} + N_{[\mathbf{c},0]}$. If we prove that $L(K_{[\mathbf{c},0]}(V_2))$ is a simple eigenvalue of $H^{\times 1}_{\mathbf{c},V_2}(\theta_{\mathbf{c}})\oplus H_{[\mathbf{c},0],V_2}(\theta_{[\mathbf{c},0]})$, then Lemma 4.7 yields $\lambda_{\mathbf{o}} = L(K_{[\mathbf{c},0]}(V_2)) > L(I^1_{[\mathbf{c},1]}(V_2)) = \mu_{\mathbf{o}}$, a contradiction.

By Lemma 4.5 (e), simplicity of the eigenvalue $\lambda_{\mathbf{o}}$ holds if it is not an eigenvalue of $H^{\times 1}_{\mathbf{c},V_2}(\theta_{\mathbf{c}}) = H_{\mathbf{c},V_2}(\theta_{\mathbf{c}})$. Using Lemma 4.5 (e), $L(I_{\mathbf{c}}(V_2))$ is the only eigenvalue of $H_{\mathbf{c},V_2}(\theta_{\mathbf{c}})$ in $I_{\mathbf{c}}(V_2)$. Thus, our working assumption, $L(I_{\mathbf{c}}(V_2)) < L(I^1_{[\mathbf{c},1]}(V_2)) = L(K_{[\mathbf{c},0]}(V_2))$ implies that $L(K_{[\mathbf{c},0]}(V_2))$ is not an eigenvalue of $H_{\mathbf{c},V_2}(\theta_{\mathbf{c}})$.

□

4.8. **Backward implies forward type property.** This section is devoted to show that if every spectral band in $\sigma_{\mathbf{c}}$ is either of backward type A or B, then each such spectral band is also of forward type A respectively B. This is the content of Proposition 4.20. This proposition is a crucial tool in proving Theorem 2.15 which is done in Section 6. We recall (Definition 2.14) that a spectral band is of m-type A (respectively B) if it is of backward type A (B) and of m-forward type A (B). A spectral band is of type A (respectively B) if it is of m-type A (B) for all $m \in \mathbb{N}$. With that we introduce a useful notation via which Proposition 4.20 is stated and proved:

$$V_{\text{crit}}([\mathbf{c}, m]) := \sup\left([0,\infty) \setminus \left\{ V \in \mathbb{R} \ : \ \begin{matrix} \text{each spectral band in } \sigma_{\mathbf{c}}(V) \text{ is either} \\ \text{of } m\text{-type } A \text{ or of } m\text{-type } B \end{matrix} \right\}\right)$$

for all $\mathbf{c} \in \mathcal{C}$ and $m \in \mathbb{N}$ such that $\varphi(\mathbf{c}) \notin \{-1, \infty\}$ and $[\mathbf{c}, m] \in \mathcal{C}$.

Proposition 4.20 (Backward implies forward type). *Let $\mathbf{c} \in \mathcal{C}$ and $\varphi(\mathbf{c}) \in (0,1)$. If each spectral band $I_{\mathbf{c}}(V)$ in $\sigma_{\mathbf{c}}(V)$ is either of backward type A for all $V > 0$ or of backward type B for all $V > 0$, then $V_{\text{crit}}([\mathbf{c}, m]) = 0$ for all $m \in \mathbb{N}$.*

We will state and prove two lemmas and two corollaries with the aid of which Proposition 4.20 is proven at the end of this subsection. But, before starting this, we will need to somewhat relax the notion of m-forward type and of the notation V_{crit}.

Definition 4.21. Let $V > 0$, $m \in \mathbb{N}$ and $\mathbf{c} \in \mathcal{C}$ be such that $\varphi(\mathbf{c}) \notin \{-1, \infty\}$ and $[\mathbf{c}, m] \in \mathcal{C}$. A spectral band $I_{\mathbf{c}}(V)$ in $\sigma_{\mathbf{c}}(V)$ is of *quasi m-type* A (respectively of *quasi m-type* B) if $I_{\mathbf{c}}(V)$ is of m-type A (resp. m-type B) but the property (A2) does not necessarily hold, i.e. the associated spectral bands $I^i_{[\mathbf{c},m]}(V)$ of $\sigma_{[\mathbf{c},m]}(V)$ for some $1 \leq i \leq M$ may be also of weak backward type B. With this notion at hand, define

$$V^{quasi}_{\text{crit}}([\mathbf{c}, m]) := \sup\left([0,\infty) \setminus \left\{ V \in \mathbb{R} \ : \ \begin{matrix} \text{each spectral band in } \sigma_{\mathbf{c}}(V) \text{ is either} \\ \text{of quasi } m\text{-type } A \text{ or of quasi } m\text{-type } B \end{matrix} \right\}\right).$$

Noting that

$$V^{quasi}_{\text{crit}}([\mathbf{c}, m]) \leq V_{\text{crit}}([\mathbf{c}, m]),$$

the strategy for proving Proposition 4.20 is to first show $V^{quasi}_{\text{crit}}([\mathbf{c}, m]) = 0$ and afterwards $V_{\text{crit}}([\mathbf{c}, m]) = 0$.

Lemma 4.22. *Let $m \in \mathbb{N}$, $\mathbf{c} \in \mathcal{C}$ be such $\varphi(\mathbf{c}) \notin \{-1, \infty\}$ and $[\mathbf{c}, m] \in \mathcal{C}$, and $V_0 \geq V^{quasi}_{\text{crit}}([\mathbf{c}, m])$. Let $I_{\mathbf{c}}$ be a spectral band in $\sigma_{\mathbf{c}}$ such that*

(a) $I_{\mathbf{c}}(V)$ is either of backward type A for all $V > 0$ or of backward type B for all $V > 0$, and
(b) $I^1_{[\mathbf{c},m,1]}(V)$, $I^{M+1}_{[\mathbf{c},m,1]}(V) \subseteq_{\text{str}} I_{\mathbf{c}}(V)$ holds for all $V \geq V_0$, where

$$M := \begin{cases} m-1 & I_{\mathbf{c}}(V) \text{ is of backward type } A \text{ for all } V > 0, \\ m & I_{\mathbf{c}}(V) \text{ is of backward type } B \text{ for all } V > 0. \end{cases}$$

If $V_0 > 0$, then there is a $\delta > 0$ such that either $I_{\mathbf{c}}(V)$ is of quasi m-type A for all $V > V_0 - \delta$ or $I_{\mathbf{c}}(V)$ is of quasi m-type B for all $V > V_0 - \delta$.

Remark. The δ in the statement of the lemma only depends on the values $\big|L\big(I^1_{[\mathbf{c},m,1]}(V_0)\big) - L\big(I_{\mathbf{c}}(V_0)\big)\big|$, $\big|R\big(I^{M+1}_{[\mathbf{c},m,1]}(V_0)\big) - R\big(I_{\mathbf{c}}(V_0)\big)\big|$ and V_0. Here we use that the Lipschitz continuity in Corollary 3.2 is independent in $\mathbf{c} \in \mathcal{C}$.

Proof. Since for $V > 4$, $I_{\mathbf{c}}(V)$ is either of m-type A or of m-type B (by Theorem 2.16), we may proceed assuming that $V_0 \leq 4$. Since $I_{\mathbf{c}}(V)$ is either of backward type A or backward type B for all $V > 0$, it is sufficient to prove the existence of a $\delta > 0$ such that

$$\text{all } I^i_{[\mathbf{c},m]} \text{ and } I^j_{[\mathbf{c},m,n]} \text{ satisfy properties (A1), (B1), (B2) and (I) for all } n \in \mathbb{N} \tag{4.24}$$

for all $V > V_0 - \delta$. Since by the assumptions of the lemma, $V_0 \geq V^{quasi}_{\text{crit}}$, we get that (4.24) holds for all $V > V_0$. In particular, we infer that

$$I^i_{[\mathbf{c},m]}(V), I^j_{[\mathbf{c},m,n]}(V) \subseteq_{\text{str}} \Big[L\big(I^1_{[\mathbf{c},m,1]}(V)\big), R\big(I^{M+1}_{[\mathbf{c},m,1]}(V)\big)\Big], \tag{4.25}$$

for all $V > V_0$, $1 \leq i \leq M$, $1 \leq j \leq M+1$ and $n \in \mathbb{N}$.

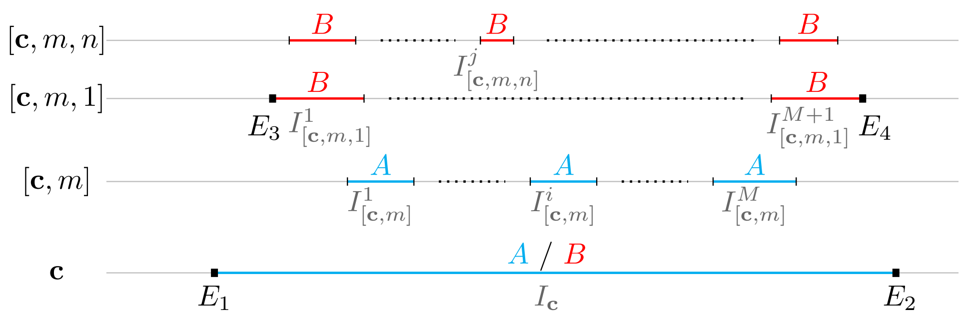

FIGURE 4.5. A sketch of the spectral bands considered in the proof of Lemma 4.22.

Define

$$E_1(V_0) := L\big(I_{\mathbf{c}}(V_0)\big), \qquad E_2(V_0) := R\big(I_{\mathbf{c}}(V_0)\big),$$

and

$$E_3(V_0) := L\big(I^1_{[\mathbf{c},m,1]}(V_0)\big), \qquad E_4(V_0) := R\big(I^{M+1}_{[\mathbf{c},m,1]}(V_0)\big),$$

confer Figure 4.5. Let $V \mapsto E(V)$ be a spectral band edge of $I^i_{[\mathbf{c},m]}(V)$ or $I^j_{[\mathbf{c},m,n]}(V)$ for some $1 \leq i \leq M$ or $1 \leq j \leq M+1$. By Corollary 3.2, the spectral band edges vary continuously in V, namely $V \mapsto E(V)$ is continuous, Thus, Equation (4.25) and assumption (b) yield

$$E_1(V_0) < E_3(V_0) \leq E(V_0) \leq E_4(V_0) < E_2(V_0).$$

Hence, $\min_{k\in\{1,2\}} |E(V_0) - E_k(V_0)| \geq 3\delta$ where

$$\delta := \frac{1}{3}\min\big\{|E_1(V_0) - E_3(V_0)|,\ |E_4(V_0) - E_2(V_0)|,\ V_0\big\} > 0.$$

Now, we use

$$\max\{|E(V) - E(V_0)|,\ |E_i(V) - E_i(V_0)|\} \leq |V - V_0|, \qquad i \in \{1,2,3,4\},$$

which holds by Corollary 3.2, to conclude

$$E_3(V) < E(V) < E_4(V), \quad V > V_0, \qquad \Longrightarrow \qquad E_1(V) < E(V) < E_2(V), \qquad V > V_0 - \delta.$$

We note that $V_0 - \delta > 0$ holds, by the definition of δ.

Since $E(V)$ was an arbitrary spectral edge of $I^j_{[\mathbf{c},m]}(V)$ or $I^j_{[\mathbf{c},m,n]}(V)$ for $n \in \mathbb{N}$, we deduce for all $V > V_0 - \delta$, $1 \leq i \leq M$, $1 \leq j \leq M+1$, and $n \in \mathbb{N}$,

$$I^j_{[\mathbf{c},m]}(V) \subseteq_{\text{str}} I_{\mathbf{c}}(V) \qquad \text{and} \qquad I^j_{[\mathbf{c},m,n]}(V) \subseteq_{\text{str}} I_{\mathbf{c}}(V).$$

Now, we apply Corollary 4.16 which implies that $I_{\mathbf{c}}(V)$ satisfies the forward properties (A1), (B2) and (I) for all $V > V_0 - \delta$. Since $I^j_{[\mathbf{c},m,n]}(V) \subseteq_{\text{str}} I^j_{[\mathbf{c},m,n-1]}(V)$ holds for all $n \in \mathbb{N}$ and $V > V_0 - \delta$, Corollary 4.18 implies that $I_{\mathbf{c}}(V)$ satisfies (B1) for all $V > V_0 - \delta$. □

We now apply Lemma 4.22 for all spectral bands $I_{\mathbf{c}}$ in $\sigma_{\mathbf{c}}$. Using that the number of spectral bands in $\sigma_{\mathbf{c}}$ is finite and taking the minimum δ among all $I_{\mathbf{c}}$ in $\sigma_{\mathbf{c}}$ (the δ which is provided by Lemma 4.22), we get:

Corollary 4.23. *Let $V_1 > 0$, $m \in \mathbb{N}$ and $\mathbf{c} \in \mathcal{C}$ be such that $\varphi(\mathbf{c}) \notin \{-1, \infty\}$ and $[\mathbf{c}, m] \in \mathcal{C}$. Suppose for each spectral band $I_{\mathbf{c}}$ in $\sigma_{\mathbf{c}}$, we have*

(a) $I_{\mathbf{c}}(V)$ is either of backward type A for all $V > 0$ or of backward type B for all $V > 0$, and
(b) $I^1_{[\mathbf{c},m,1]}(V),\ I^{M+1}_{[\mathbf{c},m,1]}(V) \subseteq_{\text{str}} I_{\mathbf{c}}(V)$ holds for all $V \geq V_1$, where

$$M := \begin{cases} m-1 & I_{\mathbf{c}}(V) \text{ is of backward type A for all } V > 0, \\ m & I_{\mathbf{c}}(V) \text{ is of backward type B for all } V > 0. \end{cases}$$

Then $V^{quasi}_{\text{crit}}([\mathbf{c}, m]) < V_1$. In particular, if (b) holds for all $V_1 > 0$, then $V^{quasi}_{\text{crit}}([\mathbf{c}, m]) = 0$.

Proof. Set $V_0 := V^{quasi}_{\text{crit}}([\mathbf{c}, m])$. First of all note that $V_0 \leq 4$ by Theorem 2.16. We seek to prove $V_0 < V_1$. Assume by contradiction $V_0 \geq V_1$.

Let $I_{\mathbf{c}}(V)$ be a spectral band in $\sigma_{\mathbf{c}}(V)$, which by (a) is either of backward type A for all $V > 0$ or of backward type B for all $V > 0$. Since $V_0 \geq V_1$, (b) implies

$$I^1_{[\mathbf{c},m,1]}(V_0),\ I^{M+1}_{[\mathbf{c},m,1]}(V_0) \subseteq_{\text{str}} I_{\mathbf{c}}(V_0).$$

Due to Lemma 4.22 and $V_0 \geq V_1 > 0$, there exists a $\delta := \delta(I_{\mathbf{c}}(V)) > 0$ such that $I_{\mathbf{c}}(V)$ is of quasi m-type A or quasi m-type B for $V > V_0 - \delta$. Since there are at most finitely spectral bands in $\sigma_{\mathbf{c}}(V)$, we can take the minimum of all these $\delta(I_{\mathbf{c}}(V))$'s and denote it by $\delta' > 0$. Then for all $V > V_0 - \delta'$, we have that every spectral band $I_{\mathbf{c}}(V)$ in $\sigma_{\mathbf{c}}(V)$ is either of quasi m-type A or quasi m-type B. Hence, by definition of $V^{quasi}_{\text{crit}}([\mathbf{c}, m])$, we conclude

$$V_0 := V^{quasi}_{\text{crit}}([\mathbf{c}, m]) \leq V_0 - \delta',$$

a contradiction. □

By adding an additional condition to the assumption of Lemma 4.22 we may get a stronger implication (showing m-type rather than just quasi m-type), which is done in the following lemma.

Lemma 4.24. *Let $m \in \mathbb{N}$, $\mathbf{c} \in \mathcal{C}$ with $\varphi(\mathbf{c}) \notin \{-1, \infty\}$ and $[\mathbf{c}, m] \in \mathcal{C}$, and $V_0 \geq V_{\text{crit}}([\mathbf{c}, m])$. Let $I_{\mathbf{c}}$ be a spectral band in $\sigma_{\mathbf{c}}$ such that*

(a) $I_{\mathbf{c}}(V)$ is either of backward type A for all $V > 0$ or of backward type B for all $V > 0$, and
(b) $I^1_{[\mathbf{c},m,1]}(V),\ I^{M+1}_{[\mathbf{c},m,1]}(V) \subseteq_{\text{str}} I_{\mathbf{c}}(V)$ holds for all $V \geq V_0$, where

$$M := \begin{cases} m-1 & I_{\mathbf{c}}(V) \text{ is of backward type A for all } V > 0, \\ m & I_{\mathbf{c}}(V) \text{ is of backward type B for all } V > 0, \end{cases}$$

and
(c) if $m = 1$, then $\varphi(\mathbf{c}) \neq 1$,
if $m \geq 2$, then $V^{quasi}_{\text{crit}}([\mathbf{c}, m-1]) = 0$.

If $V_0 > 0$, then there is a $\delta > 0$ such that $I_{\mathbf{c}}(V)$ is of m-type A for all $V > V_0 - \delta$ or $I_{\mathbf{c}}(V)$ is of m-type B for all $V > V_0 - \delta$.

Proof. Applying Lemma 4.22, which is justified by assumptions (a) and (b) here, there exists $\delta > 0$ such that for $V > V_0 - \delta$, $I_{\mathbf{c}}(V)$ is either of quasi m-type A or quasi m-type B. Thus, we only have to show that $I_{\mathbf{c}}(V)$ also satisfies the forward property (A2) for all $V > V_0 - \delta$. We consider the following two cases:

Case 1: $(m = 1)$ If $m = 1$ and $I_{\mathbf{c}}(V)$ is of backward type A then it does not contain any spectral band of $\sigma_{[\mathbf{c},1]}(V)$ for all $V > 4$ by Theorem 2.16. Hence, there are no $I^i_{[\mathbf{c},1]}$ spectral bands (see Definition 3.3) and there is nothing to prove in this case. We need only to deal with the case $m = 1$ when $I_{\mathbf{c}}(V)$ is of backward type B. Towards doing this, notice that if $\varphi(\mathbf{c}) = 0$ then $\mathbf{c} = [0, 0]$ since $[\mathbf{c}, m] \in \mathcal{C}$ is assumed. But, $\sigma_{[0,0]}(V) = [-2, 2]$ only consists of a backward type A band, see Example 2.3. Hence, when checking the case that $I_{\mathbf{c}}(V)$ is of backward type B, we may further assume $\varphi(\mathbf{c}) \neq 0$.

Combining this with condition (c) of the lemma, we may now assume that $\varphi(\mathbf{c}) \in (0, 1)$ and $I_{\mathbf{c}}(V)$ is of backward type B. By Definition 3.3, since $m = 1$, we have exactly one spectral band $I^1_{[\mathbf{c},1]}(V)$ for which we need to show that it is not of weak backward type B for all $V > V_0 - \delta$. Indeed, Lemma 4.19 implies that for all $V > V_0 - \delta$, $I^1_{[\mathbf{c},1]}(V) \subseteq_{\text{str}} I_{\mathbf{c}}(V)$ and $I^1_{[\mathbf{c},1]}(V)$ is not of weak backward type B.

Case 2: $(m \geq 2)$ We need to show that $I^i_{[\mathbf{c},m]}(V)$ is not of weak backward type B for all $1 \leq i \leq M$ and $V > V_0 - \delta$. Let $1 \leq i \leq M$. We know by Theorem 2.16 that $I^i_{[\mathbf{c},m]}(V)$ is of backward type A in $\sigma_{[\mathbf{c},m]}(V)$ for $V > 4$. Denoting $m' := m - 1 \geq 1$, Lemma 2.10 implies that $I^i_{[\mathbf{c},m]}$ equals to the spectral band $I^i_{[\mathbf{c},m',1]}$ in $\sigma_{[\mathbf{c},m',1]}$, which is of backward type B for $V > 4$. Using Lemma 2.10 again, it suffices to show that $I^i_{[\mathbf{c},m',1]}(V)$ is not of weak backward type A in $\sigma_{[\mathbf{c},m',1]}(V)$ for all $V > V_0 - \delta$.

By assumption (c) for $m \geq 2$, we have $V^{quasi}_{\text{crit}}([\mathbf{c}, m']) = V^{quasi}_{\text{crit}}([\mathbf{c}, m - 1]) = 0$. Hence, for all $V > 0$, $I_{\mathbf{c}}(V)$ is either of quasi $m-1$-type A or $I_{\mathbf{c}}(V)$ is of quasi $m-1$-type B. This implies by (B2) that $I^i_{[\mathbf{c},m',1]}(V)$ is not of weak backward type A for all $V > 0$. □

We now apply Lemma 4.24 for all spectral bands $I_{\mathbf{c}}$ in $\sigma_{\mathbf{c}}$. Using that the number of spectral bands in $\sigma_{\mathbf{c}}$ is finite and taking the minimum δ among all $I_{\mathbf{c}}$ in $\sigma_{\mathbf{c}}$ (the δ which is provided by Lemma 4.24), we get:

Corollary 4.25. *Let $V_1 > 0$, $m \in \mathbb{N}$ and $\boldsymbol{c} \in \mathcal{C}$ be such that $\varphi(\boldsymbol{c}) \notin \{-1, \infty\}$ and $[\boldsymbol{c}, m] \in \mathcal{C}$. Suppose that each spectral band $I_{\mathbf{c}}$ in $\sigma_{\mathbf{c}}$ satisfies*

(a) $I_{\boldsymbol{c}}(V)$ is either of backward type A for all $V > 0$ or of backward type B for all $V > 0$, and

(b) $I^1_{[\boldsymbol{c},m,1]}(V)$, $I^{M+1}_{[\boldsymbol{c},m,1]}(V) \subseteq_{\text{str}} I_{\boldsymbol{c}}(V)$ holds for all $V \geq V_1$, where

$$M := \begin{cases} m-1 & I_{\boldsymbol{c}}(V) \text{ is of backward type } A \text{ for all } V > 0, \\ m & I_{\boldsymbol{c}}(V) \text{ is of backward type } B \text{ for all } V > 0, \end{cases}$$

and

(c) if $m = 1$, then $\varphi(\boldsymbol{c}) \neq 1$,
if $m \geq 2$, then $V^{quasi}_{\text{crit}}([\boldsymbol{c}, m-1]) = 0$.

Then $V_{\text{crit}}([\boldsymbol{c}, m]) < V_1$. In particular, if (b) holds for all $V_1 > 0$, then $V_{\text{crit}}([\boldsymbol{c}, m]) = 0$.

Proof. Similarly as in Corollary 4.23, this follows immediately from Lemma 4.24 and the fact that $\sigma_{\mathbf{c}}(V)$ consists only of finitely many spectral bands independent of $V > 0$. □

Finally, we are ready to prove Proposition 4.20.

Proof of Proposition 4.20. Since $\varphi(\mathbf{c}) \in (0,1)$ and each spectral band $I_{\mathbf{c}}(V)$ in $\sigma_{\mathbf{c}}(V)$ is either of backward type A or B for all $V > 0$, Lemma 4.17 implies

$$I^1_{[\mathbf{c},m,1]}(V), I^{M+1}_{[\mathbf{c},m,1]}(V) \subseteq_{\mathrm{str}} I_{\mathbf{c}}(V), \qquad \text{for all } m \in \mathbb{N},\ V > 0, \tag{4.26}$$

where $M = m-1$ if $I_{\mathbf{c}}$ is of backward type A and $M = m$ if $I_{\mathbf{c}}$ is of backward type B. Now $V_{\mathrm{crit}}([\mathbf{c},m]) = 0$ is proven by induction over $m \in \mathbb{N}$.

For the induction base, let $m = 1$. Since Equation (4.26) holds for $m = 1$ and $\varphi(\mathbf{c}) \neq 1$, Corollary 4.25 (for $m = 1$) implies $V_{\mathrm{crit}}([\mathbf{c},1]) = 0$.

For the induction step, let $m \in \mathbb{N}$ be such that $V_{\mathrm{crit}}([\mathbf{c},m]) = 0$. Thus, $V^{quasi}_{\mathrm{crit}}([\mathbf{c},m]) = 0$ follows as $0 \leq V^{quasi}_{\mathrm{crit}}([\mathbf{c},m]) \leq V_{\mathrm{crit}}([\mathbf{c},m])$. Since Equation (4.26) holds for $m+1$ and $V^{quasi}_{\mathrm{crit}}([\mathbf{c},m]) = 0$, Corollary 4.25 (for $m+1 \geq 2$) implies $V_{\mathrm{crit}}([\mathbf{c},m+1]) = 0$. □

5. Towards the proof of Theorem 2.15 - the induction base

Theorem 2.15 is proven by induction. The induction base is proven in this section. Specifically, we show in this section that for all $V \neq 0$, the spectral bands in $\sigma_{[0,0]}(V)$ and $\sigma_{[0,0,1]}(V)$ are either of type A or B. For this proof we express the transfer matrices, $M_{\mathbf{c}}(E,V)$, and their traces, $t_{\mathbf{c}}(E,V)$, (see Section 2.2) using the dilated Chebyshev polynomials of the second kind $S_l : \mathbb{R} \to \mathbb{R}$, $l \in \mathbb{N}_0$. These polynomials are defined by

$$S_{-1}(x) := 0, \quad S_0(x) := 1, \quad S_l(x) := xS_{l-1}(x) - S_{l-2}(x),$$

see Appendix II for more details and properties of these polynomials.

Lemma 5.1. *For all $m \in \mathbb{N}$ and $V \in \mathbb{R}$, we have*

$$M^m_{[0,0]}(E,V) = \begin{pmatrix} S_m(E) & -S_{m-1}(E) \\ S_{m-1}(E) & -S_{m-2}(E) \end{pmatrix}, \qquad E \in \mathbb{R}.$$

Proof. We prove this by induction on m. The induction base ($m = 1$) follows just by definition as

$$M^1_{[0,0]}(E,V) = \begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} S_m(E) & -S_{m-1}(E) \\ S_{m-1}(E) & -S_{m-2}(E) \end{pmatrix}$$

using that $S_1(E) = S_0(E)E - S_{-1}(E) = E$. For the induction step, suppose the statement is true for m. Then

$$\begin{aligned} M^{m+1}_{[0,0]}(E,V) =& M_{[0,0]}(E,V) M^m_{[0,0]}(E,V) \\ =& \begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} S_m(E) & -S_{m-1}(E) \\ S_{m-1}(E) & -S_{m-2}(E) \end{pmatrix} \\ =& \begin{pmatrix} ES_m(E) - S_{m-1}(E) & -ES_{m-1}(E) + S_{m-2}(E) \\ S_m(E) & -S_{m-1}(E) \end{pmatrix} \\ =& \begin{pmatrix} S_{m+1}(E) & -S_m(E) \\ S_m(E) & -S_{m-1}(E) \end{pmatrix} \end{aligned}$$

proving the statement. □

Lemma 5.2. *For all $E, V \in \mathbb{R}$ and $m \in \mathbb{N}$ the following holds:*

(a) $t_{[0,0,m]}(E,V) = S_m(E) - VS_{m-1}(E) - S_{m-2}(E)$ *for all* $E \in \mathbb{R}$.
(b) $t_{[0,0,1,m]}(E,V) = ES_m(E-V) - 2S_{m-1}(E-V)$ *for all* $E \in \mathbb{R}$.
(c) $t_{[0,0,1,m,1]}(E,V) = ES_{m+1}(E-V) - 2S_m(E-V)$ *for all* $E \in \mathbb{R}$.

Proof. We recall (Section 2.2) that the transfer matrices are recursively defined by

$$M_{[0,0,c_1,\dots,c_k]}(E,V) := M_{[0,0,c_1\dots,c_{k-2}]}(E,V)M_{[0,0,c_1,\dots,c_{k-1}]}(E,V)^{c_k}.$$

(a) Using Lemma 5.1, we get

$$\begin{aligned} t_{[0,0,m]}(E,V) =& tr\left(M_{[0,0,m]}(E,V)\right) = tr\left(M_{[0]}(E,V)M_{[0,0]}^m(E,V)\right) \\ =& tr\left(\begin{pmatrix}1 & -V\\ 0 & 1\end{pmatrix}\begin{pmatrix}S_m(E) & -S_{m-1}(E)\\ S_{m-1}(E) & -S_{m-2}(E)\end{pmatrix}\right) \\ =& S_m(E) - VS_{m-1}(E) - S_{m-2}(E). \end{aligned}$$

(b) We first observe that

$$\begin{aligned} M_{[0,0,1]}(E,V) =& M_{[0]}(E,V)M_{[0,0]}(E,V) = \begin{pmatrix}1 & -V\\ 0 & 1\end{pmatrix}\begin{pmatrix}E & -1\\ 1 & 0\end{pmatrix} \\ =& \begin{pmatrix}E-V & -1\\ 1 & 0\end{pmatrix} = M_{[0,0]}(E-V,V). \end{aligned}$$

Thus, Lemma 5.1 leads to

$$\begin{aligned} t_{[0,0,1,m]} =& tr\left(M_{[0,0]}(E,V)M_{[0,0,1]}(E,V)^m\right) \\ =& tr\left(M_{[0,0]}(E,V)M_{[0,0]}(E-V,V)^m\right) \\ =& tr\left(\begin{pmatrix}E & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}S_m(E-V) & -S_{m-1}(E-V)\\ S_{m-1}(E-V) & -S_{m-2}(E-V)\end{pmatrix}\right) \\ =& ES_m(E-V) - 2S_{m-1}(E-V). \end{aligned}$$

(c) This follows from (b) and Proposition II.2 (a) asserting $t_{[0,0,1,m,1]} = t_{[0,0,1,m+1]}$. □

Recalling Example 2.3 (see also Figure 2.1), we have $\sigma_{[0,0]}(V) = [-2,2]$ and $\sigma_{[0,0,1]}(V) = [-2+V, 2+V]$ for all $V > 0$. Next, we prove two lemmas. The first lemma states that the spectral band $I_{[0,0]}(V) := [-2,2]$ (in $\sigma_{[0,0]}(V)$) is of type A. The second lemma states that the spectral band $I_{[0,0,1]}(V) := [-2+V, 2+V]$ (in $\sigma_{[0,0]}(V)$) is of type B. Hence, both lemmas provide the induction base needed to prove Theorem 2.15.

Lemma 5.3. *Let $I_{[0,0]}(V) := [-2,2]$ be the unique spectral band of $\sigma_{[0,0]}(V)$ for $V > 0$. The following assertions hold for all $V > 0$.*

- *(a) $I_{[0,0]}(V)$ is of backward type A and not of weak backward type B,*
- *(b) For all $m \in \mathbb{N}$, $I_{[0,0]}(V)$ is of m-type A, namely $V_{\text{crit}}([0,0,m]) = 0$,*
- *(c) For all $m \in \mathbb{N}$, $\sigma_{[0,0,m]}(V)$ consists of m spectral bands satisfying*
  - *the left-most $m-1$ spectral bands are of backward type A and not of weak backward type B. These spectral bands are strictly contained in $I_{[0,0]}(V)$.*
  - *the right-most spectral band, which we denote $K_{[0,0,m]}(V)$, is of backward type B but not of weak backward type A. The spectral bands $K_{[0,0,m]}(V)$ (one for each $m \in \mathbb{N}$) satisfy*

    $$I_{[0,0]}(V) \prec K_{[0,0,m]}(V)$$

    and

    $$K_{[0,0,m]}(V) \subseteq_{\text{str}} K_{[0,0,m-1]}(V) \subseteq_{\text{str}} \dots \subseteq_{\text{str}} K_{[0,0,1]}(V) \subseteq_{\text{str}} K_{[0]}(V),$$

    with the notational convention $K_{[0]}(V) := \mathbb{R} = \sigma_{[0]}(V)$.

Remark. Note that Lemma 5.3 (a) and (b) are enough for the purpose of providing the induction base needed in the proof of Theorem 2.15. But, (c) is needed later in the proof of Proposition 7.1, and anyway parts of (c) are already shown during the proof of (b) below.

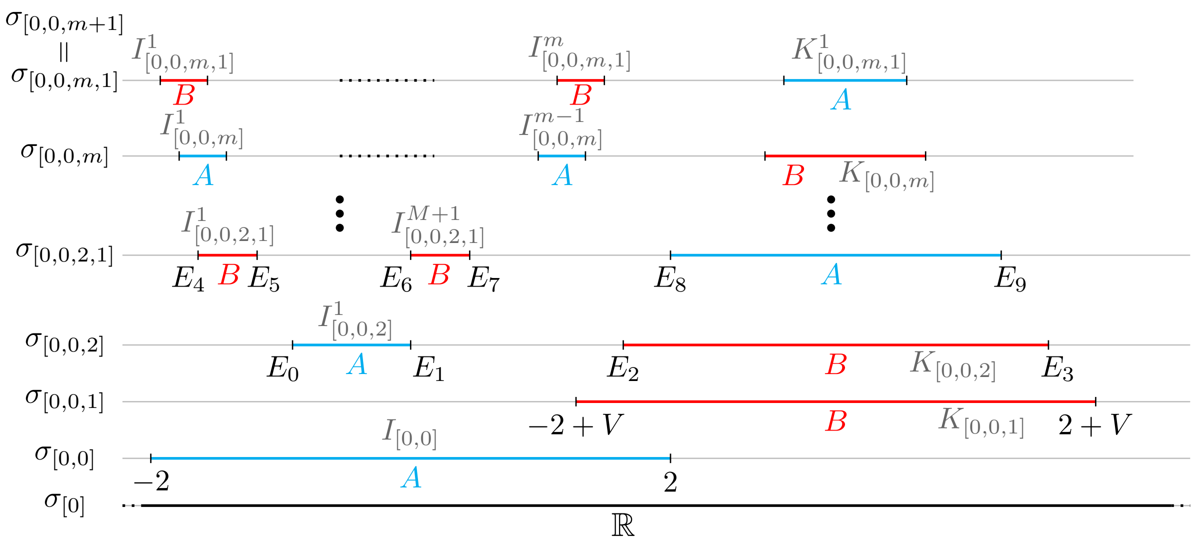

FIGURE 5.1. A sketch of the spectral bands considered in the proof of Lemma 5.3.

For the convenience of the reader, we sketch in Figure 5.1 the spectral bands mentioned in the proof of Lemma 5.3 and their relations ($\prec$ and $\subseteq_{\text{str}}$).

Proof. Part (a) of the lemma is already proven in Example 2.9 (see also Figure 2.1).

Parts (b) and (c) of the lemma are proven together by induction over $m \in \mathbb{N}$. Towards proving them we denote $M := m - 1$, since we are trying to prove that $I_{[0,0]}$ is of m-forward type A, confer Definition 2.12.

Induction base: The induction base consists of $m = 1$ and $m = 2$. We start with proving (b) and (c) for $m = 1$ and $m = 2$.

Let $m = 1$. Then $\sigma_{[0,0,1]}(V) = [-2 + V, 2 + V]$ and its unique spectral band $K_{[0,0,1]} := [-2 + V, 2 + V]$ is of backward type B but not of weak backward type A (as it is strictly contained in $\sigma_{[0]}(V) = \mathbb{R} = K_{[0]}(V)$ and $\sigma_{[0,0]}(V) = I_{[0,0]}(V) \prec K_{[0,0,1]}(V)$). This proves (c) for $m = 1$.

Let $m = 2$. We have $t_{[0,0,2]}(E, V) = E^2 - EV - 2$ (see Example 2.3) implying

$$t_{[0,0,2]}(E, V) = 2 \iff E = \frac{V}{2} \pm \sqrt{\frac{V^2}{4} + 4},$$
$$t_{[0,0,2]}(E, V) = -2 \iff E \in \{0, V\}.$$

This motivates to denote

$$E_0(V) := \frac{V}{2} - \sqrt{\frac{V^2}{4} + 4}, \quad E_1(V) := 0, \quad E_2(V) := V, \quad E_3(V) := \frac{V}{2} + \sqrt{\frac{V^2}{4} + 4},$$

so that $E_0(V) < E_1(V) < E_2(V) < E_3(V)$ holds for all $V > 0$. Thus, $I_{[0,0,2]}(V) := [E_0(V), E_1(V)]$ and $K_{[0,0,2]}(V) := [E_2(V), E_3(V)]$ are the two spectral bands in $\sigma_{[0,0,2]}(V)$. Clearly $I_{[0,0,2]}(V)$ is of backward type A for all $V > 0$ since $I_{[0,0,2]}(V) \subseteq_{\text{str}} [-2, 2] = I_{[0,0]}(V)$. In addition $E_0(V) < -2 + V$ for $V > 0$ and so $I_{[0,0,2]}(V)$ is not contained in $K_{[0,0,1]}(V) = [-2+V, 2+V]$. Hence, $I_{[0,0,2]}(V)$ is not of weak backward type B for all $V > 0$. The spectral band $K_{[0,0,2]}(V)$ is of backward type B since $K_{[0,0,2]}(V) \subseteq_{\text{str}} [-2 + V, 2 + V] = K_{[0,0,1]}(V)$ for all $V > 0$. In addition, $E_3(V) \geq \frac{V}{2} + 2 > 2$ for all $V > 0$ leading to $I_{[0,0]}(V) \prec K_{[0,0,2]}(V)$. Thus, $K_{[0,0,2]}(V)$ is not of weak backward type A.

Summing up, we have proven (b) and (c), for $m = 1$ and $m = 2$. It is left to show (b) for $m = 1$ and $m = 2$, namely that $V_{\text{crit}}([0, 0, 1]) = 0$ and $V_{\text{crit}}([0, 0, 2]) = 0$.

$V_{\text{crit}}([0,0,1]) = 0$: We first aim to apply Corollary 4.23 for $\mathbf{c} = [0,0]$ and $m = 1$. Applying Corollary 4.23 would give that $V_{\text{crit}}^{quasi}([0,0,1]) = 0$ (see Definition 4.21) and then one needs only to show property (A2) in order to conclude $V_{\text{crit}}([0,0,1]) = 0$. But, in this case property (A2) is an empty statement since $M = m-1 = 0$.

In order to apply Corollary 4.23 for $\mathbf{c} = [0,0]$ and $m = 1$ we note the following. The spectrum $\sigma_{\mathbf{c}}$ has only one spectral band $I_{[0,0]}$ that is of backward type A and not of weak backward type B for all $V > 0$, which is assumption (a) of Corollary 4.23. To check assumption (b) of Corollary 4.23 we need to prove that $I^1_{[0,0,1,1]}(V) \subseteq_{\text{str}} I_{[0,0]}(V)$ for all $V > 0$.

By Lemma 2.10, $\sigma_{[0,0,1,1]} = \sigma_{[0,0,2]}$. Consider the spectral band $I_{[0,0,2]}(V) = [E_0(V), E_1(V)]$, which we calculated above. In particular, we have seen above that $I_{[0,0,2]}(V) \subseteq_{\text{str}} I_{[0,0]}(V)$ holds for all $V > 0$. Therefore $I_{[0,0,2]}(V)$ equals to the unique spectral band $I^1_{[0,0,1,1]}(V)$ by Definition 3.3 . Thus, $I^1_{[0,0,1,1]}(V) \subseteq_{\text{str}} I_{[0,0]}(V)$ for all $V > 0$, which verifies all the assumptions of Corollary 4.23. As explained above, we conclude $V_{\text{crit}}([0,0,1]) = 0$.

$V_{\text{crit}}([0,0,2]) = 0$: We aim to apply Corollary 4.25 for $\mathbf{c} = [0,0]$ and $m = 2$ in order to conclude $V_{\text{crit}}([0,0,2]) = 0$. Condition (a) of Corollary 4.25 was already verified above, as the spectral band $I_{[0,0]}$ that is of backward type A and not of weak backward type B for all $V > 0$. We have also proved above $V_{\text{crit}}^{quasi}([0,0,1]) = 0$, which verifies condition (c) of Corollary 4.25. We only have to check condition (b) for all $V > 0$. Specifically, it is sufficient to prove $I^1_{[0,0,2,1]}(V),\ I^{M+1}_{[0,0,2,1]}(V) \subseteq_{\text{str}} I_{[0,0]}(V)$ for all $V > 0$. Note that $M + 1 = m = 2$.

Using Proposition II.2 (a) and Lemma 5.2 we conclude

$$t_{[0,0,2,1]}(E,V) = t_{[0,0,3]}(E,V) = S_3(E) - VS_2(E) - S_1(E)$$

We use this to express all the E values for which $t_{[0,0,2,1]}(E,V) \in \{-2,2\}$:

$$\begin{aligned}
E_4(V) :=& \frac{V-1}{2} - \frac{\sqrt{V^2+2V+9}}{2}, && E_5(V) := -1,\\
E_6(V) :=& \frac{V+1}{2} - \frac{\sqrt{V^2-2V+9}}{2}, && E_7(V) := 1,\\
E_8(V) :=& \frac{V-1}{2} + \frac{\sqrt{V^2+2V+9}}{2}, && E_9(V) := \frac{V+1}{2} + \frac{\sqrt{V^2-2V+9}}{2},
\end{aligned}$$

where $E_4(V) < E_5(V) < E_6(V) < E_7(V) < E_8(V) < E_9(V)$. Now, it is straightforward to check that the three spectral bands in $\sigma_{[0,0,2,1]}$ are

$$I^1_{[0,0,2,1]}(V) = [E_4(V), E_5(V)], \qquad I^{M+1}_{[0,0,2,1]}(V) = [E_6(V), E_7(V)]$$

and

$$K_{[0,0,3]}(V) = [E_8(V), E_9(V)].$$

Furthermore, $I^1_{[0,0,2,1]}(V), I^{M+1}_{[0,0,2,1]}(V) \subseteq_{\text{str}} I_{[0,0]}(V)$ for all $V > 0$. Thus, Corollary 4.25 implies $V_{\text{crit}}([0,0,2]) = 0$, hence statement (b) of the current lemma holds for $m = 2$, and this finishes the proof of the induction base.

Induction step: (see Figure 5.1) Let $m \geq 2$ and suppose (induction hypothesis) that $V_{\text{crit}}([0,0,m]) = 0$ and $\sigma_{[0,0,m]}(V)$ satisfies (c) for all $V > 0$.

We have $\varphi([0,0,m+1]) = \frac{1}{m+1}$ and so $\sigma_{[0,0,m+1]}(V)$ consists of exactly $m+1$ spectral bands by Proposition 1.2 and Lemma 2.5. Since $K_{[0,0,m]}(V)$ is of backward type B (and not of weak backward type A) for all $V > 0$, we conclude that $K_{[0,0,m]}(V)$ is of type B for $V > 4$, see Theorem 2.16. Then property (A1) of $K_{[0,0,m]}$ implies that for $V > 4$, there is a spectral band $K_{[0,0,m,1]}(V)$ in $\sigma_{[0,0,m,1]}(V)$ of backward type A such that $K_{[0,0,m,1]}(V) \subseteq_{\text{str}} K_{[0,0,m]}(V)$.

Furthermore (referring again to Theorem 2.16), $I_{[0,0]}(V) = [-2,2]$ is of type A for $V > 4$ and so it strictly contains m spectral bands of type B in $\sigma_{[0,0,m,1]}(V)$ for $V > 4$. Since $\sigma_{[0,0,m,1]}(V)$ has $m+1$ spectral bands, the spectral band $K_{[0,0,m,1]}(V)$ mentioned above satisfies the following: for $V > 4$, it is the unique spectral band in $\sigma_{[0,0,m,1]}(V)$ of backward type A; in addition $K_{[0,0,m,1]}(V) \subseteq_{\text{str}} K_{[0,0,m]}(V)$ (as seen above) and $I_{[0,0]}(V) \prec K_{[0,0,m,1]}(V)$ for $V > 4$ (since $I_{[0,0]}(V) \prec K_{[0,0,m]}(V)$ by the induction hypothesis). Furthermore, the left-most m spectral bands in $\sigma_{[0,0,m,1]}(V)$ are strictly contained in $\sigma_{[0,0]}(V)$ for $V > 4$. By Lemma 2.10, $\sigma_{[0,0,m,1]}(V) = \sigma_{[0,0,m+1]}(V)$, and in particular, we can identify $K_{[0,0,m,1]}(V)$ with a spectral band $K_{[0,0,m+1]}(V)$ in $\sigma_{[0,0,m+1]}(V)$ and $K_{[0,0,m+1]}(V)$ is of backward type B for $V > 4$.

We will show that

(d) $K_{[0,0,m+1]}(V) \subseteq_{\text{str}} K_{[0,0,m]}(V)$ and $K_{[0,0,m+1]}(V) \nsubseteq I_{[0,0]}(V)$ for all $V > 0$,
(e) $I_{[0,0]}(V) \prec K_{[0,0,m+1]}(V)$ for all $V > 0$,
(f) $V_{\text{crit}}([0,0,m+1]) = 0$.

Observe that these statements imply that parts (b) and (c) of the lemma hold for $m+1$. These implications are rather straightforward, and one just needs to notice that to get the first bullet of (c) for $m+1$, one needs also to employ $V_{\text{crit}}([0,0,m+1]) = 0$, which provides the (A1) and (A2) properties of $I_{[0,0]}(V)$ for all $V > 0$.

Proof of (d): Since $m \geq 2$, we have $\varphi([0,0,m]) \in (0,1)$ and by induction hypothesis $K_{[0,0,m]}(V)$ is of backward type B for all $V > 0$. Lemma 2.10 implies $\sigma_{[0,0,m,1]} = \sigma_{[0,0,m+1]}$. Thus, Lemma 4.19 applied to the spectral band $K_{[0,0,m]}(V)$ implies $K_{[0,0,m+1]}(V) = K^1_{[0,0,m,1]}(V) \subseteq_{\text{str}} K_{[0,0,m]}(V)$ for all $V > 0$. Moreover, Lemma 4.19 asserts that $K^1_{[0,0,m,1]}(V)$ is not of weak backward type B for all $V > 0$, namely $K^1_{[0,0,m,1]}(V)$ is not contained in a spectral band of $\sigma_{[0,0,m,1,-1]} = \sigma_{[0,0]} = [-2,2]$. Since $K_{[0,0,m+1]}(V) = K^1_{[0,0,m,1]}(V)$ holds for all $V > 0$, we conclude $K_{[0,0,m+1]}(V) \nsubseteq [-2,2]$ for $V > 0$.

Proof of (e): For $V > 0$, (d) and $[-2,2] = I_{[0,0]}(V) \prec K_{[0,0,m]}(V)$ imply

$$-2 = L(I_{[0,0]}(V)) < L(K_{[0,0,m]}(V)) < L(K_{[0,0,m+1]}(V)).$$

Furthermore, (d) asserts $K_{[0,0,m+1]}(V) \nsubseteq [-2,2]$ for all $V > 0$ implying $R(I_{[0,0]}(V)) < R(K_{[0,0,m+1]}(V))$ for all $V > 0$. Thus, $I_{[0,0]}(V) \prec K_{[0,0,m+1]}(V)$ follows for $V > 0$.

Proof of (f): Since $\sigma_{[0,0]}(V)$ consists only of the spectral band $I_{[0,0]}(V)$ we need to show that $I_{[0,0]}(V)$ is of $(m+1)$-type A for all $V > 0$. Since $V_{\text{crit}}([0,0,m]) = 0$ and $I_{[0,0]}(V)$ if of backward type A but not of weak backward type B for all $V > 0$, Corollary 4.25 (for $m \geq 2$) asserts that we only have to prove

$$I^1_{[0,0,m+1,1]}(V), I^{M+1}_{[0,0,m+1,1]}(V) \subseteq_{\text{str}} I_{[0,0]}(V)$$

for all $V > 0$ where $M = (m+1) - 1 = m$. Lemma II.1 (b) asserts $S_l(\pm 2) = (\pm 1)^l (l+1)$ for $l \in \mathbb{N}$. Thus, Lemma 5.2 (a) leads to

$$\begin{aligned} \left| t_{[0,0,l+1]}(\pm 2, V) \right| &= \left| S_{l+1}(\pm 2) - V S_l(\pm 2) - S_{l-1}(\pm 2) \right| \\ &= \left| (\pm 1)^{l+1}(l+2) - (\pm 1)^l (l+1) V - (\pm 1)^{l-1} l \right| \\ &= \left| (\pm 1)^{l+1} 2 \mp (\pm 1)^{l+1}(l+1) V \right| \\ &= \left| 2 \mp (l+1) V \right|. \end{aligned} \tag{5.1}$$

Hence, we conclude from Proposition II.2 (a) that

$$\left| t_{[0,0,m+1,1]}(-2, V) \right| = \left| t_{[0,0,m+2]}(-2, V) \right| = 2 + (m+2)V > 2, \qquad V > 0.$$

This means that for all $V > 0$, $E = -2$ is not a spectral edge of any spectral band in $\sigma_{[0,0,m+1,1]}(V)$, see Lemma 4.10 (a). Since $E = -2$ is a spectral band edge of $I_{[0,0]}(V)$, and since $I^j_{[0,0,m+1,1]}(V) \subseteq_{\rm str} I_{[0,0]}(V)$ for $1 \le j \le M+1$ and $V > 4$ (by Theorem 2.16), we conclude that $I^j_{[0,0,m+1,1]}(V) \subseteq_{\rm str} I_{[0,0]}(V)$ may be violated if and only if $R\left(I^j_{[0,0,m+1,1]}(V)\right) \ge 2$. Since $I^j_{[0,0,m+1,1]} \prec I^{M+1}_{[0,0,m+1,1]}$ holds by definition for all $1 \le j < M+1$, it suffices to show that for all $V > 0$,

$$R\left(I^{M+1}_{[0,0,m+1,1]}(V)\right) < 2. \tag{5.2}$$

For $V \ge \frac{4}{m+1}$, Equation (5.1) and Proposition II.2 (a) lead to

$$\left|t_{[0,0,m+1,1]}(2,V)\right| = \left|t_{[0,0,m+2]}(2,V)\right| = \left|(m+2)V-2\right| \ge 4\frac{m+2}{m+1} - 2 > 2.$$

Hence, (5.2) holds for all $V \ge \frac{4}{m+1}$, proving $I^1_{[0,0,m+1,1]}(V), I^{M+1}_{[0,0,m+1,1]}(V) \subseteq_{\rm str} I_{[0,0]}(V)$ for all $V \ge \frac{4}{m+1}$. Recalling also the induction hypothesis, $V_{\rm crit}([0,0,m]) = 0$, we apply Lemma 4.24 for $m+1 \ge 2$ and $\mathbf{c} = [0,0]$ and conclude that there exists $\delta > 0$ such that $I_{[0,0]}(V)$ is of $(m+1)$-type A for $V > \frac{4}{m+1} - \delta$. Since, $I_{[0,0]}$ is the only spectral band in $\sigma_{[0,0]}$ this implies $V_{\rm crit}([0,0,m+1]) \le \frac{4}{m+1} - \delta$. Thus, it is left to prove (5.2) for $0 < V \le \frac{4}{m+1} - \delta$. Equation (5.1) together with $0 < V \le \frac{4}{m+1} - \delta$ implies

$$\left|t_{[0,0,m+1]}(2,V)\right| = \left|2-(m+1)V\right| < 2,$$

so that there is a spectral band of $\sigma_{[0,0,m+1]}(V)$ which contains $E = 2$, for $0 < V \le \frac{4}{m+1} - \delta$. But, since $\sigma_{[0,0,m,1]} = \sigma_{[0,0,m+1]}$ (Lemma 2.10) and by the induction hypothesis, $I^m_{[0,0,m,1]}(V) \subseteq_{\rm str} I_{[0,0]}(V)$ for $V > 0$, the only spectral band which can contain $E = 2$ is $K^1_{[0,0,m,1]}(V) = K_{[0,0,m+1]}(V)$, and so

$$L\left(K_{[0,0,m+1]}(V)\right) < R\left(I_{[0,0]}(V)\right) = 2 \qquad \text{for } 0 < V \le \frac{4}{m+1} - \delta.$$

In order to conclude (5.2) for $0 < V \le \frac{4}{m+1} - \delta$, we will apply Lemma 4.7 for $[0,0], [0,0,m+1], [0,0,m+1,1] \in \mathcal{C}$ and $\lambda_{\mathbf{o}} = R(I_{[0,0]}(V))$ and $\mu_{\mathbf{o}} = R(I^{M+1}_{[0,0,m+1,1]}(V))$. A direct computation invoking Lemma 4.4 and Lemma 4.5 yields

$$R(I_{[0,0]}(V)) \in \sigma\left(H_{[0,0],V}(0)\right), \; L(K_{[0,0,m+1]}(V)) \in \sigma\left(H^{\times 1}_{[0,0,m+1],V}(\pi)\right)$$

and

$$R(I^{M+1}_{[0,0,m+1,1]}(V)) \in \sigma\left(H_{[0,0,m+1,1],V}(\pi)\right).$$

Thus, $\theta_{[0,0]} = 0$, $\theta_{[0,0,m+1]} = \pi$, $\theta_{[0,0,m+1,1]} = \pi$ and these spectral edges are admissible (Definition 3.5). Moreover, we can directly compute the values of the counting function

$$N_{[0,0]} := N(R(I_{[0,0]}(V));\; H_{[0,0],V}(0)) = 0,$$
$$N_{[0,0,m+1,1]} := N(R(I^{M+1}_{[0,0,m+1,1]}(V);\; H_{[0,0,m+1,1],V}(\pi)) = m,$$

and using $L\left(K_{[0,0,m+1]}(V)\right) < R\left(I_{[0,0]}(V)\right)$for $0 < V \le \frac{4}{m+1} - \delta$, we get

$$N_{[0,0,m+1]} := N(R(I_{[0,0]}(V));\; H^{\times 1}_{[0,0,m+1],V}(\pi)) = m+1.$$

Hence, $N_{[0,0]} + N_{[0,0,m+1]} > N_{[0,0,m+1,1]}$ follows. Moreover, using that we found $L\left(K_{[0,0,m+1]}(V)\right) < R\left(I_{[0,0]}(V)\right) = \lambda_{\mathbf{o}}$ for $0 < V \le \frac{4}{m+1} - \delta$ implies that $\lambda_{\mathbf{o}}$ is a simple eigenvalue in $H^{\times 1}_{[0,0,m+1],V}(\pi) \oplus H_{[0,0],V}(0)$. Thus, Lemma 4.7 (a) yields

$$2 = R(I_{[0,0]}(V)) = \lambda_{\mathbf{o}} > R(I^{M+1}_{[0,0,m+1,1]}(V)) = \mu_{\mathbf{o}}.$$

Hence, we have proven (5.2) for all $0 < V \le \frac{4}{m+1} - \delta$. Thus, Corollary 4.25 (for $m \ge 2$) implies $V_{\rm crit}([0,0,m+1]) = 0$ proving (f). □

Next we prove that the unique spectral band $I_{[0,0,1]}(V) = [-2+V, 2+V]$ of $\sigma_{[0,0,1]}(V)$ is of type B for all $V > 0$. This is demonstrated in Figure 5.2.

Lemma 5.4. *Let $I_{[0,0,1]}(V) = [-2+V, 2+V]$ be the unique spectral band of $\sigma_{[0,0,1]}(V)$ for $V \in \mathbb{R}$. The following holds for all $V > 0$:*

(a) $I_{[0,0,1]}(V)$ is of backward type B but not of weak backward type A,
(b) For all $m \in \mathbb{N}$, $I_{[0,0,1]}(V)$ is of m-type B, i.e. $V_{\text{crit}}([0,0,1,m]) = 0$.

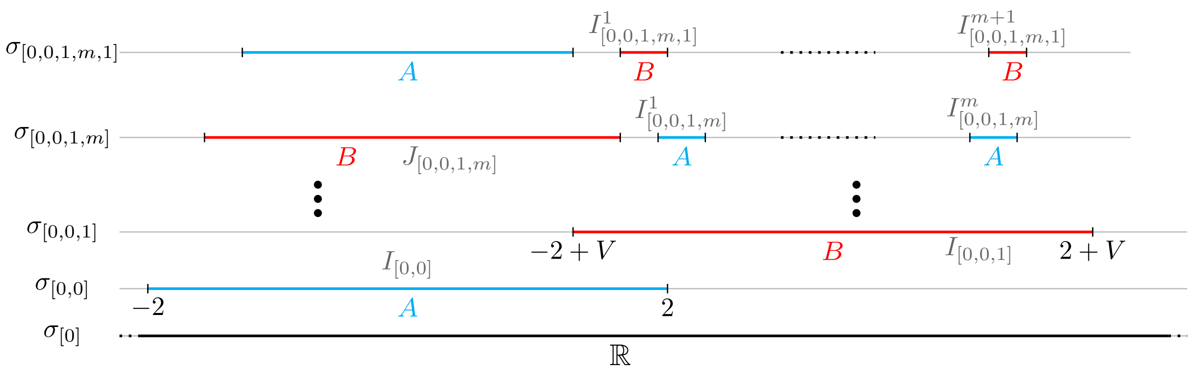

FIGURE 5.2. A sketch of the spectral bands considered in the proof of Lemma 5.4.

Proof. Statement (a) was already proven in Example 2.9 for all $V > 0$.

In order to prove (b), let $m \in \mathbb{N}$ and $M = m$ since we aim to show that $I_{[0,0,1]}$ is of m-type B, see Definition 2.12. This will be done in two steps: first by applying Lemma 4.22 to show $V^{quasi}_{\text{crit}}([0,0,1,m]) = 0$ for all $m \in \mathbb{N}$; then applying Corollary 4.25 to show $V_{\text{crit}}([0,0,1,m]) = 0$ for all $m \in \mathbb{N}$. In order to apply both corollaries we need to show that

$$I^1_{[0,0,1,m,1]}(V), I^{M+1}_{[0,0,1,m,1]}(V) \subseteq_{\text{str}} I_{[0,0,1]}(V), \qquad V > 0. \tag{5.3}$$

Step 1: We prove $V^{quasi}_{\text{crit}}([0,0,1,m]) = 0$. Let $V \geq \frac{4}{m+1}$. Lemma II.1 (b) asserts $S_l(\pm 2) = (\pm 1)^l(l+1)$. Thus, Lemma 5.2 (c) leads to

$$\begin{aligned}\left|t_{[0,0,1,m,1]}(\pm 2+V, V)\right| &= \left|(\pm 2+V)S_{m+1}(\pm 2) - 2S_m(\pm 2)\right| \\ &= \left|(\pm 2+V)(m+2) \mp 2(m+1)\right| \\ &= \left|V(m+2) \pm 2\right|.\end{aligned}$$

Thus, $\left|t_{[0,0,1,m,1]}(\pm 2+V, V)\right| > 2$ follows whenever $V \geq \frac{4}{m+1}$. The values $\pm 2+V$, at which the trace is evaluated, are the spectral band edges of $I_{[0,0,1]}(V)$. By Theorem 2.16, the strict inclusions (5.3) hold for $V > 4$. Thus, by continuity of the spectral edges (Corollary 3.2) and $\left|t_{[0,0,1,m,1]}(\pm 2+V, V)\right| > 2$, (5.3) holds for all $V \geq \frac{4}{m+1}$. Hence, Lemma 4.22 implies that there is a $\delta > 0$ such that $I_{[0,0,1]}(V)$ is of quasi m-type for all $V > \frac{4}{m+1} - \delta$, namely $V^{quasi}_{\text{crit}}([0,0,1,m]) \leq \frac{4}{m+1} - \delta$.

Now let us consider the range $0 < V \leq \frac{4}{m+1} - \delta$. First, for all $V > 0$, the trace computation above gives

$$\left|t_{[0,0,1,m,1]}(2+V, V)\right| = V(m+2) + 2 > 2.$$

Hence, $R(I^{M+1}_{[0,0,1,m,1]}(V)) < 2+V$ follows for all $V > 0$, since it holds for $V > 4$ by Theorem 2.16, and using continuity (Corollary 3.2). Since $I^1_{[0,0,1,m,1]}(V) \prec I^{M+1}_{[0,0,1,m,1]}(V)$ holds for all $V > 0$, it thus suffices to prove

$$-2+V < L(I^1_{[0,0,1,m,1]}(V)) \qquad \text{for all } 0<V\leq \tfrac{4}{m+1}-\delta. \tag{5.4}$$

Let $J_{[0,0,1,m]}$ be the spectral band associated with $I_{[0,0,1]}$ of Definition 4.8, see also Figure 5.2. Hence, it is the spectral band satisfying $\operatorname{ind}(J_{[0,0,1,m]}) = \operatorname{ind}(I^1_{[0,0,1,m]}) - 1$. Lemma 5.2 (b), Lemma II.1 (b) and $0 < V \leq \frac{4}{m+1} - \delta$ lead to

$$\begin{aligned}
\left|t_{[0,0,1,m]}(-2+V,V)\right| =& \left|(-2+V)S_m(-2) - 2S_{m-1}(-2)\right| \\
=& \left|(-2+V)(-1)^m(m+1) - 2(-1)^{m-1}m\right| \\
=& \left|(-2+V)(m+1) + 2m\right| \\
=& \left|V(m+1) - 2\right| < 2.
\end{aligned}$$

By Lemma 2.5, we conclude that $-2+V$ is in the interior of a spectral band of $\sigma_{[0,0,1,m]}(V)$. On the other hand, $-2+V < L(I^1_{[0,0,1,m]}(V))$ holds for $V > \frac{4}{m+1} - \delta$. Thus, the continuity of the spectral edges in V implies $-2+V < R(J_{[0,0,1,m]})$ for $0 \leq V \leq \frac{4}{m+1} - \delta$.

With this at hand, we prove (5.4) by applying Lemma 4.7 with $\mathbf{c} := [0,0,1]$, $[\mathbf{c},m] = [0,0,1,m]$, $[\mathbf{c},m,n] = [0,0,1,m,1]$, $\lambda_{\mathbf{o}} = L(I_{[0,0,1]}(V)) = -2+V$ and $\mu_{\mathbf{o}} = L(I^1_{[0,0,1,m,1]}(V))$. Let $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,1]} \in \{0,\pi\}$ be such that

$$L(I_{[0,0,1]}(V)) \in \sigma\left(H_{[0,0,1],V}(\theta_{\mathbf{c}})\right),\ R(J_{[0,0,1,m]}(V)) \in \sigma\left(H^{\times 1}_{[0,0,1,m],V}(\theta_{[\mathbf{c},m]})\right)$$

and

$$L(I^1_{[0,0,1,m,1]}(V)) \in \sigma\left(H_{[0,0,1,m,1],V}(\theta_{[\mathbf{c},m,1]})\right).$$

A direct computation yields $\operatorname{ind}(I_{[0,0,1]}) = 0$, $\operatorname{ind}(J_{[0,0,1,m]}) = 0$ and $\operatorname{ind}(I^1_{[0,0,1,m,1]}) = 1$, see Figure 5.2. Inserting these indices into the characterization of admissibility in Lemma 4.6, we conclude that these spectral edges are admissible. Furthermore, we can directly compute the values of the counting function

$$\begin{aligned}
N_{\mathbf{c}} &:= N(L(I_{[0,0,1]}(V));\ H_{[0,0,1],V}(\theta_{\mathbf{c}})) = 0, \\
N_{[\mathbf{c},m,1]} &:= N(L(I^1_{[0,0,1,m,1]}(V);\ H_{[0,0,1,m,1],V}(\theta_{[0,0,1,m,1]})) = 1,
\end{aligned}$$

and using $L(I_{[0,0,1]}(V)) = -2+V < R(J_{[0,0,1,m]})$ for $0 < V \leq \frac{4}{m+1} - \delta$, we get

$$N_{[\mathbf{c},m]} := N(L(I_{[0,0,1]}(V));\ H^{\times 1}_{[0,0,1,m],V}(\theta_{[\mathbf{c},m]})) = 0.$$

Hence, $N_{\mathbf{c}} + N_{[\mathbf{c},m]} < N_{[\mathbf{c},m,1]}$ follows. Moreover, $\lambda_{\mathbf{o}} = L(I_{[0,0,1]}(V)) < R(J_{[0,0,1,m]}(V))$ for $0 < V \leq \frac{4}{m+1} - \delta$ implies that $\lambda_{\mathbf{o}}$ is a simple eigenvalue in $H^{\times 1}_{[0,0,1,m],V}(\theta_{[\mathbf{c},m]}) \oplus H_{[0,0,1],V}(\theta_{\mathbf{c}})$. Thus, Lemma 4.7 yields $-2+V = \lambda_{\mathbf{o}} < \mu_{\mathbf{o}} = L(I^1_{[0,0,1,m,1]}(V))$ proving (5.4) for all $0 < V \leq \frac{4}{m+1} - \delta$. Hence, $V^{quasi}_{\text{crit}}([0,0,1,m]) = 0$ follows.

Step 2: We prove $V_{\text{crit}}([0,0,1,m]) = 0$. Let $m \geq 2$. We aim to apply Corollary 4.25 and need to check its assumptions (a), (b) and (c). We have seen above that $\sigma_{[0,0,1]}(V)$ has exactly one spectral band $I_{[0,0,1]}$ which is of backward type B but not of weak backward type A, so that assumption (a) of Corollary 4.25 holds. By step 1 of the proof, we have $V^{quasi}_{\text{crit}}([0,0,1,m]) = 0$ and $V^{quasi}_{\text{crit}}([0,0,1,m-1]) = 0$. The first implies that assumption (b) in Corollary 4.25 holds and the second implies that assumption (c) in Corollary 4.25 holds using $m \geq 2$. Hence, $V_{\text{crit}}([0,0,1,m]) = 0$ follows for all $m \geq 2$.

Let $m = 1$. Since $\varphi([0,0,1]) = 1$, assumption (c) in Corollary 4.25 does not hold for $m = 1$, and we cannot apply that corollary. Instead, we directly verify that $V_{\text{crit}}([0,0,1,1]) = 0$. Recall

$t_{[0,0,1,1]} = t_{[0,0,2]} = E^2 - EV - 2$, see Example 2.3. Thus, $\sigma_{[0,0,1,1]}(V) = [E_0(V), E_1(V)] \cup [E_2(V), E_3(V)]$ with

$$E_0(V) := \frac{V}{2} - \sqrt{\frac{V^2}{4} + 4}, \quad E_1(V) := 0, \quad E_2(V) := V, \quad E_3(V) := \frac{V}{2} + \sqrt{\frac{V^2}{4} + 4}.$$

Thus, $I_{[0,0,1,1]}(V) = [E_2(V), E_3(V)] \subseteq_{\text{str}} I_{[0,0,1]}(V)$ and $I_{[0,0,1,1]}(V)$ is the unique spectral band in $\sigma_{[0,0,1,1]}(V)$ of backward type A. Since $2 < E_3(V)$ for all $V > 0$, we conclude that $I_{[0,0,1,1]}(V)$ is not included in $I_{[0,0]}(V)$ and hence it is not of weak backward type B for all $V > 0$. Thus, $I_{[0,0,1]}(V)$ satisfies (A2) for all $V > 0$. Since $V_{\text{crit}}^{quasi}([0,0,1,1]) = 0$ by step 1, we conclude $V_{\text{crit}}([0,0,1,1]) = 0$. □

6. Proof of Theorem 2.15 - using vertical and horizontal induction steps

Recall the definition of the critical value for $m \in \mathbb{N}$ and $\mathbf{c} \in \mathcal{C}$ with $\varphi(\mathbf{c}) \notin \{-1, \infty\}$,

$$V_{\text{crit}}([\mathbf{c}, m]) := \sup [0, \infty) \setminus \left\{ V \in \mathbb{R} \;\middle|\; \begin{array}{c} \text{each spectral band in } \sigma_{\mathbf{c}}(V) \text{ is either} \\ \text{of } m\text{-type } A \text{ or of } m\text{-type } B \end{array} \right\}.$$

In order to prove Theorem 2.15, that every spectral band is either of type A or B, we show that $V_{\text{crit}}([\mathbf{c}, m]) = 0$ for all $\mathbf{c} \in \mathcal{C}$, $m \in \mathbb{N}$ with $\varphi(\mathbf{c}) \notin \{-1, \infty\}$ and $[\mathbf{c}, m] \in \mathcal{C}$. To prove $V_{\text{crit}}([\mathbf{c}, m]) = 0$, we use an inductive argument over the space of finite continued fractions $\mathcal{C}$. For the induction step we need to extend $[\mathbf{c}, m] \in \mathcal{C}$, both in terms of number of digits (we call it a horizontal step) and also in showing that every digit $m \in \mathbb{N}$ is valid (vertical step). Towards this we supply the following two lemmas. Throughout this section we use the notational conventions of Definition 3.3. In particular, we use the notations of the spectral bands $\{I^i_{[\mathbf{c},m]}\}_{i=1}^M$ in $\sigma_{[\mathbf{c},m]}$ and the spectral bands $\{I^j_{[\mathbf{c},m,n]}\}_{j=1}^{M+1}$ in $\sigma_{[\mathbf{c},m,n]}$ with $M \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ that are associated with a given spectral band $I_{\mathbf{c}}$ in $\sigma_{\mathbf{c}}$.

Lemma 6.1 (horizontal induction step). *Let $m \in \mathbb{N}$ and $\mathbf{c} \in \mathcal{C}$ be such that $\varphi(\mathbf{c}) \notin \{-1, \infty\}$ and $[\mathbf{c}, m] \in \mathcal{C}$. If $V_{\text{crit}}([\mathbf{c}, m]) = 0$ and $V_{\text{crit}}([\mathbf{c}, m, 1]) = 0$, then $V_{\text{crit}}([\mathbf{c}, m, 1, n]) = 0$ for all $n \in \mathbb{N}$.*

Proof. Let $\mathbf{c}' := [\mathbf{c}, m, 1]$. We have to show that $V_{\text{crit}}([\mathbf{c}', n]) = 0$ for all $n \in \mathbb{N}$. Since $m \in \mathbb{N}$, we conclude $\varphi(\mathbf{c}') \in (0, 1)$. Thus, Proposition 4.20 implies that it suffices to prove that each spectral band in $\sigma_{\mathbf{c}'}(V)$ is either of backward type A for all $V > 0$ or of backward type B for all $V > 0$. Let $I_{\mathbf{c}'}$ be a spectral band in $\sigma_{\mathbf{c}'}$. By Theorem 2.16, we already have that $I_{\mathbf{c}'}(V)$ is either of backward type A for all $V > 4$ or of backward type B for all $V > 4$. We treat each of these two cases separately.

<u>Case 1:</u> (For all $V > 4$, $I_{\mathbf{c}'}(V)$ is of backward type A). In this case, using $\sigma_{[\mathbf{c}',0]}(V) = \sigma_{[\mathbf{c},m]}(V)$ (as $c' := [\mathbf{c}, m, 1]$ and by Lemma 2.5) we conclude that $I_{\mathbf{c}'}(V)$ is strictly included in a spectral band of $\sigma_{[\mathbf{c},m]}(V)$ for all $V > 4$. By Theorem 2.16, there is a unique spectral band $I_{[\mathbf{c},m]}(V)$ such that $I_{\mathbf{c}'}(V) = I^i_{[\mathbf{c},m,1]}(V) \subseteq_{\text{str}} I_{[\mathbf{c},m]}(V)$, for all $V > 4$. Since $V_{\text{crit}}([\mathbf{c}, m, 1]) = 0$, we conclude $I_{\mathbf{c}'}(V) = I^i_{[\mathbf{c},m,1]}(V) \subseteq_{\text{str}} I_{[\mathbf{c},m]}(V)$ for all $V > 0$ implying that $I_{\mathbf{c}'}(V)$ is of backward type A for all $V > 0$.

<u>Case 2:</u> (For all $V > 4$, $I_{\mathbf{c}'}(V)$ is of backward type B). In this case, using $\sigma_{[\mathbf{c}',-1]}(V) = \sigma_{[\mathbf{c},m,0]}(V) = \sigma_{\mathbf{c}}(V)$ (as $c' := [\mathbf{c}, m, 1]$ and by Lemma 2.5) we conclude that $I_{\mathbf{c}'}(V)$ is strictly included in a spectral band of $\sigma_{\mathbf{c}}(V)$ for all $V > 4$. By Theorem 2.16, there is a unique spectral band $I_{\mathbf{c}}(V)$ such that $I_{\mathbf{c}'}(V) = I^j_{[\mathbf{c},m,1]}(V) \subseteq_{\text{str}} I_{\mathbf{c}}(V)$, for all $V > 4$. Since $V_{\text{crit}}([\mathbf{c}, m]) = 0$, we conclude $I_{\mathbf{c}'}(V) = I^j_{[\mathbf{c},m,1]}(V) \subseteq_{\text{str}} I_{\mathbf{c}}(V)$ for all $V > 0$ implying that $I_{\mathbf{c}'}(V)$ is of backward type B for all $V > 0$. □

Lemma 6.2 (vertical induction step)**.** *Let $\boldsymbol{c} \in \mathcal{C}$ be such that $\varphi(\boldsymbol{c}) \notin \{-1, \infty\}$ and $[\boldsymbol{c}, m] \in \mathcal{C}$ for all $m \in \mathbb{N}$. If $V_{\text{crit}}([\boldsymbol{c}, m]) = 0$ for all $m \in \mathbb{N}$ and $V_{\text{crit}}([\boldsymbol{c}, 1, n]) = 0$ for all $n \in \mathbb{N}$, then $V_{\text{crit}}([\boldsymbol{c}, m, n]) = 0$ for all $m, n \in \mathbb{N}$.*

Proof. We denote by $T(m)$ the statement that $V_{\text{crit}}([\mathbf{c}, m, n]) = 0$ for all $n \in \mathbb{N}$. We use induction over $m \in \mathbb{N}$ to prove that $T(m)$ holds for all $m \in \mathbb{N}$ and the lemma follows. The induction base $T(1)$ is true by the assumption in the lemma.

Suppose $T(m)$ holds. Denote $\mathbf{c}' := [\mathbf{c}, m + 1]$. We need to show $V_{\text{crit}}([\mathbf{c}, m + 1, n]) = V_{\text{crit}}([\mathbf{c}', n]) = 0$ for all $n \in \mathbb{N}$. Since $m \in \mathbb{N}$, we have $m + 1 \geq 2$ and so $\varphi(\mathbf{c}') \in (0, 1)$. Thus, Proposition 4.20 implies that it suffices to prove that each spectral band in $\sigma_{\mathbf{c}'}(V)$ is either of backward type A for all $V > 0$ or of backward type B for all $V > 0$. Let $I_{\mathbf{c}'}$ be a spectral band in $\sigma_{\mathbf{c}'}$. By Theorem 2.16, $I_{\mathbf{c}'}(V)$ is either of backward type A for all $V > 4$ or of backward type B for all $V > 4$. We treat each of these two cases separately.

Case 1: (For all $V > 4$, $I_{\mathbf{c}'}(V)$ is of backward type A). In this case, using $\sigma_{[\mathbf{c}',0]}(V) = \sigma_{\mathbf{c}}(V)$ (as $c' := [\mathbf{c}, m+1]$ and by Lemma 2.5) we conclude that $I_{\mathbf{c}'}(V)$ is strictly included in a spectral band of $\sigma_{\mathbf{c}}(V)$ for all $V > 4$. By Theorem 2.16 there is a unique spectral band $I_{\mathbf{c}}(V)$ such that $I_{\mathbf{c}'}(V) = I^i_{[\mathbf{c},m+1]}(V) \subseteq_{\text{str}} I_{\mathbf{c}}(V)$, for all $V > 4$. Since $V_{\text{crit}}([\mathbf{c}, m + 1]) = 0$, we conclude $I_{\mathbf{c}'}(V) = I^i_{[\mathbf{c},m+1]}(V) \subseteq_{\text{str}} I_{\mathbf{c}}(V)$ for all $V > 0$ implying that $I_{\mathbf{c}'}(V)$ is of backward type A for all $V > 0$.

Case 2: (For all $V > 4$, $I_{\mathbf{c}'}(V)$ is of backward type B). In this case, using $\sigma_{[\mathbf{c}',-1]}(V) = \sigma_{[\mathbf{c},m]}(V)$ (as $c' := [\mathbf{c}, m + 1]$ and by Lemma 2.5) we conclude that $I_{\mathbf{c}'}(V)$ is strictly included in a spectral band of $\sigma_{[\mathbf{c},m]}(V)$ for all $V > 4$. Recalling that $\sigma_{\mathbf{c}'}(V) = \sigma_{[\mathbf{c},m,1]}(V)$ (again by Lemma 2.5) and applying Theorem 2.16, we conclude that there is a unique spectral band $I_{[\mathbf{c},m]}(V)$ such that $I_{\mathbf{c}'}(V) = I^i_{[\mathbf{c},m,1]}(V) \subseteq_{\text{str}} I_{[\mathbf{c},m]}(V)$, for all $V > 4$. Since by the induction hypothesis $V_{\text{crit}}([\mathbf{c}, m, 1]) = 0$, we conclude $I_{\mathbf{c}'}(V) = I^i_{[\mathbf{c},m,1]}(V) \subseteq_{\text{str}} I_{[\mathbf{c},m]}(V)$ for all $V > 0$ implying that $I_{\mathbf{c}'}(V)$ is of backward type B for all $V > 0$. □

We are ready to prove Theorem 2.15.

Proof of Theorem 2.15. In order to prove the theorem (that every spectral band is either of type A or B), we show that $V_{\text{crit}}([\mathbf{c}, m]) = 0$ for all $m \in \mathbb{N}$ and $\mathbf{c} \in \mathcal{C}$ with $\varphi(\mathbf{c}) \notin \{-1, \infty\}$. For $l \in \mathbb{N}$, we denote by $T(l)$ the statement that

$$V_{\text{crit}}([0, 0, c_1, \dots, c_l]) = 0 \qquad \text{and} \qquad V_{\text{crit}}([0, 0, c_1, \dots, c_{l+1}]) = 0, \tag{6.1}$$

for all $[0, 0, c_1, \dots, c_l, c_{l+1}] \in \mathcal{C}$ with $c_{l+1} \in \mathbb{N}$. We prove by induction that $T(l)$ holds for all $l \in \mathbb{N}$.

We start from the induction base, $T(1)$. By Lemma 5.3 and Lemma 5.4,

$$V_{\text{crit}}([0, 0, c_1]) = 0 \qquad \text{and} \qquad V_{\text{crit}}([0, 0, 1, c_2]) = 0,$$

hold for all $c_1, c_2 \in \mathbb{N}$. Using this, applying Lemma 6.2 (with $\mathbf{c} = [0, 0]$, $m = c_1$, $n = c_2$) gives that $V_{\text{crit}}([0, 0, c_1, c_2]) = 0$ for all $c_1, c_2 \in \mathbb{N}$, proving the induction base.

Now, suppose $T(l)$ is true for $l \in \mathbb{N}$. In order to prove $T(l + 1)$, it suffices to show that $V_{\text{crit}}([0, 0, c_1, \dots, c_{l+1}, c_{l+2}]) = 0$ for all $c_{l+2} \in \mathbb{N}$. Apply Lemma 6.1 (for $\mathbf{c} = [0, 0, c_1, \dots, c_{l-1}]$ and $m = c_l$) to the induction hypothesis (6.1) with $c_{l+1} = 1$ to conclude

$$V_{\text{crit}}([0, 0, c_1, \dots, c_l, 1, c_{l+2}]) = 0,$$

for all $c_{l+2} \in \mathbb{N}$. Using this and the induction hypothesis (6.1), we apply Lemma 6.2 (for $\mathbf{c} = [0, 0, c_1, \dots, c_l]$, $m = c_{l+1}$, $n = c_{l+2}$) and get that

$$V_{\text{crit}}([0, 0, c_1, \dots, c_l, c_{l+1}, c_{l+2}]) = 0,$$

for all $c_{l+1}, c_{l+2} \in \mathbb{N}$. Hence, $T(l + 1)$ holds. □

7. Proofs of Theorem 1.7 and Theorem 1.9

In this section, we prove Theorem 1.7 and Theorem 1.9. In addition, in the last subsection we provide a useful characterization of type A and B spectral bands (which was conveniently used in [BBL23]).

7.1. Proof of Theorem 1.7.

Theorem 1.7 connects the spectra of all the rational approximants σ_{α_k} to the vertices of the spectral α-tree.

Proof of Theorem 1.7. The statement straightforwardly follows from the next Proposition 7.1 once one observes $\sigma_{\alpha_k} = \sigma_{[0,0,c_1,\ldots,c_k]}$ using Lemma 2.5 and $\varphi([0,0,c_1,\ldots,c_k]) = \alpha_k$. □

In fact, the following proposition contains one additional property (d) (connecting the A, B types of the spectral bands to the A, B labels of the vertices) which is insightful, but was not included in the statement of Theorem 1.7, since the A, B types of the spectral bands were not defined yet in Section 1.

Proposition 7.1. *Let $\alpha \in [0,1] \setminus \mathbb{Q}$ with infinite continued fraction expansion $(0, c_1, c_2, c_3, \ldots)$. Then there exists a unique bijection Ψ between all the vertices of spectral α-tree $\mathcal{T}_\alpha$ and all spectral bands (viewed as maps on $(0,\infty)$) of $\left\{\sigma_{[0,0,c_1,\ldots,c_k]}\right\}_{k\in\mathbb{N}\cup\{-1,0\}}$ such that*

- *(a) For each $k \in \mathbb{N} \cup \{-1, 0\}$, Ψ bijectively maps each vertex in level k to a spectral band of $\sigma_{[0,0,c_1,\ldots,c_k]}$.*
- *(b) If u, w are two vertices such that $u \to w$ then $\Psi(w) \subseteq_{\mathrm{str}} \Psi(u)$.*
- *(c) If u_1, u_2 are vertices in levels k_1, k_2 (respectively) such that $|k_1 - k_2| \leq 1$, then*
$$u_1 \prec u_2 \quad \Longleftrightarrow \quad \Psi(u_1) \prec \Psi(u_2).$$
- *(d) A vertex u is labeled A (respectively B) if and only if the spectral band $(\Psi(u))\,(V)$ is of type A (correspondingly B) for all $V > 0$.*

Example 7.2. As mentioned in Example 1.8, the order relation in Proposition 7.1 (c) is only preserved by Ψ for vertices that are in the same level or in consecutive levels. In general this may fail: Consider the middle spectral band $(\Psi(w))\,(V) = J_{[0,0,1,2]}(V)$ in $\sigma_{[0,0,1,2]}(V)$, as sketched in Figure 1.2 (level 2). From Example 2.3, we have
$$t_{[0,0,1,2]}(E,V) = E^3 - 2E^2V + E(V^2 - 3) + 2V.$$
Thus, for $V = 1$, we have $J_{[0,0,1,2]}(1) = [1-\sqrt{3}, \sqrt{2}]$, which is strictly contained one level below in $I_{[0,0,1]}(1) = [-1,3]$ and two level below in $I_{[0,0]}(1) = [-2,2]$, confer also Figure 1.2 (dashed lines). On the one hand, we have $u := (\Psi^{-1}(I_{[0,0]})) \prec (\Psi^{-1}(J_{[0,0,1,2]})) =: w$ for the corresponding vertices in the spectral α-tree $\mathcal{T}_\alpha$ (here u is in level 0 and w is in level 2). On the other hand, the previous considerations show that $\Psi(u) \not\prec \Psi(w)$.

Proof of Proposition 7.1. We will omit in the following the V dependence unless it is crucial. Let $\alpha \in [0,1] \setminus \mathbb{Q}$ with an infinite continued fraction expansion $(0, c_1, c_2, c_3, \ldots)$. For each $k \in \mathbb{N} \cup \{0\}$, denote $\alpha_k := \varphi([0,0,c_1,\ldots c_k])$ and $\frac{p_k}{q_k} := \alpha_k$ with co-prime p_k, q_k, see also (1.5). By standard properties of continued fractions (see e.g. Lemma I.1 (b)), we have for $k \geq 0$,
$$q_{-1} = 0,\ q_0 = 1,\ q_k = c_k q_{k-1} + q_{k-2}, \qquad k \in \mathbb{N}. \tag{7.1}$$
Let $\mathcal{T}_\alpha$ be the spectral α-tree with edge set $\mathcal{E}_\alpha$. Observe that any bijection satisfying the properties (a) and (c) must be unique since the vertices within a level k are totally ordered by $\prec$, and correspondingly the spectral bands within $\sigma_{[0,0,c_1,\ldots,c_k]}$ are totally ordered by $\prec$.

We start by constructing the bijection Ψ between all vertices of $\mathcal{T}_\alpha$ and the spectral bands of all $\left\{\sigma_{[0,0,c_1,\ldots,c_k]}\right\}_{k\in\mathbb{N}\cup\{-1,0\}}$. We do so inductively in k and start by handling the levels $k \in \{-1, 0, 1\}$ in the induction base, sketched in Figure 7.1 (3).

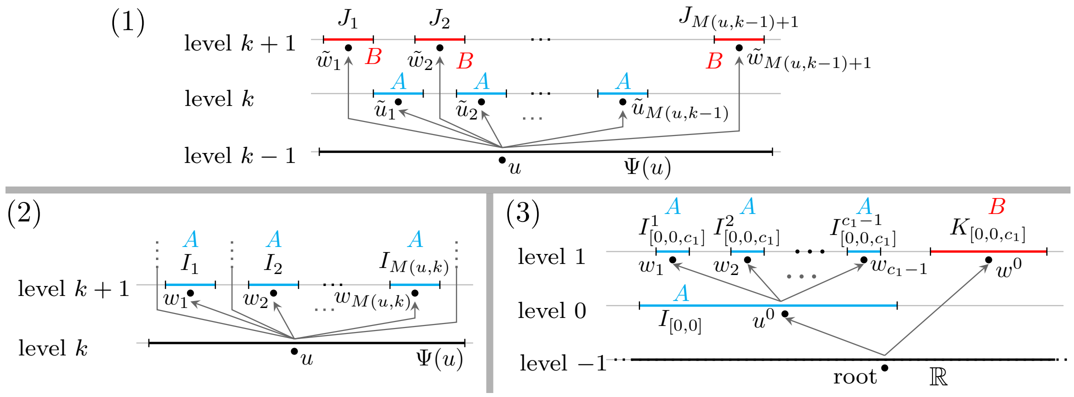

FIGURE 7.1. In (1) and (2), the recursive construction of Ψ is sketched. In (3), the map Ψ is sketched for the levels $k \in \{-1, 0, 1\}$. The plotted vertices are mapped to the spectral bands next to them.

Induction base. Level $k = -1$ of the graph $\mathcal{T}_\alpha$ contains only its root. The corresponding spectrum is $\sigma_{[0]} = \mathbb{R}$. We set Ψ to map the root of the graph to the "spectral band" $\mathbb{R}$ in $\sigma_{[0]}$.

In level $k = 0$, the graph $\mathcal{T}_\alpha$ has a single vertex (connected by an edge to the root), which is labeled A (it is vertex u^0 in Figure 1.1 (a)). The corresponding spectrum $\sigma_{[0,0]}$ consists of a single spectral band $I_{[0,0]} = [-2, 2]$ which is of type A by Lemma 5.3. We then set Ψ to map the vertex u^0 in level $k = 0$ to the spectral band $[-2, 2]$ of $\sigma_{[0,0]}$.

In level $k = 1$, the graph $\mathcal{T}_\alpha$ has c_1 vertices which are totally ordered by the order relation $\prec$. The right-most vertex is labeled B, which is the vertex w^0 in Figure 7.1 (3). This vertex is directly connected to the root. By construction, the left-most $c_1 - 1$ vertices $w_1 \prec \ldots \prec w_{c_1-1}$ are labeled A, which are directly connected by an edge to the vertex u^0 in level $k = 0$, i.e. $(u^0, w_j) \in \mathcal{E}_\alpha$. The corresponding spectrum $\sigma_{[0,0,c_1]}$ consists of $q_1 = c_1$ spectral bands using Equation (7.1). By Lemma 5.3 and Theorem 2.15, the left-most $c_1 - 1$ spectral bands $I^1_{[0,0,c_1]}, \ldots, I^{c_1-1}_{[0,0,c_1]}$ of $\sigma_{[0,0,c_1]}$ are of type A and the rightmost spectral band $K_{[0,0,c_1]}$ is of type B satisfying $I^j_{[0,0,c_1]} \prec I^{j+1}_{[0,0,c_1]}$ for $1 \le j \le c_1 - 2$ and $I^{c_1-1}_{[0,0,c_1]} \prec K_{[0,0,c_1]}$. Moreover, Lemma 5.3 asserts $I^j_{[0,0,c_1]} \subseteq_{\text{str}} I_{[0,0]}$ for $1 \le j \le c_1 - 1$ and $I_{[0,0]} \prec K_{[0,0,c_1]} \subseteq_{\text{str}} \mathbb{R}$. Hence, we define Ψ on level $k = 1$ by $\Psi(w_j) = I^j_{[0,0,c_1]}$ and $\Psi(w^0) = K_{[0,0,c_1]}$, sketched in Figure 7.1 (3).

By construction, Ψ satisfies the claimed properties (a), (b), (c), (d), for the levels $k \in \{-1, 0, 1\}$. We continue recursively constructing the map Ψ and inductively proving that all properties of the proposition statement hold for all $k > 1$. But, before doing so, we provide an enumeration of the vertices in level k of $\mathcal{T}_\alpha$ and of the spectral bands in $\sigma_{[0,0,c_1,\ldots,c_k]}$.

Counting vertices in level k of $\mathcal{T}_\alpha$. For $k \ge 0$, define

$$\mathcal{N}_k^{(A)} := \text{number of vertices labeled } A \text{ in level } k,$$
$$\mathcal{N}_k^{(B)} := \text{number of vertices labeled } B \text{ in level } k.$$

We will inductively prove

$$\mathcal{N}_k^{(A)} = q_k - q_{k-1} \quad \text{and} \quad \mathcal{N}_k^{(B)} = q_{k-1} \tag{7.2}$$

using the construction of the tree $\mathcal{T}_\alpha$, as described in Section 1.3 and Figure 1.1.

For the induction base, we check $k \in \{0, 1\}$. If $k = 0$, then there is exactly $1 = q_0 - q_{-1}$ vertex with label A and $0 = q_{-1}$ vertices with label B. If $k = 1$, then we have seen at the beginning of the proof that there are exactly $c_1 - 1 = q_1 - q_0$ vertices with label A and $1 = q_0$ vertex with label B.

By the recursive definition of the tree we conclude

$$\mathcal{N}_{k+1}^{(A)} = (c_{k+1} - 1) \cdot \mathcal{N}_k^{(A)} + c_{k+1} \cdot \mathcal{N}_k^{(B)} \quad \text{and} \quad \mathcal{N}_{k+1}^{(B)} = c_k \cdot \mathcal{N}_{k-1}^{(A)} + (c_k + 1) \cdot \mathcal{N}_{k-1}^{(B)}.$$

If we assume by induction hypothesis that (7.2) holds for k and $k-1$, then the previous equations combined with (7.1) lead to $\mathcal{N}_{k+1}^{(A)} = q_{k+1} - q_k$ and $\mathcal{N}_{k+1}^{(B)} = q_k$.

Counting spectral bands in $\sigma_{[0,0,c_1,\ldots,c_k]}$. For $k \geq 0$, define

$$\widetilde{\mathcal{N}}_k^{(A)} := \text{number of spectral bands of type } A \text{ in } \sigma_{[0,c_0,c_1,\ldots,c_k]},$$
$$\widetilde{\mathcal{N}}_k^{(B)} := \text{number of spectral bands of type } B \text{ in } \sigma_{[0,c_0,c_1,\ldots,c_k]}.$$

We will inductively prove

$$\widetilde{\mathcal{N}}_k^{(A)} = q_k - q_{k-1} \quad \text{and} \quad \widetilde{\mathcal{N}}_k^{(B)} = q_{k-1}. \tag{7.3}$$

The induction base for $k \in \{0, 1\}$ follows similarly as for the vertices. For the induction step suppose (7.3) holds for k and $k-1$. By the forward property of type A and type B spectral bands (see Definition 2.12), we conclude with the induction hypothesis that

$$\widetilde{\mathcal{N}}_{k+1}^{(A)} \geq (c_{k+1} - 1) \cdot \widetilde{\mathcal{N}}_k^{(A)} + c_{k+1} \cdot \widetilde{\mathcal{N}}_k^{(B)} = c_{k+1} q_k + q_{k-1} - q_k,$$
$$\widetilde{\mathcal{N}}_{k+1}^{(B)} \geq c_k \cdot \widetilde{\mathcal{N}}_{k-1}^{(A)} + (c_k + 1) \cdot \widetilde{\mathcal{N}}_{k-1}^{(B)} = c_k q_{k-1} + q_{k-2}.$$

Note that we currently get above only inequalities (rather than equalities). The reason is that we have not (yet) proven that counting the spectral bands according to Definition 2.12 is exhaustive. Namely, we have not proven yet that every spectral band may be written as $I_{[\mathbf{c},m]}^i$ or $I_{[\mathbf{c},m,n]}^j$ (as defined in Definition 2.12) for appropriate choices of $\mathbf{c}, m, n, i, j$. This is actually true and follows from the next argument. Applying (7.1), the estimates $\widetilde{\mathcal{N}}_{k+1}^{(A)} \geq q_{k+1} - q_k$ and $\widetilde{\mathcal{N}}_{k+1}^{(B)} \geq q_k$ follow. These are actually equalities since $\widetilde{\mathcal{N}}_{k+1}^{(A)} + \widetilde{\mathcal{N}}_{k+1}^{(B)}$ is bounded from above by the total number of spectral bands in $\sigma_{[0,c_0,c_1,\ldots,c_{k+1}]}$, which equals to q_{k+1}. □

Induction step for constructing Ψ and proving its properties. Let $k \geq 1$ be such that there is a map Ψ satisfying up to (and including) level k all the claimed properties (a), (b), (c) and (d). We show that we can extend Ψ to a map satisfying these properties also in level $k+1$.

For each vertex w in level $k+1$ with label A, there is a vertex u in k such that there is an edge from u to w, i.e. $(u, w) \in \mathcal{E}_\alpha$. Then the associated spectral band $\Psi(u)$ in $\sigma_{[0,0,c_1,\ldots,c_k]}$ has (by the induction hypothesis) the label of u. Define

$$M(u, k) := \begin{cases} c_{k+1} - 1 & u \text{ is of label } A, \\ c_{k+1} & u \text{ is of label } B. \end{cases} \tag{7.4}$$

By the definition of $\mathcal{T}_\alpha$ (see Section 1.3), u is connected to $M(u,k)$ vertices $w_1 \prec \ldots \prec w_{M(u,k)}$ of label A in level $k+1$. Similarly, by Theorem 2.15 and Definition 2.12, $\Psi(u) \subseteq \sigma_{[0,0,c_1,\ldots,c_k]}$ strictly contains $M(u,k)$ spectral bands $I_1 \prec \ldots \prec I_{M(u,k)}$ of type A in $\sigma_{[0,0,c_1,\ldots,c_{k+1}]}$. We define Ψ to map these $M(u,k)$ vertices to these $M(u,k)$ spectral bands such that the $\prec$ order between them is preserved (i.e. $\Psi(w_j) = I_j$), see a sketch in Figure 7.1 (2). We repeat this for all vertices u in level k. Doing so, we have bijectively mapped all A labeled vertices in level $k+1$ to all spectral bands in $\sigma_{[0,0,c_1,\ldots,c_{k+1}]}$ of type A since they have the same cardinality by the previous considerations (Equations (7.2) and (7.3)).

The vertices with label B are treated similarly: For each vertex w in level $k+1$ with label B, there is a vertex u in $k-1$ such that there is an edge from u to w. Then the associated spectral band $\Psi(u)$ in $\sigma_{[0,0,c_1,\ldots,c_{k-1}]}$ has (by the induction hypothesis) the label of u. As before u is connected to $M(u,k-1)+1$ vertices $\tilde{w}_1 \prec \ldots \prec \tilde{w}_{M(u,k-1)+1}$ with label B (i.e. , i.e. $(u,\tilde{w}_j) \in \mathcal{E}_\alpha$) and $\Psi(u) \subseteq \sigma_{[0,0,c_1,\ldots,c_{k-1}]}$ contains $M(u,k-1)+1$ spectral bands $J_1 \prec \ldots \prec J_{M(u,k-1)+1}$ of type B in $\sigma_{[0,0,c_1,\ldots,c_{k+1}]}$. We define Ψ to map these $M(u,k-1)+1$ vertices to these $M(u,k-1)+1$ spectral bands such that the $\prec$ order between them is preserved (i.e. $\Psi(\tilde{w}_j) = J_j$), see a sketch in Figure 7.1 (1). We repeat this for all vertices u in level $k-1$. By construction Ψ maps bijectively all B labeled vertices in level $k+1$ to all spectral bands in $\sigma_{[0,0,c_1,\ldots,c_{k+1}]}$ of type B since they have the same cardinality by the previous considerations.

We emphasize at this point that by construction and the interlacing property (I) of the spectral band $\Psi(u)$

$$\Psi(\tilde{w}_1) \prec \Psi(\tilde{u}_1) \prec \ldots \prec \Psi(\tilde{w}_{M(u,k-1)}) \prec \Psi(\tilde{u}_{M(u,k-1)}) \prec \Psi(\tilde{w}_{M(u,k-1)+1}), \tag{7.5}$$

borrowing the notation of Figure 7.1 (1).

Since we have proven above that the number of vertices (distinguished by their labels) coincides with the number of spectral bands in $\sigma_{[0,0,c_1,\ldots,c_{k+1}]}$ (distinguished by their types), we conclude that Ψ satisfies (a) and (d). Property (b) follows also immediately from the construction and the induction hypothesis. Thus, it is left to prove (c).

We recall that we prove by induction, and that the induction base was already proven above. It remains to prove that if $k \geq 1$ and Ψ satisfies (c) for all vertices up to (and including) level k, then it also satisfies (c) for all vertices up to and including level $k+1$. Explicitly, let w and $\tilde{w}$ be two different vertices, which are either both in level $k+1$, or such that one of them is in level k and the other in level $k+1$. We need to show that

$$w \prec \tilde{w} \quad \Longleftrightarrow \quad \Psi(w) \prec \Psi(\tilde{w}). \tag{7.6}$$

Suppose $w \to \tilde{w}$. Since (b) holds, $w \to \tilde{w}$ implies $\Psi(\tilde{w}) \subseteq_{\text{str}} \Psi(w)$ so that $\Psi(\tilde{w}) \not\prec \Psi(w)$ and $\Psi(w) \not\prec \Psi(\tilde{w})$. Hence, the right relation in (7.6) above does not hold. By definition of $\prec$ on the vertex set of $\mathcal{T}_\alpha$ one can show that $w \to \tilde{w}$ implies $w \not\prec \tilde{w}$ and $\tilde{w} \not\prec w$, so that also the left relation in (7.6) does not hold. Therefore, the equivalence (7.6) is valid in this case, and we may therefore proceed with the proof assuming that w and $\tilde{w}$ are not connected by an edge.

Proof. If w and $\tilde{w}$ are not connected by an edge, then we show next that either $w \prec \tilde{w}$ or $\tilde{w} \prec w$. To see this, choose a path γ from the root to w and a path $\tilde{\gamma}$ from the root to $\tilde{w}$ (both paths are unique). We denote by v the vertex of maximal level which appears in both paths (such a vertex exists since the root appears in both paths). We denote by u the vertex to which v is connected in γ and denote by $\tilde{u}$ the vertex to which v is connected in $\tilde{\gamma}$. By the construction of the tree and the order relation $\prec$ on its vertices (Definition 1.4 and Figure 1.1) we have that either $u \prec \tilde{u}$ or $\tilde{u} \prec u$. Since $u \to w$ and $\tilde{u} \to \tilde{w}$ we get (by Definition 1.4) that either $w \prec \tilde{w}$ or $\tilde{w} \prec w$.

We will now show that $w \prec \tilde{w}$ implies $\Psi(w) \prec \Psi(\tilde{w})$. This actually proves (7.6) which can be seen as follows. Assume by contradiction that $\Psi(w) \prec \Psi(\tilde{w})$ and $\tilde{w} \prec w$ hold. Then the previous implication leads to $\Psi(\tilde{w}) \prec \Psi(w)$ contradicting $\Psi(w) \prec \Psi(\tilde{w})$.

We continue proving that $w \prec \tilde{w}$ implies $\Psi(w) \prec \Psi(\tilde{w})$. We now go over all the possible configurations in which $w, \tilde{w}$ are either both in level $k+1$, or one of them is in level k and the other in level $k+1$ and they are not connected by an edge. There are 11 such cases and we verify that $\Psi(w) \prec \Psi(\tilde{w})$ holds in all these cases.

We start by checking the cases in which w and $\tilde{w}$ are different vertices in level $k+1$. Since $w \neq \tilde{w}$, the injectivity of Ψ implies $\Psi(w) \neq \Psi(\tilde{w})$. Thus, $(\Psi(w))(V) \cap (\Psi(\tilde{w}))(V) = \emptyset$ follows from Proposition 1.2 (different spectral bands in the same level do not touch). Hence, we have

$$\text{either} \quad \Psi(w) \prec \Psi(\tilde{w}) \quad \text{or} \quad \Psi(\tilde{w}) \prec \Psi(w). \tag{7.7}$$

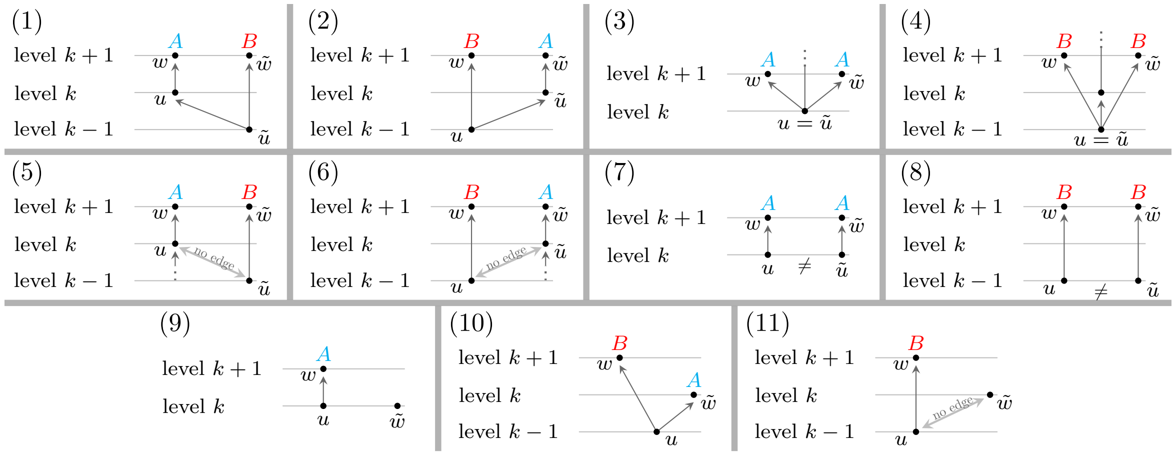

FIGURE 7.2. The different cases in the proof of Proposition 7.1 (c) are outlined.

By the construction of the tree, there are vertices u and $\tilde{u}$ in level k or $k-1$, which are connected to w and $\tilde{w}$, i.e., $(u,w),(\tilde{u},\tilde{w}) \in \mathcal{E}_\alpha$. Then property (b) implies $\Psi(\tilde{w}) \subseteq_{\text{str}} \Psi(\tilde{u})$ and $\Psi(w) \subseteq_{\text{str}} \Psi(u)$. To show $\Psi(w) \prec \Psi(\tilde{w})$, one needs to treat 8 different cases. These cases are plotted in Figure 7.2, (1) to (8) and analyzed below.

In case (1), $u \prec \tilde{w}$ follows from $w \prec \tilde{w}$. Recall that Ψ is defined by sending the vertices which are connected to $\tilde{u}$ to the spectral bands associated with $\Psi(\tilde{u})$ according to Definition 3.3. In particular, the map Ψ preserves the interlacing property (I) (see discussion around Equation (7.5)) implying $\Psi(u) \prec \Psi(\tilde{w})$. By $\Psi(w) \subseteq_{\text{str}} \Psi(u)$ and (7.7), we conclude $\Psi(w) \prec \Psi(\tilde{w})$. The case (2) is treated similarly. The cases (3) and (4), also follow directly from the definition of Ψ and the interlacing property of $\Psi(u) = \Psi(\tilde{u})$.

In the cases (5) to (8), there is neither an edge between u and $\tilde{u}$ nor these vertices coincide. By the definition of the order on the tree $\mathcal{T}_\alpha$, we conclude $u \prec \tilde{u}$ since $w \prec \tilde{w}$, $(u,w) \in \mathcal{E}_\alpha$ and $(\tilde{u},\tilde{w}) \in \mathcal{E}_\alpha$. Since in these cases u and $\tilde{u}$ are in level k or $k-1$, we get by the induction hypothesis that $u \prec \tilde{u}$ implies $\Psi(u) \prec \Psi(\tilde{u})$. In the cases (7) and (8), since $u, \tilde{u}$ are in the same level, so are $\Psi(u), \Psi(\tilde{u})$ and hence we have even $\Psi(u) \prec_{\text{str}} \Psi(\tilde{u})$. Since $\Psi(w) \subseteq_{\text{str}} \Psi(u)$ and $\Psi(\tilde{w}) \subseteq_{\text{str}} \Psi(\tilde{u})$, we conclude $\Psi(w) \prec_{\text{str}} \Psi(\tilde{w})$. However, in cases (5) and (6), we cannot directly conclude from $\Psi(u) \prec \Psi(\tilde{u})$, $\Psi(\tilde{w}) \subseteq_{\text{str}} \Psi(\tilde{u})$ and $\Psi(w) \subseteq_{\text{str}} \Psi(u)$ that $\Psi(w) \prec \Psi(\tilde{w})$. A-priori, it is possible in these conditions that $\Psi(\tilde{w}) \prec \Psi(w)$ if $\Psi(w)$ and $\Psi(\tilde{w})$ are both included in $\Psi(u) \cap \Psi(\tilde{u})$. However, this leads to a contradiction as we explain now. Either $\tilde{w}$ is of type B (in case (5)) or w is of type B (in case (6)). Since Ψ preserves the labels by (d) either $\Psi(\tilde{w})$ (in case (5)) or $\Psi(w)$ (in case (6)) is of type B. A spectral band which is of type B cannot be of weak backward type A using Lemma 5.3, Lemma 5.4, Proposition 4.20 and Theorem 2.15. But this contradicts that both $\Psi(\tilde{w})$ and $\Psi(w)$ are included one level below (since they are included in both $\Psi(u)$ and $\Psi(\tilde{u})$ and one of them is in level k). This finishes the proof for the cases (5) and (6).

Next, let w be a vertex in level $k+1$ and $\tilde{w}$ be a vertex in level k. The symmetric case of w being in level k and $\tilde{w}$ being in level $k+1$ follows similarly. Recall that we want to show that $w \prec \tilde{w}$ implies $\Psi(w) \prec \Psi(\tilde{w})$. Let u be the vertex in level k (if w has label A) or $k-1$ (if w has label B) such that there is an edge $(u,w) \in \mathcal{E}_\alpha$.

If w has label A, then $u \neq \tilde{w}$ follows since otherwise $w \prec \tilde{w}$ is not defined. Since $w \prec \tilde{w}$, we conclude $u \prec \tilde{w}$, which are both in level k, see Figure 7.2 (9). Thus, the induction hypothesis asserts that $u \prec \tilde{w}$ implies $\Psi(u) \prec \Psi(\tilde{w})$. Since $\Psi(u)$ and $\Psi(\tilde{w})$ are both spectral bands in $\sigma_{[0,0,c_1,\dots,c_k]}$, they cannot intersect, i.e. $(\Psi(u))(V) \cap (\Psi(\tilde{w}))(V) = \emptyset$ for all $V > 0$. Thus,

$\Psi(u) \prec_{\text{str}} \Psi(\tilde{w})$ follows. Since $(u, w) \in \mathcal{E}_\alpha$ implies $\Psi(w) \subseteq_{\text{str}} \Psi(u)$ by (b), $\Psi(u) \prec_{\text{str}} \Psi(\tilde{w})$ implies $\Psi(w) \prec \Psi(\tilde{w})$.

If w has label B, then we have two cases sketched in Figure 7.2 (10) and (11). In case (10), $\Psi(w) \prec \Psi(\tilde{w})$ follows from $w \prec \tilde{w}$ since Ψ preserves the interlacing property (I) (see discussion around Equation (7.5)) of the spectral band $\Psi(u)$ by construction. In case (11), there is no edge between u and $\tilde{w}$. By the definition of the order on the tree $\mathcal{T}_\alpha$, we conclude $u \prec \tilde{w}$ since $w \prec \tilde{w}$ and $(u, w) \in \mathcal{E}_\alpha$. Thus, the induction hypothesis (u and $\tilde{w}$ are in level k and $k-1$) asserts that $u \prec \tilde{w}$ implies $\Psi(u) \prec \Psi(\tilde{w})$. Thus, either $\Psi(w) \subseteq \Psi(\tilde{w})$ or $\Psi(w) \prec \Psi(\tilde{w})$. However, $\Psi(w)$ is of type B since w has label B and Ψ preserves the labels by (d). Thus, $\Psi(w)$ is not of weak backward type A (as explained above when treating cases (5) and (6)) and so it is not contained in any spectral band of $\sigma_{[0,0,c_1,\ldots,c_k]}$. In particular, $\Psi(w)$ is not contained in $\Psi(\tilde{w})$ implying $\Psi(w) \prec \Psi(\tilde{w})$ by the previous considerations. □

In the course of proving Proposition 7.1, we have actually shown the following auxiliary statement. We phrase it here as a Lemma, which is interesting on its own right.

Lemma 7.3. *Let $\alpha \in [0,1]\setminus\mathbb{Q}$ with an infinite continued fraction expansion given by the natural number coefficients, $(0, c_1, c_2, c_3, \ldots)$. For all $k \in \mathbb{N}$, denote $\frac{p_k}{q_k} := \alpha_k$ with co-prime p_k, q_k as in (1.5) and $q_{-1} = 0$, $q_0 = 1$.*

(a) For all $k \in \mathbb{N} \cup \{0\}$, the level k of the spectral α-tree, $\mathcal{T}_\alpha$, contains $q_k - q_{k-1}$ vertices of label A, and q_{k-1} vertices of label B (so a total number of q_k in level k).
(b) For all $k \in \mathbb{N} \cup \{0\}$, the spectrum $\sigma_{[0,0,c_1,\ldots,c_k]}$ contains $q_k - q_{k-1}$ spectral bands of type A, and q_{k-1} spectral bands of type B.

7.2. **Proof of Theorem 1.9.** We first recall the terminology used in Theorem 1.9. The boundary $\partial\mathcal{T}_\alpha$ of the ordered tree $\mathcal{T}_\alpha$ with edge set $\mathcal{E}_\alpha$ is

$$\partial\mathcal{T}_\alpha := \{\gamma = (u_0, u_1, u_2 \ldots) : u_0 \text{ is the root of } \mathcal{T}_\alpha \text{ and } (u_{m-1}, u_m) \in \mathcal{E}_\alpha \text{ for all } m \in \mathbb{N}\},$$

i.e. the set of all infinite paths which start from the root. Given $\gamma \in \partial\mathcal{T}_\alpha$, we consider the (infinite) intersection of all spectral bands which correspond to γ's vertices, i.e., $\cap_{m\in\mathbb{N}} (\Psi(u_m))(V)$. By Proposition 7.1 (b) (or equivalently Theorem 1.7 (b)), we have $(\Psi(u_{m+1}))(V) \subseteq_{\text{str}} (\Psi(u_m))(V)$. With this at hand, we have argued in Section 1.4 that this intersection $\cap_{m\in\mathbb{N}} (\Psi(u_m))(V)$ contains a single point in $\sigma(H_{\alpha,V})$, which we denoted by $E_\alpha(\gamma; V)$ (see also [BBB+, lem. 5.11] for the case $V > 4$). This defines the map

$$E_\alpha(\,\cdot\,; V) : \partial\mathcal{T}_\alpha \to \sigma(H_{\alpha,V}),$$

whose properties are in the focus of Theorem 1.9.

In this section, we split the statement of Theorem 1.9 into various lemmas (Lemmas 7.4, 7.5, 7.10, and 7.16), which are proven separately.

Lemma 7.4. *[also Theorem 1.9 (c)] Let $\alpha \in [0,1] \setminus \mathbb{Q}$. For all $\gamma \in \partial\mathcal{T}_\alpha$, the map $E_\alpha(\gamma;\,\cdot\,) : (0,\infty) \to \mathbb{R}$ is Lipschitz continuous.*

Proof. Fix $\gamma \in \partial\mathcal{T}_\alpha$. To prove the Lipschitz continuity (in V) of $E_\alpha(\gamma; V) : \partial\mathcal{T}_\alpha \to \sigma(H_{\alpha,V})$, we use the Lipschitz continuity of the spectral bands proven in Corollary 3.2. Denoting $\gamma = (u_0, u_1, u_2, \ldots)$ we have that $\{(\Psi(u_m))(V)\}_{m\in\mathbb{N}}$ is an infinite nested sequence of compact intervals such that $\bigcap_{m\in\mathbb{N}} (\Psi(u_m))(V) = \{E_\alpha(\gamma; V)\}$. In particular, this means that $\lim_{m\to\infty} L((\Psi(u_m))(V)) = E_\alpha(\gamma; V)$. Hence, we get for all $V_1, V_2 > 0$ that

$$\begin{aligned}|E_\alpha(\gamma; V_1) - E_\alpha(\gamma; V_2)| &= \lim_{m\to\infty} |L((\Psi(u_m))(V_1)) - L((\Psi(u_m))(V_2))| \\ &\leq |V_1 - V_2|,\end{aligned}$$

where the last estimate follows from Corollary 3.2. □

Lemma 7.5. *[also Theorem 1.9 (d), (e)] Let $\alpha \in [0,1] \setminus \mathbb{Q}$. Then there exists a function $N_\alpha : \partial\mathcal{T}_\alpha \to [0,1]$ such that for all $\gamma \in \partial\mathcal{T}_\alpha$ and all $V > 0$,*

$$N_{\alpha,V}\left(E_\alpha(\gamma;V)\right) = N_\alpha(\gamma).$$

In addition, we have the following spectral properties connecting between negative and positive values of V:

(a) For all $V \in \mathbb{R}$ and all $k \in \mathbb{N} \cup \{0\}$,
$$\sigma(H_{\alpha_k,V}) = -\sigma(H_{\alpha_k,-V}).$$

(b) For all $V \in \mathbb{R}$,
$$\sigma(H_{\alpha,V}) = -\sigma(H_{\alpha,-V}).$$

(c) For all $\gamma \in \partial\mathcal{T}_\alpha$ and all $V < 0$,
$$N_{\alpha,V}\left(-E_\alpha(\gamma;-V)\right) = 1 - N_\alpha(\gamma).$$

Proof. We start by proving the existence of the function N_α. Let $\gamma \in \partial\mathcal{T}_\alpha$ and $V > 0$. Denote $\gamma = (u_0, u_1, u_2, \ldots)$, and for each $j \geq 0$ denote by $k(u_j)$ the level of the vertex u_j. For example, $k(u_0) = -1$, since u_0 is the root of $\mathcal{T}_\alpha$. Note that $\{k(u_j)\}_{j\geq 0}$ is an increasing sequence and that $1 \leq k(u_{j+1}) - k(u_j) \leq 2$. In order to prove the lemma, we show that the value of the IDS $N_{\alpha,V}(E)$ is related to the spectral bands of the associated periodic approximations. We also consider $\{E\} = [E,E]$ as an interval and so we can use the notation $I \prec \{E\}$ if I is another interval.

By the definition of the integrated density of states, (1.2), we have

$$\begin{aligned}
N_{\alpha,V}(E_\alpha(\gamma;V)) &= \lim_{k\to\infty} \frac{\#\left\{\lambda \in \sigma\left(H_{\alpha,V}|_{[0,q_k-1]}\right) \,:\, \lambda \leq E_\alpha(\gamma;V)\right\}}{q_k} \qquad (7.8)\\
\begin{pmatrix}\text{periodic boundary}\\ \text{condition}\end{pmatrix} \quad &= \lim_{k\to\infty} \frac{\#\left\{\lambda \in \sigma\left(H_{\alpha,V}|_{[0,q_k-1]}(0)\right) \,:\, \lambda \leq E_\alpha(\gamma;V)\right\}}{q_k}\\
(\text{Lem. } I.1) \quad &= \lim_{k\to\infty} \frac{\#\left\{\lambda \in \sigma\left(H_{\alpha_k,V}|_{[0,q_k-1]}(0)\right) \,:\, \lambda \leq E_\alpha(\gamma;V)\right\}}{q_k}\\
&= \lim_{k\to\infty} \frac{\#\left\{I \text{ is a spectral band of } \sigma(H_{\alpha_k,V}) \,:\, I \prec \{E_\alpha(\gamma;V)\}\right\}}{q_k}\\
&= \lim_{j\to\infty} \frac{\#\left\{I \text{ is a spectral band of } \sigma(H_{\alpha_{k(u_j)},V}) \,:\, I \prec \Psi(u_j)(V)\right\}}{q_{k(u_j)}}\\
(\text{Prop. 7.1 } (c)) \quad &= \lim_{j\to\infty} \frac{\#\left\{w \text{ is in level } k(u_j) \,:\, w \prec u_j\right\}}{q_{k(u_j)}} =: N_\alpha(\gamma),
\end{aligned}$$

where, moving to the second line, we used that $H_{\alpha,V}|_{[0,q_k-1]}(0) = H_{\alpha,V}|_{[0,q_k-1]} + \mathbb{I}_{1,q_k} + \mathbb{I}_{q_k,1}$ is a rank two perturbation of $H_{\alpha,V}|_{[0,q_k-1]}$ and hence it might change the numerator of the fraction above by at most two (applying twice Proposition III.1) and does not affect the limit. When moving from the second line to the third, we replace the matrix $H_{\alpha,V}|_{[0,q_k-1]}$ by $H_{\alpha_k,V}|_{[0,q_k-1]}$. According to Lemma I.1, these matrices differ by at most one entry (the last entry of the diagonal, when k is even), so once again (by Proposition III.1) such rank one perturbation might change the numerator by one, but does not affect the limit. Similarly, the numerator in the fourth line might differ than the numerator in the third line by one using Lemma 4.5 (e) (for $n = 1$), but this does not affect the limit. When moving to the fifth line we use that the limit exists and henceforth may take a subsequence, $\{k(u_j)\}_{j\in\mathbb{N}}$ of $\{k\}_{k\in\mathbb{N}}$. In addition, we used $E_\alpha(\gamma;V) \in (\Psi(u_j))(V)$ for $j \in \mathbb{N}$, which holds by construction of E_α.

The last line clearly is independent of V and depends only on the embedding of the path γ within $\mathcal{T}_\alpha$, so we may denote it by $N_\alpha(\gamma)$.

We continue with treating the case $V < 0$ and proving the second part of the lemma. We recall that by Floquet-Bloch theory (see Proposition 1.2 and Equation (3.1)), we have

$$\sigma(H_{\alpha_k,V}) = \bigcup_{\theta\in[0,\pi]} \sigma\Big(H_{\alpha,V}|_{[0,q_k-1]}(\theta)\Big), \tag{7.9}$$

and contains exactly q_k spectral bands. Using the $q_k \times q_k$ diagonal matrix, $D := \operatorname{diag}\{1,-1,1,-1,\ldots\}$, one can compute that

- $D^{-1}\, H_{\alpha,-V}|_{[0,q_k-1]}(\theta)\, D = -\, H_{\alpha,V}|_{[0,q_k-1]}(\theta)$ if q_k is even.
- $D^{-1}\, H_{\alpha,-V}|_{[0,q_k-1]}(\theta)\, D = -\, H_{\alpha,V}|_{[0,q_k-1]}(\theta+\pi)$ if q_k is odd.

The unitary equivalence between the operators above together with (7.9) yields that for all $k \in \mathbb{N}\cup\{0\}$ (no matter whether q_k is even or odd) $\sigma(H_{\alpha_k,V}) = -\sigma(H_{\alpha_k,-V})$, which proves property (a).

By Proposition 1.3 we have

$$\sigma(H_{\alpha,V}) = \lim_{k\to\infty}\left(\sigma(H_{\alpha_k,V})\cup\sigma(H_{\alpha_{k+1},V})\right).$$

Combining this with $\sigma(H_{\alpha_k,V}) = -\sigma(H_{\alpha_k,-V})$ yields $\sigma(H_{\alpha,V}) = -\sigma(H_{\alpha,-V})$ and proves property (b). In order to prove (c), we return to the calculation (7.8) and write, for all $V < 0$,

$$\begin{aligned}
N_{\alpha,V}(E) &= \lim_{k\to\infty}\frac{\#\{I \text{ is a spectral band of } \sigma(H_{\alpha_k,V}) \,:\, I \prec \{E\}\}}{q_k}\\
&\overset{(a)}{=} \lim_{k\to\infty}\frac{\#\{I \text{ is a spectral band of } \sigma(H_{\alpha_k,-V}) \,:\, I \succ \{-E\}\}}{q_k}\\
&= \lim_{k\to\infty}\frac{q_k - \#\{I \text{ is a spectral band of } \sigma(H_{\alpha_k,-V}) \,:\, I \prec \{-E\}\}}{q_k}\\
&= 1-\lim_{k\to\infty}\frac{\#\{I \text{ is a spectral band of } \sigma(H_{\alpha_k,-V}) \,:\, I \prec \{-E\}\}}{q_k} = 1 - N_{\alpha,-V}(-E),
\end{aligned}$$

where in moving to the third line the numerators might differ by at most one, but this does not affect the limit. Now, for $V < 0$ and $\gamma \in \partial\mathcal{T}_\alpha$, we substitute above $E = -E_\alpha(\gamma; -V)$ and so Equation (7.8) implies statement (c). □

Remark 7.6. We note that the tree $\mathcal{T}_\alpha$ was constructed to encode the spectral information for positive values of V. This is reflected in Theorem 1.7 (which is proven in Proposition 7.1) and Theorem 1.9; all phrased for $V > 0$. In light of Lemma 7.5 one may wonder whether it is possible to provide an analogous tree graph to reflect the spectral properties of $\sigma(H_{\alpha_k,V})$ and $\sigma(H_{\alpha,V})$ for $V < 0$. Indeed, Lemma 7.5 shows that such a tree will be a reflection of the original tree graph, $\mathcal{T}_\alpha$. Returning to the construction of $\mathcal{T}_\alpha$ as described in Section 1.3, one may construct the analogous $V < 0$ tree by just replacing the order of the two vertices u^0, w^0 which are connected to the root, such that $w^0 \prec u^0$. Obviously, the bijection Ψ in Theorem 1.7 and Proposition 7.1 should be reflected accordingly.

Lemma 7.5 also allows us also to provide the analogue of Theorem 2.15 for $V < 0$.

Corollary 7.7. *For all $V < 0$ and $\boldsymbol{c} \in \mathcal{C}$ with $\varphi(\boldsymbol{c}) \notin \{-1,\infty\}$, every spectral band in $\sigma_{\boldsymbol{c}}(V)$ is either of type A or B and its type is invariant for all $V < 0$.*

Proof. This follows directly from Lemma 7.5 (a) and Theorem 2.15. □

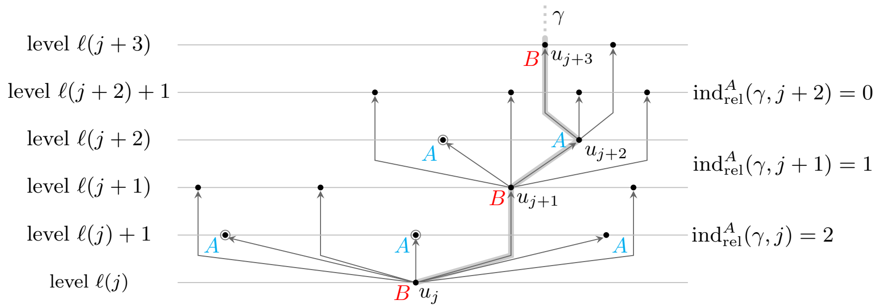

FIGURE 7.3. The figure demonstrating the notation introduced in Remark 7.8. The circled vertices are the ones that are counted for the corresponding relative A-index. Note that not all vertices are plotted in each level.

Remark 7.8. As mentioned in the introduction, the representation of the the IDS $N_{\alpha,V}$ via $\partial\mathcal{T}_\alpha$ allows to provide an explicit expression of the IDS. This is done in [Ray95a] using a coding scheme and we shortly present here an adaptation of this expression using infinite paths, i.e., elements of $\partial\mathcal{T}_\alpha$. Towards this, we define for $\gamma = (u_0, u_1, \dots) \in \partial\mathcal{T}_\alpha$, the *level function* $\ell : \mathbb{N}_0 \to \mathbb{N}_{-1}$ by setting $\ell(j)$ to be the level of u_j (so that $\ell(j) = k(u_j)$, in the notation used in the proof of Lemma 7.5). Next define the *relative A-index* $\mathrm{ind}^A_{\mathrm{rel}}(\gamma, j)$ to be the number of vertices w in level $\ell(j) + 1$ admitting an edge $(u_j, w) \in \mathcal{E}_\alpha$ (i.e. w has label A) such that $w \prec u_{j+1}$, see Figure 7.3. Note that by the tree construction

$$\mathrm{ind}^A_{\mathrm{rel}}(\gamma, j) \in \begin{cases} \{0, \dots, c_{\ell(j)+1} - 1\} & \text{if } u_j \text{ has label } A, \\ \{0, \dots, c_{\ell(j)+1}\} & \text{if } u_j \text{ has label } B, \end{cases}$$

and $\mathrm{ind}^A_{\mathrm{rel}}(\gamma, j) = 0$ if and only if u_{j+1} is either the leftmost vertex with label B connected to u_j or the leftmost vertex with label A connected to u_j. Finally, set

$$\delta_A(\gamma, j) := \begin{cases} 1 & \text{if } u_j \text{ has label } A, \\ 0 & \text{if } u_j \text{ has label } B. \end{cases}$$

With these notations at hand, the function $N_\alpha : \partial\mathcal{T}_\alpha \to [0, 1]$ is written explicitly as

$$N_\alpha(\gamma) = -\alpha + \sum_{j\in\mathbb{N}_0} (-1)^{\ell(j)} \left(\mathrm{ind}^A_{\mathrm{rel}}(\gamma, j) + \delta_A(\gamma, j)\right) \left(q_{\ell(j)}\alpha - p_{\ell(j)}\right),$$

where p_k, q_k are coprime such that $\alpha_k = \frac{p_k}{q_k}$, see Equation (1.5). This equality is an immediate consequence of Lemma 7.5 and [Ray95a, Theorem 4.7]. Note that this explicit representation is a crucial ingredient in Raymond's work to prove that all gaps are there for $V > 4$. We refer the reader also a more detailed discussion on that in [BBB$^+$, Section 5.3, Proposition 5.21].

Lemma 7.9. *[the surjectivity part of Theorem 1.9 (a)] Let $\alpha \in [0, 1] \setminus \mathbb{Q}$ and $V > 0$. The map $E_\alpha(\,\cdot\,; V) : \partial\mathcal{T}_\alpha \to \sigma(H_{\alpha,V})$ is surjective.*

Proof. Let $V > 0$ and $E \in \sigma(H_{\alpha,V})$. By Proposition 1.3, we have $E \in \sigma_{\alpha_k}(V) \cup \sigma_{\alpha_{k+1}}(V)$ for all $k \in \mathbb{N}$ and so E is contained in a spectral band of $\sigma_{\alpha_k}(V)$ or $\sigma_{\alpha_{k+1}}(V)$. By Proposition 7.1, there is a bijection Ψ between spectral bands and vertices of the spectral α-tree $\mathcal{T}_\alpha$. We conclude that we can choose a sequence of vertices, $\{w_j\}_{j\in\mathbb{N}}$ in $\mathcal{T}_\alpha$, such that $E \in \Psi(w_j)(V)$. In particular this sequence of vertices may be chosen such that for all $j \in \mathbb{N}$, w_j is in level k_j and $k_j < k_{j+1}$.

By construction, $\mathcal{T}_\alpha$ is a directed connected tree. Thus, we can choose for each $j \in \mathbb{N}$ an infinite path $\gamma_j \in \partial\mathcal{T}_\alpha$ passing through the vertex w_j, see a sketch in Figure 7.4 (1). We note that $\partial\mathcal{T}_\alpha$ is a compact space (using the product topology on spheres from the root, or equivalently noticing that this space is a Gromov boundary). Thus, $\{\gamma_j\}_{j\in\mathbb{N}}$ admits a convergent subsequence $\{\gamma_{j_l}\}_{l\in\mathbb{N}}$ with limit $\gamma = (u_0, u_1, u_2, \dots) \in \partial\mathcal{T}_\alpha$. We claim that $E_\alpha(\gamma\ ; V) = E$, which proves surjectivity.

By the convergence of $\{\gamma_{j_l}\}_{l\in\mathbb{N}}$ to γ, there is for each $N \in \mathbb{N}$, an $l_N \in \mathbb{N}$ such that for all $l \geq l_N$, we have $\gamma|_{[0,N]} = \gamma_{j_l}|_{[0,N]}$ and $k_{j_l} > N$ (i.e. the vertex w_{j_l} does not appear on the first N vertices of the path γ_{j_l}). Thus, for each $N \in \mathbb{N}$ and $l \geq l_N$, we have $u_N \to w_{j_l}$ (a path from u_N to the vertex w_{j_l}). Hence, Proposition 7.1 (b) implies $\Psi(w_{j_l})(V) \subsetneq_{\text{str}} \Psi(u_N)(V)$. Since by construction $E \in \Psi(w_{j_l})(V)$, we conclude $E \in \Psi(u_N)(V)$ for all $N \in \mathbb{N}$. Hence, $E = E_\alpha(\gamma\ ; V)$ follows from the definition of E_α by the intersection of all $\Psi(u_N)(V)$. □

Next, we show that the map $E_\alpha(\ \cdot\ ; V) : \partial\mathcal{T}_\alpha \to \sigma(H_{\alpha,V})$ is order preserving. Towards this, recall the order relation on $\partial\mathcal{T}_\alpha$. Let $\gamma_1 = (u_0, u_1, \dots), \gamma_2 = (w_0, w_1, \dots) \in \partial\mathcal{T}_\alpha$. If $\gamma_1 = \gamma_2$, we set both $\gamma_1 \preceq \gamma_2$ and $\gamma_2 \preceq \gamma_1$ (so that the order is reflexive). Otherwise, there exists a unique $k \in \mathbb{N}_0$ such that $u_{k-1} = w_{k-1}$ and $u_k \neq w_k$. By Definition 1.4, either

- $u_l \prec w_l$ for all $l \geq k$ and so $\gamma_1 \preceq \gamma_2$, or
- $w_l \prec u_l$ for all $l \geq k$ and so $\gamma_2 \preceq \gamma_1$.

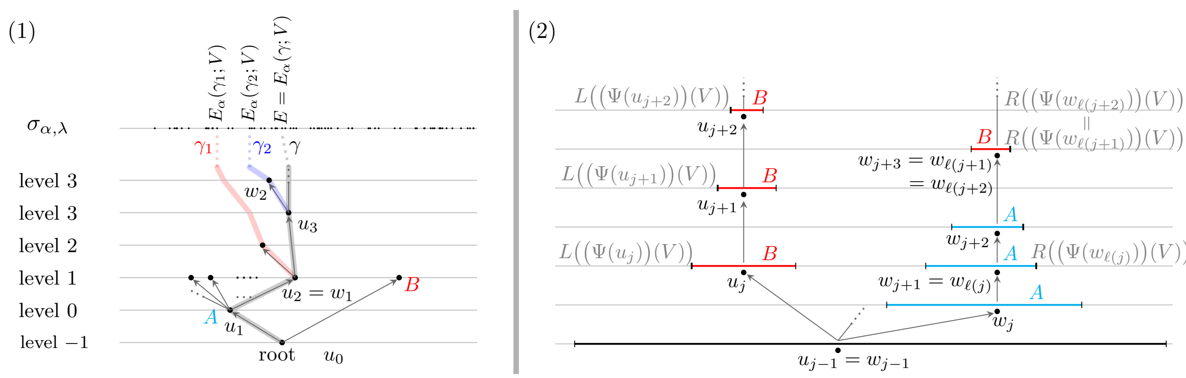

FIGURE 7.4. In (1), the sequence of paths and vertices constructed in Lemma 7.9 are outlined. In (2), the paths γ_1 and γ_2 and their associated spectral bands in the proof of Lemma 7.10 are sketched.

Lemma 7.10. *[also Theorem 1.9 (b)] Let $\alpha \in [0,1] \setminus \mathbb{Q}$. If $\gamma_1, \gamma_2 \in \partial\mathcal{T}_\alpha$ satisfy $\gamma_1 \preceq \gamma_2$, then $E_\alpha(\gamma_1; V) \leq E_\alpha(\gamma_2; V)$ for all $V > 0$.*

Proof. Let $V > 0$ and $\gamma_1 = (u_0, u_1, \dots), \gamma_2 = (w_0, w_1, \dots) \in \partial\mathcal{T}_\alpha$ be such that $\gamma_1 \preceq \gamma_2$. If $\gamma_1 = \gamma_2$ then $E_\alpha(\gamma_1; V) = E_\alpha(\gamma_2; V)$ by definition, and we may proceed assuming $\gamma_1 \neq \gamma_2$. Let $k \in \mathbb{N}_0$ be such that $u_i \prec w_j$ for all $i, j \geq k$. It is worth pointing out that neither the vertex u_j (respectively w_j) is necessarily in level j nor both u_j, w_j are in the same level.

However, since two vertices connected by an edge differ at most by two levels, we conclude that we can choose a map $\ell : \mathbb{N} \to \mathbb{N}$ with $\lim_{j\to\infty} \ell(j) = \infty$ such that u_j and $w_{\ell(j)}$ are at most one level apart (so, they are either in the same level or in consecutive levels). Such a map is in general not unique - an example is depicted in Figure 7.4 (2). For k as above, there is a $j_0 \in \mathbb{N}$ such that for $j \geq j_0$, we have $j \geq k$ and $\ell(j) \geq k$. Thus, $u_j \prec w_{\ell(j)}$ holds for $j \geq j_0$. Then Proposition 7.1 (c) implies $(\Psi(u_j))(V) \prec \left(\Psi(w_{\ell(j)})\right)(V)$ for $j \geq j_0$ using that u_j and $w_{\ell(j)}$ are at most one level apart. Hence, $L\left((\Psi(u_j))(V)\right) < R\left(\left(\Psi(w_{\ell(j)})\right)(V)\right)$ for $j \geq j_0$ follows from Definition 2.6. By construction of E_α, we have

$$E_\alpha(\gamma_1; V) = \lim_{j\to\infty} L\left((\Psi(u_j))(V)\right) \quad \text{and} \quad E_\alpha(\gamma_2; V) = \lim_{j\to\infty} R\left(\left(\Psi(w_{\ell(j)})\right)(V)\right).$$

Thus, $E_\alpha(\gamma_1; V) \leq E_\alpha(\gamma_2; V)$ is concluded. □

We note that the statement of Lemma 7.10 can be strengthened: If $\gamma_1 \preceq \gamma_2$ and $\gamma_1 \neq \gamma_2$, then $E_\alpha(\gamma_1; V) < E_\alpha(\gamma_2; V)$ (not equal!) follows for all $V > 0$. This is an immediate consequence of the injectivity of $E_\alpha(\,\cdot\,; V) : \partial\mathcal{T}_\alpha \to \sigma(H_{\alpha,V})$, which is proven in Lemma 7.16 (later within this subsection). Towards proving this injectivity, we introduce the notion of a spectral band's central point (which we call Zentrum) and develop a couple of auxiliary lemmas.

We recall the notion of the trace functions (Section 2.2), $t_{\mathbf{c}}(E, V)$. In what follows, whenever $E : (0, \infty) \to \mathbb{R}$ is taken to be a V dependent map, we abbreviate notation and use also $t_{\mathbf{c}}(E(V))$ or even $t_{\mathbf{c}}(E)$. Adopting this notation, for any $\mathbf{c} \in \mathcal{C}$ and $I_{\mathbf{c}}$ a spectral band of $\sigma_{\mathbf{c}}$, we have that $|t_{\mathbf{c}}(L(I_{\mathbf{c}}))| = |t_{\mathbf{c}}(R(I_{\mathbf{c}}))| = 2$, $t_{\mathbf{c}}(L(I_{\mathbf{c}})) = -t_{\mathbf{c}}(R(I_{\mathbf{c}}))$ and that $t_{\mathbf{c}}|_{I_{\mathbf{c}}} : I_{\mathbf{c}} \to [-2, 2]$ is strictly monotone and continuous (see Definition 2.4 and Section 3.2). Hence, $t_{\mathbf{c}}$ vanishes exactly once on $I_{\mathbf{c}}$. Explicitly, for each $V \in (0, \infty)$, there is a unique $Z(I_{\mathbf{c}}(V))$ such that $t_{\mathbf{c}}\left(Z\left(I_{\mathbf{c}}(V)\right), V\right) = 0$. We call $Z(I_{\mathbf{c}})$, viewed as a map $(0, \infty) \ni V \mapsto Z(I_{\mathbf{c}}(V))$, the *Zentrum* of the spectral band $I_{\mathbf{c}}$, even though it is only the center in terms of the image of $t_{\mathbf{c}}$, and not necessarily equal to $\frac{1}{2}(R(I_{\mathbf{c}}) - L(I_{\mathbf{c}}))$. Note that $(0, \infty) \ni V \mapsto Z(I_{\mathbf{c}}(V))$ is continuous by construction.

Reading the following lemma may be aided by looking at Figure 7.5.

Lemma 7.11 (Monotonicity of $t_{[\mathbf{c},m,n]}$ in n)**.** *Let $\mathbf{c} \in \mathcal{C}$ and $m \in \mathbb{N}$ be such that $[\mathbf{c}, m] \in \mathcal{C}$. Let $I_{\mathbf{c}}$ be a spectral band of type B with the unique associated spectral band $I^1_{[\mathbf{c},m,1]}$ of type B as introduced in Definition 3.3. Furthermore, let $J_{[\mathbf{c},m]}$ be the associated spectral band with $I_{\mathbf{c}}$ as defined in Definition 4.8. If $E(V) := R(J_{[\mathbf{c},m]}(V)) \in I_{\mathbf{c}}(V)$ for $V > 0$, then the following statements hold.*

(a) We have $E(V) < Z(I_{\mathbf{c}}(V))$ and $\operatorname{sign}\left(t_{\mathbf{c}}(E(V))\right) = \operatorname{sign}\left(t_{\mathbf{c}}\left(L(I_{\mathbf{c}}(V))\right)\right)$.

(b) For all $n \in \mathbb{N}$, we have $E(V) < Z\left(I^1_{[\mathbf{c},m,n]}(V)\right)$ and

$$\operatorname{sign}\left(t_{[\mathbf{c},m,n]}(E(V))\right) = \operatorname{sign}\left(t_{[\mathbf{c},m,n]}\left(L(I^1_{[\mathbf{c},m,n]}(V))\right)\right).$$

(c) If $E(V) \in I^1_{[\mathbf{c},m,1]}(V)$, then for all $n \in \mathbb{N}$,

$$\left|t_{[\mathbf{c},m,n]}(E(V))\right| > \left|t_{[\mathbf{c},m,n-1]}(E(V))\right| > \ldots > \left|t_{[\mathbf{c},m,1]}(E(V))\right| > 0. \tag{7.10}$$

(d) If $E(V) \in I^1_{[\mathbf{c},m,1]}(V)$, then for all $n \in \mathbb{N}$,

$$\operatorname{sign}\left(t_{[\mathbf{c},m]}(E(V)) \cdot t_{[\mathbf{c},m,n-1]}(E(V)) \cdot t_{[\mathbf{c},m,n]}(E(V))\right) = +1.$$

Proof. Let $V > 0$ be such that $E(V) \in I_{\mathbf{c}}(V)$. For brevity, we remove the V dependence from the notations unless we want to emphasize its dependence. For convenience of the reader, a sketch of the involved spectral bands is provided in Figure 7.5. The order in which we prove the sections of the lemma is (a), (c), (b) and (d).

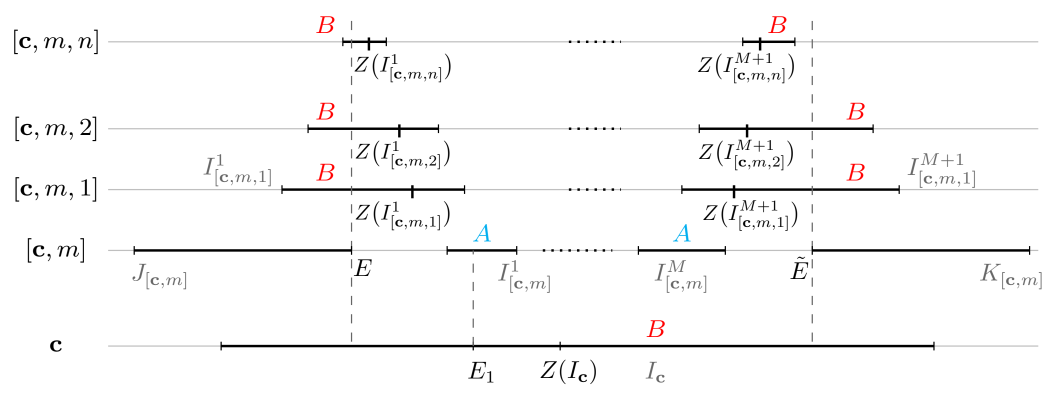

FIGURE 7.5. A sketch of the situation in Lemma 7.11.

(a) Since $I_{\mathbf{c}}$ is of type B, it is proven in [Ray95a, Proof of Lemma 3.3] (see also [BBB$^+$, Corollary 4.15]), that there exists a unique value $E_1 \in I_{\mathbf{c}}$ such that

$$t_{\mathbf{c}}(E_1) := 2\cos\left(\frac{\pi}{m+1}\right) \cdot \operatorname{sign}(t_{\mathbf{c}}(L(I_{\mathbf{c}}))) \tag{7.11}$$

and $E_1 \in I^1_{[\mathbf{c},m]}$, where $I^1_{[\mathbf{c},m]}$ is the left-most spectral band of type A which is contained in $I_{\mathbf{c}}$, see Definition 3.3. Recalling that $I^1_{[\mathbf{c},m]} \subseteq_{\text{str}} I_{\mathbf{c}}$ and $J_{[\mathbf{c},m]} \prec I_{\mathbf{c}}$ we get $J_{[\mathbf{c},m]} \prec_{\text{str}} I^1_{[\mathbf{c},m]}$ (the relation must be strict since both spectral bands are in $\sigma_{[\mathbf{c},m]}$). Note that we used the fact that $I_{\mathbf{c}}$ is of type B. If $I_{\mathbf{c}}$ is of type A and $m = 1$, then $I_{\mathbf{c}}$ would not contain such a spectral band $I^1_{[\mathbf{c},m]}$ and the argument fails, consult Example 7.13.

Thus, $J_{[\mathbf{c},m]} \prec_{\text{str}} I^1_{[\mathbf{c},m]}$ implies $E < E_1$. Recalling that $t_{\mathbf{c}}|_{I_{\mathbf{c}}}$ is strictly monotone and that $t_{\mathbf{c}}(Z(I_{\mathbf{c}})) = 0$, we get from (7.11) that if $m = 1$, then $E_1 = Z(I_{\mathbf{c}})$ and if $m > 1$, then $E_1 < Z(I_{\mathbf{c}})$. In either of these cases, we conclude $E < Z(I_{\mathbf{c}})$. This estimate immediately implies $\operatorname{sign}(t_{\mathbf{c}}(E(V))) = \operatorname{sign}(t_{\mathbf{c}}(L(I_{\mathbf{c}}(V))))$ since $t_{\mathbf{c}}|_{I_{\mathbf{c}}}$ is strictly monotone and continuous and $E(V) \in I_{\mathbf{c}}(V)$.

(c) The statement is proven by induction over $n \in \mathbb{N}$ simultaneously for all $V > 0$ satisfying $E(V) \in I^1_{[\mathbf{c},m,1]}(V)$. By (a), we have $|t_{\mathbf{c}}(E(V)))| > 0$. Thus, it suffices for the induction base to show $\left|t_{[\mathbf{c},m,1]}(E(V)))\right| > |t_{\mathbf{c}}(E(V)))|$. In order to simplify notation, define

$$x(V) := t_{\mathbf{c}}(E(V)), \quad y(V) := t_{[\mathbf{c},m]}(E(V)) \quad \text{and} \quad z(V) := t_{[\mathbf{c},m,1]}(E(V)).$$

By [Ray95a, prop. 3.1 (iii)] (see also [BBB$^+$, prop. 4.7], we have $\sigma_{\mathbf{c}}(V) \cap \sigma_{[\mathbf{c},m]}(V) \cap \sigma_{[\mathbf{c},m,-1]}(V) = \emptyset$ for $V > 4$ implying $J_{[\mathbf{c},m]}(V) \prec_{\text{str}} I_{\mathbf{c}}(V)$ for $V > 4$. Using the continuity of the spectral band edges (Corollary 3.2), we get that there is a $V_1 > 0$ such that $E(V_1) = L(I_{\mathbf{c}}(V_1))$ and for $V > V_1$, we have $E(V) < L(I_{\mathbf{c}}(V))$ (and in particular $E(V) \notin I^1_{[\mathbf{c},m,1]}(V)$). Since all spectral bands are either of backward type A or B (strict inclusions), we conclude $E(V_1) \notin \sigma_{[\mathbf{c},m,1]}(V_1)$ implying

$$|z(V_1)| = \left|t_{[\mathbf{c},m,1]}(E(V_1)))\right| > 2 = |t_{\mathbf{c}}(E(V_1)))| = |x(V_1)|. \tag{7.12}$$

Recall that we aim to prove $|z(V)| > |x(V)|$ if $E(V) \in I^1_{[\mathbf{c},m,1]}(V)$. Therefore, we assume by contradiction that $|z(V')| \leq |x(V')|$ for some $V' > 0$ with $E(V') \in I^1_{[\mathbf{c},m,1]}(V')$. We have seen above that for $V > V_1$, we have $E(V) \notin I^1_{[\mathbf{c},m,1]}(V)$ and so $V' \leq V_1$. By continuity of these maps in V and by (7.12), there is a $V_0 \geq V' > 0$ satisfying

$$E(V_0) \in I^1_{[\mathbf{c},m,1]}(V_0), \quad |z(V_0)| = |x(V_0)| \quad \text{and} \quad |z(V)| > |x(V)| \quad \text{for } V > V_0. \tag{7.13}$$

We will show that $V_0 = 0$ contradicting $V_0 > 0$. Since $E(V_0) \in I^1_{[\mathbf{c},m,1]}(V_0) \subseteq_{\text{str}} I_{\mathbf{c}}(V_0)$, there is an $\varepsilon > 0$ such that $E(V) \in I_{\mathbf{c}}(V)$ for $V_0 < V < V_0 + \varepsilon$. Since in this case $E(V) < Z(I_{\mathbf{c}}(V))$ holds by (a), we conclude $|x(V)| > 0$ for $V_0 < V < V_0 + \varepsilon$. By the choice of V_0, we have $|z(V)| > |x(V)| > 0$ for $V_0 < V < V_0 + \varepsilon$ implying

$$\operatorname{sign}(z(V)) = \operatorname{sign}\left(t_{[\mathbf{c},m,1]}\left(L(I^1_{[\mathbf{c},m,1]}(V))\right)\right).$$

Hence, (a) together with Lemma 4.12 imply $\operatorname{sign}(x(V)y(V)z(V)) = +1$. Moreover, $|y(V)| = 2$ follows from Lemma 4.10, since $E(V) = R\left(J_{[\mathbf{c},m]}(V)\right)$. Combining these observation with the Fricke–Vogt invariant (see Proposition II.2 (b)), we conclude

$$\begin{aligned} 4 + V^2 = x(V)^2 + y(V)^2 + z(V)^2 - x(V)y(V)z(V) &= x(V)^2 + 4 + z(V)^2 - 2|x(V)z(V)| \\ &= 4 + (|x(V)| - |z(V)|)^2, \end{aligned}$$

for $V_0 < V < V_0 + \varepsilon$. If $V \searrow V_0$, then (7.13) leads to

$$|V_0|^2 = \lim_{V \searrow V_0} |V|^2 = \lim_{V \searrow V_0} (|x(V)| - |z(V)|)^2 = 0,$$

contradicting $V_0 > 0$. This proves the induction base.

For the induction step, suppose that $E \in I^1_{[\mathbf{c},m,1]}$ and $\left|t_{[\mathbf{c},m,n]}(E)\right| > \left|t_{[\mathbf{c},m,n-1]}(E)\right|$ holds for some $n \geq 1$. Using the recursive trace relation in Proposition II.2, we conclude

$$\begin{aligned}\left|t_{[\mathbf{c},m,n+1]}(E)\right| &= \left|t_{[\mathbf{c},m]}(E)t_{[\mathbf{c},m,n]}(E) - t_{[\mathbf{c},m,n-1]}(E)\right| \\ &\geq \left|t_{[\mathbf{c},m]}(E)\right| \left|t_{[\mathbf{c},m,n]}(E)\right| - \left|t_{[\mathbf{c},m,n-1]}(E)\right| \\ &= 2\left|t_{[\mathbf{c},m,n]}(E)\right| - \left|t_{[\mathbf{c},m,n-1]}(E)\right| > \left|t_{[\mathbf{c},m,n]}(E)\right|,\end{aligned}$$

where we used that $\left|t_{[\mathbf{c},m]}(E)\right| = 2$ (since $E = R(J_{[\mathbf{c},m]})$) and the induction assumption.

(b) Like in (a), it suffices to prove $E(V) < Z\left(I^1_{[\mathbf{c},m,n]}(V)\right)$. The statement of the signs of the traces follows then directly. By [Ray95a, prop. 3.1 (iii)] (see also [BBB+, prop. 4.7], we have $\sigma_{\mathbf{c}}(V) \cap \sigma_{[\mathbf{c},m]}(V) \cap \sigma_{[\mathbf{c},m,1]}(V) = \emptyset$ for $V > 4$ implying $J_{[\mathbf{c},m]}(V) \prec_{\text{str}} I^1_{[\mathbf{c},m,1]}(V)$ for $V > 4$. Since $I^1_{[\mathbf{c},m,n+1]} \subseteq_{\text{str}} I^1_{[\mathbf{c},m,n]}$ (tower property), we then conclude

$$E(V) < L\left(I^1_{[\mathbf{c},m,1]}(V)\right) < L\left(I^1_{[\mathbf{c},m,n]}(V)\right) < Z\left(I^1_{[\mathbf{c},m,n]}(V)\right), \qquad V > 4.$$

If $E(V) \in I^1_{[\mathbf{c},m,n]}(V)$, then $E(V) \in I^1_{[\mathbf{c},m,1]}(V)$ follows as $I^1_{[\mathbf{c},m,n]} \subseteq_{\text{str}} I^1_{[\mathbf{c},m,1]}$ holds by the tower property (B1). By (c), we conclude $t_{[\mathbf{c},m,n]}(E(V)) > 0$ whenever $E(V) \in I^1_{[\mathbf{c},m,1]}(V)$. Thus, $E(V) < Z\left(I^1_{[\mathbf{c},m,n]}(V)\right)$ follows.

(d) The case $n = 1$ follows from (a), (b) and Lemma 4.12. The case $n > 1$ follows from (b) and Lemma 4.13. □

Remark 7.12. We point out that statement in Lemma 7.11 (a) fails if $I_{\mathbf{c}}$ is not of type B and henceforth also the consecutive statements are not necessarily true anymore, see Example 7.13 and the short explanation in the proof. However, the statement can be extended to the case that $I_{\mathbf{c}}$ is of type A and $m > 1$.

Example 7.13. Here we show that a spectral band edge may pass the Zentrum of an adjacent spectral band one level below, if this band is not of type B. Let $\mathbf{c} = [0, 0]$ and $I_{\mathbf{c}}(V) = [-2, 2]$. Then $I_{\mathbf{c}}(V)$ is of type A (Lemma 5.3) and $K_{[\mathbf{c},m]}(V) = [-2 + V, 2 + V]$. Since $t_{\mathbf{c}}(E, V) = E$ (Example 2.3), the Zentrum satisfies $Z(I_{\mathbf{c}}(V)) = 0$. If $\tilde{E}(V) := L(K_{[\mathbf{c},m]}(V)) = -2 + V$, then $\tilde{E}(V) > Z(I_{\mathbf{c}}(V))$ if $V > 2$ and $\tilde{E}(V) < Z(I_{\mathbf{c}}(V))$ if $V < 2$. In particular, $\text{sign}(R(I_{\mathbf{c}}(V)) \neq \text{sign}(\tilde{E}(V))$ whenever $V < 2$.

Lemma 7.14 (Estimating $t_{[\mathbf{c},m,n]}$)**.** *Let $\mathbf{c} \in \mathcal{C}$ and $m \in \mathbb{N}$ be such that $[\mathbf{c}, m] \in \mathcal{C}$. Let $I_{\mathbf{c}}$ be a spectral band of type B with the unique associated spectral band $I^1_{[\mathbf{c},m,1]}$ of type B as introduced in Definition 3.3. Furthermore, let $J_{[\mathbf{c},m]}$ be the associated spectral band with $I_{\mathbf{c}}$ as defined in Definition 4.8. If $E(V) := R(J_{[\mathbf{c},m]}(V)) \in I^1_{[\mathbf{c},m,1]}(V)$ for $V > 0$, then for all $n \in \mathbb{N}$,*

$$\left|t_{[\mathbf{c},m,n]}(E(V))\right| - \left|t_{\mathbf{c}}(E(V))\right| = n \cdot V.$$

Proof. Combining the Fricke–Vogt invariant (Proposition II.2 (b)) with the fact that the product of traces is positive (Lemma 7.11 (d)) and $\left|t_{[\mathbf{c},m]}(E(V))\right| = 2$, we conclude for $j \in \mathbb{N}$,

$$\begin{aligned}4 + V^2 &= \left(t_{[\mathbf{c},m]}(E(V))\right)^2 + \left(t_{[\mathbf{c},m,j-1]}(E(V))\right)^2 + \left(t_{[\mathbf{c},m,j]}(E(V))\right)^2 \\ &\qquad - t_{[\mathbf{c},m]}(E(V)) \cdot t_{[\mathbf{c},m,j-1]}(E(V)) \cdot t_{[\mathbf{c},m,j]}(E(V)) \\ &= \left(t_{[\mathbf{c},m]}(E(V))\right)^2 + \left(t_{[\mathbf{c},m,j-1]}(E(V))\right)^2 + \left(t_{[\mathbf{c},m,j]}(E(V))\right)^2 \\ &\qquad - \left|t_{[\mathbf{c},m]}(E(V)) \cdot t_{[\mathbf{c},m,j-1]}(E(V)) \cdot t_{[\mathbf{c},m,j]}(E(V))\right| \\ &= 4 + \left(t_{[\mathbf{c},m,j-1]}(E(V))\right)^2 + \left(t_{[\mathbf{c},m,j]}(E(V))\right)^2 - 2 \cdot \left|t_{[\mathbf{c},m,j-1]}(E(V)) \cdot t_{[\mathbf{c},m,j]}(E(V))\right| \\ &= 4 + \left(\left|t_{[\mathbf{c},m,j]}(E(V))\right| - \left|t_{[\mathbf{c},m,j-1]}(E(V))\right|\right)^2.\end{aligned}$$

Hence, Lemma 7.11 (c) implies

$$\left|t_{[\mathbf{c},m,j]}(E(V))\right| - \left|t_{[\mathbf{c},m,j-1]}(E(V))\right| = V.$$

Summing the above for j ranging from 1 to n, then a telescoping sum argument and $t_{[\mathbf{c},m,0]} = t_{\mathbf{c}}$ (Proposition II.2) lead to the desired claim. □

Remark 7.15. One can prove the symmetric cases of Lemma 7.11 and Lemma 7.14: Let $K_{[\mathbf{c},m]}$ be the spectral band associated with $I_{\mathbf{c}}$ defined in Definition 4.8 and $I^{M+1}_{[\mathbf{c},m,1]}$ be the associated spectral band defined in Definition 3.3. Set $\tilde{E}(V) = L(K_{[\mathbf{c},m]}(V))$, see also Figure 7.5. Similarly as in Lemma 7.11, one can prove that for all $n \in \mathbb{N}$, that if $\tilde{E}(V) \in I^{M+1}_{[\mathbf{c},m,1]}(V)$, then

$$\operatorname{sign}\left(t_{[\mathbf{c},m]}(\tilde{E}(V)) \cdot t_{[\mathbf{c},m,n-1]}(\tilde{E}(V)) \cdot t_{[\mathbf{c},m,n]}(\tilde{E}(V))\right) = +1$$

and

$$\left|t_{[\mathbf{c},m,n]}(\tilde{E}(V)))\right| > \left|t_{[\mathbf{c},m,n-1]}(\tilde{E}(V)))\right| > \ldots > \left|t_{[\mathbf{c},m,1]}(\tilde{E}(V)))\right| > 0.$$

With this, one can similarly repeat the proof of Lemma 7.14 and get for $n \in \mathbb{N}$,

$$\left|t_{[\mathbf{c},m,n]}(\tilde{E}(V))\right| - \left|t_{\mathbf{c}}(\tilde{E}(V))\right| = n \cdot V,$$

if $\tilde{E}(V) = L(K_{[\mathbf{c},m]}(V)) \in I^{M+1}_{[\mathbf{c},m,1]}(V)$ where $I_{\mathbf{c}}$ is of type B and $K_{[\mathbf{c},m]}$ is the associated spectral band of $I_{\mathbf{c}}$ introduced in Definition 4.8.

We finally prove the injectivity of the map E_α.

Lemma 7.16. *[the injectivity part of Theorem 1.9 (a)] Let $\alpha \in [0,1] \setminus \mathbb{Q}$ and $V > 0$. Then the map $E_\alpha(\,\cdot\,;V) : \partial\mathcal{T}_\alpha \to \sigma(H_{\alpha,V})$ is injective.*

Proof. Let $\gamma_1 = (u_0, u_1, \ldots), \gamma_2 = (w_0, w_1, \ldots) \in \partial\mathcal{T}_\alpha$ be different. We have to show that $E_\alpha(\gamma_1\,; V) \neq E_\alpha(\gamma_2\,; V)$. Since $\gamma_1 \neq \gamma_2$, there is no loss of generality in assuming that $\gamma_1 \preceq \gamma_2$ and $\gamma_1 \neq \gamma_2$. Note that $u_0 = w_0$ is the root of $\mathcal{T}_\alpha$. Since $\gamma_1 \preceq \gamma_2$, there is a $k_0 \in \mathbb{N}_0$ such that $u_j = w_j$ for $1 \leq j \leq k_0$ and $u_j \prec w_j$ for $j > k_0$.

We now describe two auxiliary paths γ_L and γ_R (in step 1) such that $\gamma_1 \preceq \gamma_L \preceq \gamma_R \preceq \gamma_2$ and $\gamma_L \neq \gamma_R$. Hence, $E_\alpha(\gamma_1; V) \leq E_\alpha(\gamma_L; V) \leq E_\alpha(\gamma_R; V) \leq E_\alpha(\gamma_2; V)$ follows from Lemma 7.10. In Step 2 we show that in fact $E_\alpha(\gamma_L; V) \neq E_\alpha(\gamma_R; V)$ finishing the proof.

Step 1: The auxiliary paths γ_L and γ_R are recursively constructed, see a sketch of the main ideas in Figure 7.6. Since $u_{k_0} = w_{k_0}$ and $u_{k_0+1} \prec w_{k_0+1}$, there is a unique vertex $w^R_{k_0+1}$ satisfying

(1) $u_{k_0+1} \prec w^R_{k_0+1}$ and there is an edge $u_{k_0} \to w^R_{k_0+1}$,
(2) every vertex w satisfying (1) fulfills either $w = w^R_{k_0+1}$ or $w^R_{k_0+1} \prec w$.

Note that such a vertex exists since w_{k_0+1} satisfies (1). Define $u^L_{k_0+1} := u_{k_0+1}$. By construction and the interlacing property (Figure 1.1 (2)), we have

$$u^L_{k_0+1} \prec w^R_{k_0+1} \quad \text{and the vertices have different labels.} \tag{7.14}$$

Continue defining γ_L as follows. For $j \in \mathbb{N}$, choose $u^L_{k_0+1+j}$ to be the unique rightmost vertex such that there is an edge $u^L_{k_0+1+j-1} \to u^L_{k_0+1+j}$, i.e. for any other vertex u, for which there is an edge $u^L_{k_0+1+j-1} \to u$, we have $u \prec u^L_{k_0+1+j}$. Then, γ_L is defined by the path $(u_0, \ldots, u_{k_0}, u^L_{k_0+1}, u^L_{k_0+2}, \ldots)$.

Similarly, we define γ_R but instead of choosing the rightmost vertex, the leftmost is chosen. For $j \in \mathbb{N}$, $w^R_{k_0+1+j}$ is the unique leftmost vertex such that there is an edge $w^R_{k_0+1+j-1} \to w^R_{k_0+1+j}$,

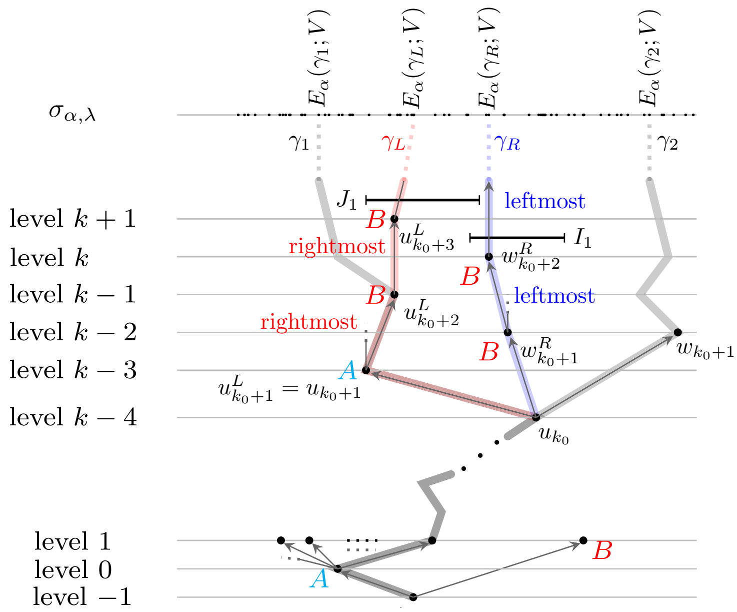

FIGURE 7.6. A sketch of the construction of the paths γ_L and γ_R.

i.e. for any other vertex w, which admits an edge $w^R_{k_0+1+j-1} \to w$, we have $w^R_{k_0+1+j} \prec w$. Then, γ_R is defined by the path $(w_0, \ldots, w_{k_0}, w^R_{k_0+1}, w^R_{k_0+2}, \ldots)$.

By construction, we get (as justified below) that for all $j \in \mathbb{N}$,

(a) the vertices $u^L_{k_0+j+1}$ and $w^R_{k_0+j+1}$ are of type B,
(b) the vertices $u^L_{k_0+j+1}$ and $w^R_{k_0+j+1}$ are in different but consecutive levels,
(c) $u^L_{k_0+j+2} \prec w^R_{k_0+j+1}$ and there exists no other vertex u with $u^L_{k_0+j+2} \prec u \prec w^R_{k_0+j+1}$.

Statement (a) follows as the leftmost (resp. rightmost) vertex u connected to some vertex w (except the root) is always labeled B by definition of the branching, see Figure 1.1 (2). By Equation (7.14), the vertices $u^L_{k_0+1}$ and $w^R_{k_0+1}$ are in consecutive levels. By (a), $u^L_{k_0+j+1}$ is two levels higher than $u^L_{k_0+j}$ and $w^R_{k_0+j+1}$ is two levels higher than $w^R_{k_0+j}$. Thus, (b) follows inductively from (a). Finally, (c) follows from construction and the definition of the order.

From (7.14) we get $\gamma_L \preceq \gamma_R$ and $\gamma_L \neq \gamma_R$. Since we choose for γ_L the rightmost vertices (and for γ_R the leftmost vertices) in the construction, we have $\gamma_1 \preceq \gamma_L$ and $\gamma_R \preceq \gamma_2$ (note that they can be equal). Thus, Lemma 7.10 implies

$$E_\alpha(\gamma_1;V) \leq E_\alpha(\gamma_L;V) \leq E_\alpha(\gamma_R;V) \leq E_\alpha(\gamma_2;V).$$

Step 2: Let $V > 0$. We show $E_\alpha(\gamma_L;V) \neq E_\alpha(\gamma_R;V)$. By definition of the map E_α, it suffices to prove

$$\{E_\alpha(\gamma_L;V)\} = \bigcap_{j\in\mathbb{N}_0} \left(\Psi(u^L_{k_0+1+j})\right)(V) \neq \bigcap_{j\in\mathbb{N}_0} \left(\Psi(w^R_{k_0+1+j})\right)(V) = \{E_\alpha(\gamma_R;V)\}\,. \tag{7.15}$$

Let $(0, c_1, c_2, \ldots)$ be the continuous fraction expansion of α. Let $\ell \in \mathbb{N}$ be the level of the vertex $w^R_{k_0+2}$, i.e. $\Psi\left(w^R_{k_0+2}\right)$ is a spectral band of $\sigma_{[0,0,c_1,\ldots,c_l]}$. We will prove that if for $m \in \mathbb{N}$,

$$\left(\bigcap_{j=1}^{m} \left(\Psi(u^L_{k_0+1+j})\right)(V)\right) \bigcap \left(\bigcap_{j=1}^{m+1} \left(\Psi(w^R_{k_0+1+j})\right)(V)\right) \neq \emptyset \tag{7.16}$$

then

$$m \leq \sum_{j=1}^{m} c_{\ell+2j} < \frac{2}{V}. \tag{7.17}$$

Note that the first inequality is trivially true since $c_i \geq 1$ for all $i \in \mathbb{N}$. After proving this statement, if we assume by contradiction that (7.15) does not hold, it means that (7.16) holds for arbitrarily large m, and hence that $0 < V < \frac{2}{m}$ for all $m \in \mathbb{N}$, which brings to a contradiction and yields $E_\alpha(\gamma_L; V) \neq E_\alpha(\gamma_R; V)$. So, it is left to prove that (7.16) implies the upper bound in (7.17).

Suppose (7.16) holds for $m \in \mathbb{N}$. In order to simplify the notation, define for $j \in \mathbb{N}_0$,

$$J_j(V) := \left(\Psi(u^L_{k_0+2+j})\right)(V) \quad \text{and} \quad I_j(V) := \left(\Psi(w^R_{k_0+1+j})\right)(V).$$

Then (a), (b) and (c) in step 1 inductively imply (using Proposition 7.1) that for $j \in \mathbb{N}$,

- J_j is a spectral band of $\sigma_{[0,0,c_1,\ldots,c_{\ell+2j-1}]}$ and $J_{j+1} \subsetneq_{\text{str}} J_j$,
- I_j is a spectral band of $\sigma_{[0,0,c_1,\ldots,c_{\ell+2j-2}]}$ and $I_{j+1} \subsetneq_{\text{str}} I_j$,
- J_j is the rightmost band $\sigma_{[0,0,c_1,\ldots,c_{\ell+2j-1}]}$ satisfying $J_j \prec I_j$, namely, set $\widetilde{\mathbf{c}} := [0, 0, c_1, \ldots, c_{\ell+2j-2}]$, $\widetilde{m} = c_{\ell+2j-1}$ and $I_{\mathbf{c}} := I_j$, then $J_j = J_{[\widetilde{\mathbf{c}},\widetilde{m}]}$ using the notation of Definition 4.8,
- I_{j+1} is the leftmost spectral band of type B strictly contained in I_j two levels above, i.e. set $\widetilde{\mathbf{c}} := [0, 0, c_1, \ldots, c_{\ell+2j-2}]$, $\widetilde{m} = c_{\ell+2j-1}$ and $\widetilde{n} = c_{\ell+2j}$, then $I_{j+1} = I^1_{[\widetilde{\mathbf{c}},\widetilde{m},\widetilde{n}]}$ using the notation of Definition 3.3.

These properties are sketched in Figure 7.7.

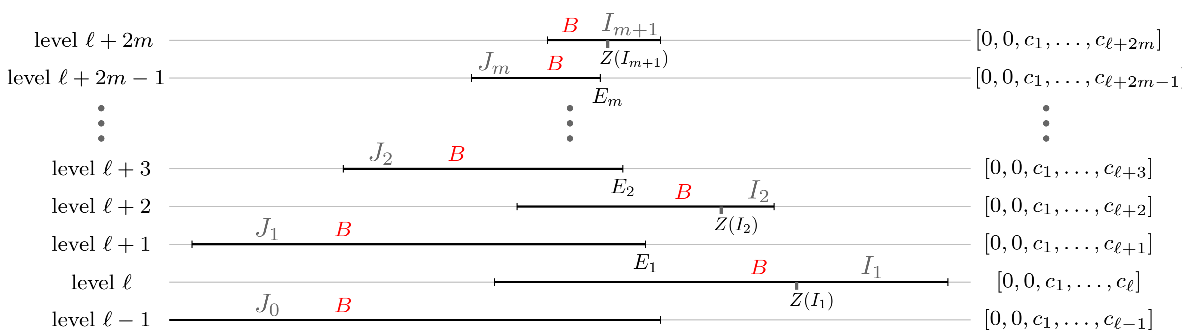

FIGURE 7.7. A sketch of the spectral bands $I_1, \ldots, I_{m+1}$ and $J_0, \ldots, J_m$ introduced in the proof of Lemma 7.16.

Set $E_j := R(J_j)$ to be the right spectral band edge of J_j. If (7.16) holds for $m \in \mathbb{N}$, then $E_j \in I_{j+1}$ for all $1 \leq j \leq m$. Thus, Lemma 7.14 leads to

$$c_{\ell+2j} \cdot V = \left| t_{[0,0,c_1,\ldots,c_{\ell+2j}]}(E_j(V)) \right| - \left| t_{[0,0,c_1,\ldots,c_{\ell+2j-2}]}(E_j(V)) \right|, \qquad 1 \leq j \leq m. \tag{7.18}$$

Moreover, $J_{j+1} \subsetneq_{\text{str}} J_j$ implies $E_{j+1} < E_j$. In addition, Lemma 7.11 (a) implies $E_{j+1} < Z(I_{j+1})$ and Lemma 7.11 (b) yields $E_j < Z(I_{j+1})$. Recall $Z(I_j)$ is the Zentrum of the interval I_j. Note that we can apply Lemma 7.11 by the previous considerations and because (7.16) for $m \in \mathbb{N}$ implies $E_j \in I_{j+1}$. Since the trace $t_{[0,0,c_1,\ldots,c_{\ell+2j}]}$ is strictly monotone on the spectral band I_{j+1}, Lemma 4.10 (a) the estimates $E_{j+1} < Z(I_{j+1})$, $E_j < Z(I_{j+1})$ and $E_{j+1} < E_j$ yield

$$\left| t_{[0,0,c_1,\ldots,c_{\ell+2j}]}(E_{j+1}(V)) \right| > \left| t_{[0,0,c_1,\ldots,c_{\ell+2j}]}(E_j(V)) \right| \quad 1 \leq j \leq m-1. \tag{7.19}$$

Summing Equation (7.18) for all $1 \leq j \leq m$ and reordering the summands leads to

$$\sum_{j=1}^{m} c_{\ell+2j} \cdot V = \sum_{j=1}^{m} \left| t_{[0,0,c_1,\ldots,c_{\ell+2j}]}(E_j(V)) \right| - \left| t_{[0,0,c_1,\ldots,c_{\ell+2j-2}]}(E_j(V)) \right|$$

$$
\begin{aligned}
&= \left|t_{[0,0,c_1,\ldots,c_{\ell+2m}]}(E_m(V))\right| - \underbrace{\left|t_{[0,0,c_1,\ldots,c_\ell]}(E_1(V))\right|}_{\geq 0} \\
&\quad + \sum_{j=1}^{m-1} \underbrace{\left|t_{[0,0,c_1,\ldots,c_{\ell+2j}]}(E_j(V))\right| - \left|t_{[0,0,c_1,\ldots,c_{\ell+2j}]}(E_{j+1}(V))\right|}_{<0 \text{ by Equation (7.19)}} \\
&< \left|t_{[0.0,c_1,\ldots,c_{\ell+2m}]}(E_m(V))\right|.
\end{aligned}
$$

Since Equation (7.16) holds for $m \in \mathbb{N}$, we conclude $E_m(V) \in I_{m+1}(V) \subseteq \sigma_{[0,0,c_1,\ldots,c_{\ell+2m}]}(V)$ and use it in the inequality above to get

$$
\sum_{j=1}^{m} c_{\ell+2j} \cdot V < \left|t_{[0,0,c_1,\ldots,c_{\ell+2m}]}(E_m(V))\right| \leq 2.
$$

This proves that (7.16) implies (7.17). □

Remark 7.17. We observe that the upper bound in the proof of Lemma 7.16 may be improved, using Remark 7.15. Specifically, it can be shown that if (7.16) holds for $m \in \mathbb{N}$ and $V > 0$, then

$$
2m \leq \sum_{j=1}^{2m} c_{\ell+j} < \frac{2}{V}.
$$

Here m dictates how many spectral bands of type B overlap (at least $4m$) and ℓ is the level of the vertex $w^R_{k_0+2}$. Thus, 4 bands of type B can only overlap if $V < 1$. Note that we only provided here a rough estimate which is enough for our purpose to prove injectivity. However, these estimates may be further refined.

Remark 7.18. Note that if $V > 4$, then proving the injectivity of $E_\alpha(\cdot; V)$ is substantially shorter. Specifically, from [Ray95a, Proposition 3.1 (iii)] (see also [BBB$^+$, Proposition 4.7]) one can deduce that if $V > 4$ and $u, w \in \mathcal{T}_\alpha$ are two vertices which are not connected by a directed path then $\Psi(u)(V) \cap \Psi(w)(V) = \emptyset$. Using this (7.15) in the proof above follows straightforwardly if $V > 4$.

Proof of Theorem 1.9. Theorem 1.9 is merely a combination of Lemma 7.4, Lemma 7.5, Lemma 7.9, Lemma 7.10 and Lemma 7.16. □

7.3. **A different characterization of the types of spectral bands.** Lemma 7.5 (a) together with Proposition 7.1 allow us to prove a characterization of the types of spectral bands for all $V \neq 0$. This justifies Remark 2.8 as well as the definitions used in [BBL23].

Proposition 7.19. *Let* $\boldsymbol{c} = [0, c_0, c_1, c_2, \ldots, c_k] \in \mathcal{C}$ *with* $\varphi(\boldsymbol{c}) \notin \{-1, \infty\}$ *and* $V \neq 0$. *For a spectral band* $I_{\boldsymbol{c}}$ *in* $\sigma_{\boldsymbol{c}}$, *we have the following equivalences*

$$
I_{\boldsymbol{c}}(V) \text{ is of type } A \quad \Leftrightarrow \quad \begin{matrix} I_{\boldsymbol{c}}(V) \text{ is of} \\ \text{backward type } A \end{matrix} \quad \Leftrightarrow \quad I_{\boldsymbol{c}}(V) \subseteq_{\text{str}} \sigma_{[0,c_0,c_1,\ldots,c_{k-1}]}(V)
$$

and

$$
I_{\boldsymbol{c}}(V) \text{ is of type } B \quad \Leftrightarrow \quad \begin{matrix} I_{\boldsymbol{c}}(V) \text{ is of} \\ \text{backward type } B \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} I_{\boldsymbol{c}}(V) \nsubseteq \sigma_{[0,c_0,c_1,\ldots,c_{k-1}]}(V) \text{ and} \\ I_{\boldsymbol{c}}(V) \subseteq_{\text{str}} \sigma_{[0,c_0,c_1,\ldots,c_{k-2}]}(V). \end{matrix}
$$

Proof. First note that we only have to treat $V > 0$ due to Lemma 7.5 (a). If $\varphi(\mathbf{c}) = 0$ (i.e. $\mathbf{c} = [0, 0]$), then $\sigma_{[0,0]}$ consists of exactly one spectral band $I_{[0,0]} = [-2, 2]$ of type A satisfying the first three equivalent statements and not the second one, see Lemma 5.3.

If $\varphi(\mathbf{c}) = 1$ (i.e. $\mathbf{c} = [0, 0, 1]$), then $\sigma_{[0,0,1]}$ consists of exactly one spectral band $I_{[0,0,1]}$ of type B satisfying the second three equivalent statements and not the first one, see Lemma 5.4.

Suppose $\mathbf{c} \in \mathcal{C}$ is such that $\varphi(\mathbf{c}) \in (0,1)$. By Theorem 2.15 a spectral band is either of type A or of type B. First note that $I_{\mathbf{c}}(V)$ is of backward type A if and only if $I_{\mathbf{c}}(V)$ is strictly contained in $\sigma_{[\mathbf{c},0]}(V) = \sigma_{[0,0,c_1,\ldots,c_{k-1}]}(V)$ by Definition 2.7. Moreover, if $I_{\mathbf{c}}(V)$ is of type A (respectively B) it is by Definition 2.12 also of backward type A (respectively B). Conversely, if $I_{\mathbf{c}}$ is of backward type A (respectively B), then it is of type A (respectively B) by Proposition 4.20 and Theorem 2.15 since $\varphi(\mathbf{c}) \in (0,1)$.

If $I_{\mathbf{c}}(V)$ satisfies $I_{\mathbf{c}}(V) \nsubseteq \sigma_{[0,0,c_1,\ldots,c_{k-1}]}(V) = \sigma_{[\mathbf{c},0]}(V)$, it follows by Definition 2.7 that $I_{\mathbf{c}}(V)$ is not of backward type A. Thus, Theorem 2.15 implies that $I_{\mathbf{c}}(V)$ is of type B. If conversely $I_{\mathbf{c}}(V)$ is of type B, it is by definition not of backward type A and so $I_{\mathbf{c}}(V) \nsubseteq \sigma_{[0,0,c_1,\ldots,c_{k-1}]}(V)$ follows. It is left to prove $I_{\mathbf{c}}(V) \subseteq_{\text{str}} \sigma_{[0,0,c_1,\ldots,c_{k-2}]}(V)$ if $I_{\mathbf{c}}$ is of type B. Therefore, let $\alpha \in [0,1] \setminus \mathbb{Q}$ be such that its first $k \geq 1$ digits in the continued fraction expansion coincide with $c_1, c_2, \ldots, c_k$. Proposition 7.1 implies that there is a vertex u in the spectral α-tree $\mathcal{T}_\alpha$ with label B (as $I_{\mathbf{c}}(V)$ is of type B) in level k in the spectral α-tree $\mathcal{T}_\alpha$ such that $\Psi(u)(V) = I_{\mathbf{c}}(V)$. By construction of $\mathcal{T}_\alpha$ (Section 1.3), there is a unique vertex w in level $k-2$ such that $w \to u$. Applying again Proposition 7.1 (b), we conclude that $I_{\mathbf{c}}(V) = \Psi(u)(V) \subseteq_{\text{str}} \Psi(w)(V)$ where $\Psi(w)(V)$ is a spectral band in $\sigma_{[0,0,c_1,\ldots,c_{k-2}]}(V)$. □

Remark. The previous statement asserts that the type of a spectral band determines whether it is contained one or two levels below as it was used in [BBL23]. We note however that a spectral band of type A may also be contained two levels below depending on the coupling constant $V > 0$. Such an example was given in Example 7.2 by the spectral band $J_{[0,0.1,2]}$ that is of type A, but it is contained one and two levels below for $V = 1$ (but not for $V > 4$). This explains the extra condition in the characterization of type B bands.

Appendix I. Sturmian dynamical systems

This appendix contains a very short description of Sturmian dynamical systems. A thorough background may be found in the books [Fog02, Lot02, DF25]. In addition, we state a lemma summarizing some basic properties of Sturmian sequences and mechanical words, which are applied in this paper.

To define the Sturmian Hamiltonian we have defined the sequences

$$\omega_\alpha(n) := \chi_{[1-\alpha,1[}(n\alpha \mod 1), \quad n \in \mathbb{N},\ \alpha \in [0,1],$$

which are called *mechanical words* [Lot02, Section 2.1.2] . If $\alpha \notin \mathbb{Q}$, ω_α is also called a *Sturmian sequence*. They naturally define a dynamical system as follows. Let $\mathcal{A} := \{0,1\}$ be equipped with the discrete topology and $\mathcal{A}^{\mathbb{Z}} := \{\omega : \mathbb{Z} \to \mathcal{A}\}$ be the compact metrizable space equipped with the product topology. Consider the shift $T : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}},\ (T\omega)(n) := \omega(n-1),\ n \in \mathbb{Z}$, being a homeomorphism. This induces a continuous group action $\mathbb{Z} \curvearrowright \mathcal{A}^{\mathbb{Z}}$ via $(n,\omega) \mapsto T^n\omega$. For $\alpha \in [0,1]$, we have $\omega_\alpha \in \{0,1\}^{\mathbb{Z}}$ and its associated orbit closure (in the product topology)

$$\Omega_\alpha := \overline{\mathrm{Orb}(\omega_\alpha)} := \overline{\{T^n\omega_\alpha \,:\, n \in \mathbb{Z}\}}$$

defines a dynamical system $\mathbb{Z} \curvearrowright \Omega_\alpha$. If $\alpha \in \mathbb{Q}$, then ω_α is periodic, i.e. there is a period $q \in \mathbb{N}$ such that $T^q\omega_\alpha = \omega_\alpha$. Note that in this case $\mathrm{Orb}(\omega_\alpha) = \Omega_\alpha$. There are various different representations of this dynamical system. For instance, the authors in [BIST89, Lemma 1] proved that

$$\omega_\alpha(n) = \lfloor \alpha(n+1) \rfloor - \lfloor \alpha\, n \rfloor, \qquad n \in \mathbb{N},\ \alpha \in [0,1] \setminus \mathbb{Q}.$$

A different approach to describe these words is via a recursive rule using the continued fraction expansion $(0, c_1, c_2, \ldots)$ of $\alpha \in [0,1] \setminus \mathbb{Q}$ is described in [Lot02, Equation 2.2.9]. The reader is also referred to [BBB$^+$] for a more detailed discussion. The following lemma provides the properties of mechanical words which are useful in our paper.

Lemma I.1. *Let $\mathbf{c} = [0,0,c_1,c_2,\ldots,c_k] \in \mathcal{C}$ for $k \in \mathbb{N}$ and $\frac{p_k}{q_k} := \varphi(\mathbf{c})$, with co-prime p_k, q_k. Then the following holds.*

(a) The sequence $\omega_{\frac{p_k}{q_k}}$ is periodic with period length q_k. Let its period $W_k \in \{0,1\}^{q_k}$ be defined by
$$W_k(i) := \omega_{\alpha_k}(i), \qquad 0 \leq i \leq q_k - 1$$
(b) For $k \in \mathbb{N}$, we have $q_k = c_k \cdot q_{k-1} + q_{k-2}$ with $q_{-1} = 0$ and $q_0 = 1$.
(c) The period of $\omega_{\frac{p_k}{q_k}}$ satisfy the following $W_0 = 0, W_1 = 0^{c_1-1}1$ and if $k \geq 2$, then
$$W_k = \begin{cases} W_{k-2}W_{k-1}^{c_k}, & k \equiv 0 \mod 2, \\ W_{k-1}^{c_k}W_{k-2}, & k \equiv 1 \mod 2. \end{cases}$$
(d) For $k \geq 1$,
- If $k \equiv 0 \mod 2$ then $\omega_\alpha(i) = W_k(i)$ for all $0 \leq i \leq q_k - 1$.
- If $k \equiv 1 \mod 2$ then $\omega_\alpha(i) = W_k(i)$ for all $0 \leq i \leq q_k - 2$.

Proof. The first two parts of the lemma are basic. The third and fourth parts appears e.g. in [BBB$^+$, Lemma 2.4]. □

The dynamical system Ω_α is minimal (namely for all $\omega \in \Omega_\alpha$, we have $\Omega_\alpha := \overline{\mathrm{Orb}(\omega)}$) and uniquely ergodic (it admits a unique shift invariant probability measure). Particular elements of Ω_α are the sequences $\omega_{\alpha,\xi} \in \{0,1\}^{\mathbb{Z}}$ for $\xi \in [0,1]$ defined by $\omega_{\alpha,\xi}(n) := \chi_{[1-\alpha,1[}(\xi + n\alpha \mod 1)$. For $\alpha, \xi \in [0,1]$ and $V \in \mathbb{R}$, consider the self-adjoint operator $H_{\alpha,V,\xi} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ defined by
$$(H_{\alpha,V,\xi}\psi)(n) := \psi(n+1) + \psi(n-1) + V\omega_{\alpha,\xi}(n)\,\psi(n).$$
If $\xi = 0$, this operator coincides with $H_{\alpha,V}$ defined in Equation (1.1). Let $\alpha \in [0,1]$ and $V \in \mathbb{R}$ be fixed. Since $\omega_{\alpha,\xi} \in \Omega_\alpha$ and Ω_α is minimal, the spectrum $\sigma(H_{\alpha,V,\xi})$ is independent of $\xi \in [0,1]$ and coincides with $\sigma(H_{\alpha,V})$. Moreover, the integrated density of states $N_{\alpha,V,\xi}$ can be defined for each $\xi \in [0,1]$ as in Equation (1.2) where $H_{\alpha,V}|_{[0,n-1]}$ is replaced by $H_{\alpha,V,\xi}|_{[0,n-1]}$. Since the dynamical system Ω_α is uniquely ergodic, the function $N_{\alpha,V,\xi}$ is independent of $\xi \in [0,1]$ and coincides with $N_{\alpha,V}$. Therefore, we set $\xi = 0$ throughout this work.

Appendix II. Chebyshev polynomials and trace identities

In this section, we provide several known identities of traces and their connection to Chebyshev polynomials, see e.g. [Cas86, Ray95a, BIST89, BBB$^+$, Ray, DF25]. Moreover, we prove Lemma 4.11.

II.1. **Chebyshev polynomials.** A crucial tool for studying the spectral theory of Sturmian Hamiltonians are the dilated Chebyshev polynomials of the second kind (see [DLMF, (18.1.3)]) $S_n : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}_{-1}$, defined by
$$S_{-1}(x) := 0, \quad S_0(x) := 1, \quad S_n(x) := xS_{n-1}(x) - S_{n-2}(x) \quad \text{for } x \in \mathbb{R}.$$
For $x \in \mathbb{R} \setminus \{0\}$, denote by $\mathrm{sign}(x) \in \{+1, -1\}$ the sign of x.

Lemma II.1. *Let $x \in \mathbb{R}$ and $n \in \mathbb{N}_0$. Then the following holds.*

(a) We have $S_{n+1}S_{n-1} - S_n^2 = -1$.
(b) If $|x| = 2$, *then* $\mathrm{sign}(x)^{n-1}S_{n-1}(x) = n$.
(c) If $|x| \geq 2$, *then* $\mathrm{sign}(x)^n S_n(x) = |S_n(x)|$ *and*
$$\mathrm{sign}(x)^n x S_{n-1}(x) \geq 2\big|S_{n-1}(x)\big|.$$
(d) If $|x| \geq 2$, *then*
$$\mathrm{sign}(x)^n\big(S_n(x) - \frac{x}{2}S_{n-1}(x)\big) \geq 1.$$

(e) If $|x| > 2$ and $n \geq 1$, then

$$\mathrm{sign}(x)^n\big(S_n(x) - \frac{x}{2}S_{n-1}(x)\big) > 1.$$

Proof. The proof follows by induction using the recursive relation, the details can be found for instance in [BBB+, Lemma III.2.]. □

II.2. **Trace identities .** This section is devoted to various trace identities and the proof of Lemma 4.11.

The following proposition is a collection of well-known identities of the traces, see e.g. [Cas86, Ray95a, BIST89, DF22, BBB+, Ray, DF25]. Recall that for $\mathbf{c} \in \mathcal{C}$, $t_{\mathbf{c}}$ is a function of $E, V \in \mathbb{R}$, but we abbreviate notation and suppress this dependencies in the following.

Proposition II.2 (trace maps)**.** *Let $m \in \mathbb{N}$, $n \in \mathbb{N}_0$ and $\boldsymbol{c} \in \mathcal{C}$ be such that $[\boldsymbol{c}, m] \in \mathcal{C}$. Then the following holds.*

(a) We have $t_{[\boldsymbol{c},m,0]} = t_{\boldsymbol{c}}$, $t_{[\boldsymbol{c},m,1]} = t_{[\boldsymbol{c},m+1]}$ and $t_{[\boldsymbol{c},m,-1]} = t_{[\boldsymbol{c},m-1]}$.
(b) We have for all $V \in \mathbb{R}$, (the Fricke–Vogt invariant)

$$V^2 + 4 = t^2_{[\boldsymbol{c},n+1]} + t^2_{[\boldsymbol{c},n]} + t^2_{\boldsymbol{c}} - t_{[\boldsymbol{c},n+1]}t_{[\boldsymbol{c},n]}t_{\boldsymbol{c}}$$

(c) For $-1 \leq l \leq n$, we have

$$t_{[\boldsymbol{c},n+1]} = S_{l+1}(t_{\boldsymbol{c}})t_{[\boldsymbol{c},n-l]} - S_l(t_{\boldsymbol{c}})t_{[\boldsymbol{c},n-l-1]}.$$

In particular, we have $t_{[\boldsymbol{c},n+1]} = t_{\boldsymbol{c}}t_{[\boldsymbol{c},n]} - t_{[\boldsymbol{c},n-1]}$ (for $l = 0$).

We will continue with two auxiliary lemmas which are needed to prove Lemma 4.11. In order to treat certain cases of the backward type A ($\ell = 0$) bands or backward type B ($\ell = -1$) bands, we need the following identity.

Lemma II.3. *Let $m \in \mathbb{N}$ and $\boldsymbol{c} \in \mathcal{C}$ be such that $[\boldsymbol{c}, m] \in \mathcal{C}$. Let $V \in \mathbb{R}$ and $E \in \mathbb{R}$ be such that $|t_{\boldsymbol{c}}(E, V)| = 2$. Then for all $n \in \mathbb{N}$ and $\ell \in \{-1, 0\}$,*

$$\begin{aligned} t_{[\boldsymbol{c},\ell]}(E,V)S_n\big(t_{[\boldsymbol{c},m]}(E,V)\big) =& z^{m-\ell}\Big(2(m-\ell)S_{n+1}\big(t_{[\boldsymbol{c},m]}(E,V)\big) - z(m-\ell)t_{[\boldsymbol{c},m,n]}(E,V) \\ &- (m-1-\ell)t_{[\boldsymbol{c},m]}(E,V)S_n\big(t_{[\boldsymbol{c},m]}(E,V)\big)\Big) \end{aligned}$$

holds where $z := \mathrm{sign}(t_{\boldsymbol{c}}(E, V))$.

Remark. This lemma is closely related to [BIST89, prop. 2].

Proof. For the sake of simplicity, we abbreviate the notation in the following and write $t_{\widetilde{\mathbf{c}}} = t_{\widetilde{\mathbf{c}}}(E, V)$ for $\widetilde{\mathbf{c}} \in \mathcal{C}$. As a direct consequence of Lemma II.1 (b), we conclude that $S_l(t_{\mathbf{c}}) \neq 0$ for all $l \geq 0$ since $|t_{\mathbf{c}}| = 2$. Proposition II.2 (c) (applied for $n = m-1$ and $l = m-2-\ell \geq -1$) leads to

$$t_{[\mathbf{c},m]} = S_{m-1-\ell}\big(t_{\mathbf{c}}\big)t_{[\mathbf{c},1+\ell]} - S_{m-2-\ell}\big(t_{\mathbf{c}}\big)t_{[\mathbf{c},\ell]}.$$

Since $S_{m-1-\ell}(t_{\mathbf{c}}) \neq 0$, we derive

$$t_{[\mathbf{c},1+\ell]} = \frac{1}{S_{m-1-\ell}\big(t_{\mathbf{c}}\big)}\Big(t_{[\mathbf{c},m]} + S_{m-2-\ell}\big(t_{\mathbf{c}}\big)t_{[\mathbf{c},\ell]}\Big). \tag{II.1}$$

Let $n \in \mathbb{N}$. Using again Proposition II.2 (a) and (c) (with $l = n$), we conclude

$$t_{[\mathbf{c},m,n]} = S_{n+1}\big(t_{[\mathbf{c},m]}\big)t_{[\mathbf{c},m,0]} - S_n\big(t_{[\mathbf{c},m]}\big)t_{[\mathbf{c},m,-1]} = S_{n+1}\big(t_{[\mathbf{c},m]}\big)t_{\mathbf{c}} - S_n\big(t_{[\mathbf{c},m]}\big)t_{[\mathbf{c},m-1]}. \tag{II.2}$$

The case $\ell = 0$ and $m = 1$ need to be treated separately. We treat this case later and first assume that if $\ell = 0$ then $m \geq 2$. Then Proposition II.2 (c) (applied for $n = m-2$ and $l = m-3-\ell \geq -1$) and Equation (II.1) leads to

$$t_{[\mathbf{c},m-1]} = S_{m-2-\ell}\big(t_{\mathbf{c}}\big)t_{[\mathbf{c},1+\ell]} - S_{m-3-\ell}\big(t_{\mathbf{c}}\big)t_{[\mathbf{c},\ell]}$$

$$=\frac{S_{m-2-\ell}(t_{\mathbf{c}})}{S_{m-1-\ell}(t_{\mathbf{c}})}t_{[\mathbf{c},m]}+t_{[\mathbf{c},\ell]}\left(\frac{S_{m-2-\ell}(t_{\mathbf{c}})^2}{S_{m-1-\ell}(t_{\mathbf{c}})}-S_{m-3-\ell}(t_{\mathbf{c}})\right)$$
$$=\frac{S_{m-2-\ell}(t_{\mathbf{c}})}{S_{m-1-\ell}(t_{\mathbf{c}})}t_{[\mathbf{c},m]}+t_{[\mathbf{c},\ell]}\left(\frac{S_{m-2-\ell}(t_{\mathbf{c}})^2-S_{m-3-\ell}(t_{\mathbf{c}})S_{m-1-\ell}(t_{\mathbf{c}})}{S_{m-1-\ell}(t_{\mathbf{c}})}\right).$$

Since $S_kS_{k-2}-S_{k-1}^2=-1$ for $k=m-1-\ell$ by Lemma II.1 (a), we conclude

$$t_{[\mathbf{c},m-1)}=\frac{S_{m-2-\ell}(t_{\mathbf{c}})}{S_{m-1-\ell}(t_{\mathbf{c}})}t_{[\mathbf{c},m]}+t_{[\mathbf{c},\ell]}\frac{1}{S_{m-1-\ell}(t_{\mathbf{c}})}.$$

Inserting the latter into Equation (II.2), we get

$$t_{[\mathbf{c},m,n]}=S_{n+1}(t_{[\mathbf{c},m]})t_{\mathbf{c}}-S_n(t_{[\mathbf{c},m]})\frac{S_{m-2-\ell}(t_{\mathbf{c}})}{S_{m-1-\ell}(t_{\mathbf{c}})}t_{[\mathbf{c},m]}-t_{[\mathbf{c},\ell]}\frac{S_n(t_{[\mathbf{c},m]})}{S_{m-1-\ell}(t_{\mathbf{c}})}. \tag{II.3}$$

We claim that the latter identity holds also if $\ell=0$ and $m=1$. Indeed, this follows immediately from Equation (II.2), $t_{[\mathbf{c},m-1)}=t_{[\mathbf{c},\ell]}$, $S_{m-1-\ell}(t_{\mathbf{c}})=1$ and $S_{m-2-\ell}(t_{\mathbf{c}})=0$.

Now we can proceed with arbitrary $\ell\in\{-1,0\}$ and $m\in\mathbb{N}$. Reorganizing Equation (II.3) leads to

$$t_{[\mathbf{c},\ell]}S_n(t_{[\mathbf{c},m]})=S_{m-1-\ell}(t_{\mathbf{c}})\Big(S_{n+1}(t_{[\mathbf{c},m]})t_{\mathbf{c}}-t_{[\mathbf{c},m,n]}\Big)-S_{m-2-\ell}(t_{\mathbf{c}})S_n(t_{[\mathbf{c},m]})t_{[\mathbf{c},m]}.$$

Since we assumed that $t_{\mathbf{c}}=t_{\mathbf{c}}(E,V)=2z$, Lemma II.1 (b) implies $S_n(t_{\mathbf{c}})=z^n(n+1)$ for $n\geq 0$. Thus,

$$\begin{aligned}t_{[\mathbf{c},\ell]}S_n(t_{[\mathbf{c},m]})=&z^{m-1-\ell}(m-\ell)\Big(S_{n+1}(t_{[\mathbf{c},m]})2z-t_{[\mathbf{c},m,n]}\Big)\\&-z^{m-2-\ell}(m-1-\ell)S_n(t_{[\mathbf{c},m]})t_{[\mathbf{c},m]}\end{aligned}$$

follows proving the desired identity. □

Lemma II.4. *Let $m,n\in\mathbb{N}$ and $\boldsymbol{c}\in\mathcal{C}$ be such that $[\boldsymbol{c},m]\in\mathcal{C}$. Let $V\in\mathbb{R}$ and $E\in\mathbb{R}$ be such that*

$$|t_{\boldsymbol{c}}(E,V)|=2 \qquad \text{and} \qquad |t_{[\boldsymbol{c},m]}(E,V)|\geq 2.$$

Then for all $n\in\mathbb{N}$ and $\ell\in\{-1,0\}$,

$$\big|t_{[\boldsymbol{c},\ell]}(E,V)S_n\big(t_{[\boldsymbol{c},m]}(E,V)\big)\big|\geq(m-\ell)\big(2-\big|t_{[\boldsymbol{c},m,n]}(E,V)\big|\big)+2\big|S_n\big(t_{[\boldsymbol{c},m]}(E,V)\big)\big|$$

holds and the estimate is strict if additionally $|t_{[\boldsymbol{c},m]}(E,V)|>2$.

Remark. The latter estimate is the general formula that we need to treat backward type A bands ($\ell=0$) or backward type B bands ($\ell=-1$).

Proof. In order to simplify notation, set $t:=t_{[\mathbf{c},m]}(E,V)$, $z_1:=\operatorname{sign}(t)$ and $z_0:=\operatorname{sign}(t_{\mathbf{c}}(E,V))$. Furthermore, we abbreviate the notation and write $t_{\widetilde{\mathbf{c}}}=t_{\widetilde{\mathbf{c}}}(E,V)$ for $\widetilde{\mathbf{c}}\in\mathcal{C}$. Due to Lemma II.3 and $z_1^{2n}=1$, we have

$$t_{[\mathbf{c},\ell]}S_n(t)=z_0^{m-\ell}z_1^{n+1}\Big(2(m-\ell)z_1^{n+1}S_{n+1}(t)-(m-1-\ell)z_1^{n+1}tS_n(t)-z_0z_1^{n+1}(m-\ell)t_{[\mathbf{c},m,n]}\Big).$$

Hence,

$$\begin{aligned}\big|t_{[\mathbf{c},\ell]}S_n(t)\big|\geq&\Big|2(m-\ell)z_1^{n+1}S_{n+1}(t)-(m-1-\ell)z_1^{n+1}tS_n(t)\Big|-\big|(m-\ell)t_{[\mathbf{c},m,n]}\big|\\=&2(m-\ell)z_1^{n+1}\Big(S_{n+1}(t)-\frac{t}{2}S_n(t)\Big)+z_1^{n+1}tS_n(t)-(m-\ell)\big|t_{[\mathbf{c},m,n]}\big|\\\geq&2(m-\ell)+2\big|S_n(t)\big|-(m-\ell)\big|t_{[\mathbf{c},m,n]}\big|\end{aligned}$$

follows by first using the triangle inequality, secondly Lemma II.1 (c) and (d) since $|t|\geq 2$ and finally Lemma II.1 (c) and (d). Note that the last estimate is strict by Lemma II.1 (e) if $|t|>2$. This leads to the desired estimate. □

Now we can prove Lemma 4.11.

Proof of Lemma 4.11. Recall the assumptions of the proposition. Let $V \in \mathbb{R}$, $m, n \in \mathbb{N}$, $\mathbf{c} \in \mathcal{C}$. Let $I(V)$ be a spectral band of $\sigma_{\mathbf{c}}(V)$ of backward type A or B. Set

$$\ell := \begin{cases} 0, & I(V) \text{ is of backward type } A, \\ -1, & I(V) \text{ is of backward type } B. \end{cases}$$

Let $E \in \{L(I(V)),\ R(I(V))\}$. Then $|t_{[\mathbf{c},\ell]}(E, V)| \leq 2$ follows from Lemma 4.10 and the estimate is strict if $\varphi(\mathbf{c}) \in (0, 1)$.

(a) If $|t_{[\mathbf{c},m]}(E, V)| \geq 2$, then Lemma II.4 and $m - \ell \geq 1$ imply

$$2\left|S_n\big(t_{[\mathbf{c},m]}\big)\right| \geq \left|t_{[\mathbf{c},\ell]} S_n\big(t_{[\mathbf{c},m]}\big)\right| \geq (m-\ell)\big(2 - \left|t_{[\mathbf{c},m,n]}\right|\big) + 2\left|S_n\big(t_{[\mathbf{c},m]}\big)\right|$$

and so $|t_{[\mathbf{c},m,n]}| \geq 2$ must hold.

(b) If $|t_{[\mathbf{c},m]}(E, V)| > 2$, then Lemma II.4 and $m - \ell \geq 1$ imply

$$2\left|S_n\big(t_{[\mathbf{c},m]}\big)\right| \geq \left|t_{[\mathbf{c},\ell]} S_n\big(t_{[\mathbf{c},m]}\big)\right| > (m-\ell)\big(2 - \left|t_{[\mathbf{c},m,n]}\right|\big) + 2\left|S_n\big(t_{[\mathbf{c},m]}\big)\right|$$

and so $|t_{[\mathbf{c},m,n]}| > 2$ must hold.

(c) Since $\varphi(\mathbf{c}) \in (0, 1)$, Lemma 4.10 asserts $|t_{[\mathbf{c},\ell]}(E, V)| < 2$. If $|t_{[\mathbf{c},m]}(E, V)| \geq 2$, then Lemma II.4 and $m - \ell \geq 1$ imply

$$2\left|S_n\big(t_{[\mathbf{c},m]}\big)\right| > \left|t_{[\mathbf{c},\ell]} S_n\big(t_{[\mathbf{c},m]}\big)\right| \geq (m-\ell)\big(2 - \left|t_{[\mathbf{c},m,n]}\right|\big) + 2\left|S_n\big(t_{[\mathbf{c},m]}\big)\right|.$$

Thus, $|t_{[\mathbf{c},m,n]}| > 2$ must hold. □

Appendix III. A perturbation argument for eigenvalue interlacing

This section is devoted to the proof of Theorem 3.4. Given an $n \times n$ hermitian matrix X, we denote its eigenvalues in non-decreasing order by

$$\lambda_0(X) \leq \lambda_1(X) \cdots \leq \lambda_{n-2}(X) \leq \lambda_{n-1}(X).$$

We first recall a well-known result on interlacing of eigenvalues using classical Weyl inequalities, see e.g. [HJ13, Corollary 4.3.3, Theorem 4.3.6].

Proposition III.1. *Let X and Q be $n \times n$ hermitian matrices, and suppose that Q is a positive semidefinite, rank one matrix.*

(a) For $j = 1, \ldots n-1$, we have

$$\lambda_{j-1}(X + Q) \leq \lambda_j(X) \leq \lambda_j(X + Q).$$

(b) For $j = 0, \ldots n-2$, we have

$$\lambda_j(X - Q) \leq \lambda_j(X) \leq \lambda_{j+1}(X - Q).$$

From these inequalities we can directly derive the following estimates for traceless rank two perturbations.

Corollary III.2. *Let X and Y be $n \times n$ hermitian matrices, and let $Q = Y - X$. If Q has rank two and trace zero, then*

$$\lambda_{j-1}(Y) \leq \lambda_j(X) \leq \lambda_{j+1}(Y), \qquad j = 1, 2, \ldots n-2.$$

Proof. Using matrix diagonalization one may verify that there exist $n \times n$ hermitian, positive semidefinite matrices Q_1, Q_2 of rank one such that $Q = Q_1 - Q_2$. Applying first Proposition III.1 (a) to X and $X + Q_1$, and then Proposition III.1 (b) to $X + Q_1$ and $Y = (X + Q_1) - Q_2$ yields the desired inequalities. □

Let $\mathbf{c} = [0, c_0, c_1, \ldots, c_k] \in \mathcal{C}$ be such that $k \in \mathbb{N}_0$ and $c_k \in \mathbb{N}$ if $k \geq 1$. Note that this implies $\varphi(\mathbf{c}) \notin \{-1, \infty\}$. Recall the notations of the matrices $H_{\mathbf{c},V}(\theta)$ and $H^{\times n}_{\mathbf{c},V}(\theta)$ for $\theta \in [0, \pi]$ as introduced in Section 3.2. We aim to apply Corollary III.2 to the matrices

$$H_{[\mathbf{c},m,n],V}\left(\theta_{[\mathbf{c},m,n]}\right) \text{ and } H^{\times n}_{[\mathbf{c},m],V}\left(\theta_{[\mathbf{c},m]}\right) \oplus H_{\mathbf{c},V}(\theta_{\mathbf{c}})$$

with appropriate choice of $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]} \in \{0, \pi\}$. It turns out that if $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]} \in \{0, \pi\}$ are admissible (see Definition 3.5), then these matrices are a rank two perturbation with trace zero of each other. To formalize this statement, we define the matrix

$$H^{\oplus}_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m]}, \theta_{\mathbf{c}}) := \begin{cases} \begin{pmatrix} H_{\mathbf{c},V}(\theta_{\mathbf{c}}) & 0 \\ 0 & H^{\times n}_{[\mathbf{c},m],V}\left(\theta_{[\mathbf{c},m]}\right) \end{pmatrix} & \text{if } k \equiv 0 \mod 2, \\ \\ \begin{pmatrix} H^{\times n}_{[\mathbf{c},m],V}\left(\theta_{[\mathbf{c},m]}\right) & 0 \\ 0 & H_{\mathbf{c},V}(\theta_{\mathbf{c}}) \end{pmatrix} & \text{if } k \equiv 1 \mod 2. \end{cases}$$

In the following statements we refer to vectors $x \in \mathbb{R}^q$ as column vectors and use the notation x^t to indicate the transpose of a vector (which is then a row vector). We also use the notation $\langle x, y \rangle$ to denote the (Euclidean) inner product between vectors.

Lemma III.3. *Let $V \in \mathbb{R}$, $m, n \in \mathbb{N}$ and $\boldsymbol{c} = [0, c_0, c_1, \ldots, c_k] \in \mathcal{C}$ be such that $k \in \mathbb{N}_0$ and $c_k \in \mathbb{N}$ if $k \geq 1$ be such that $[\boldsymbol{c}, m] \in \mathcal{C}$. Let $\frac{p_1}{q_1} = \varphi(\boldsymbol{c})$, $\frac{p_2}{q_2} = \varphi([\boldsymbol{c}, m])$ and $\frac{p_3}{q_3} = \varphi([\boldsymbol{c}, m, n])$ be such that p_i, q_i are coprime. If $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]} \in \{0, \pi\}$* are admissible*, then there are $x := x(\theta_{\boldsymbol{c}}, \theta_{[\boldsymbol{c},m]}, \theta_{[\boldsymbol{c},m,n]}), y := y(\theta_{\boldsymbol{c}}, \theta_{[\boldsymbol{c},m]}, \theta_{[\boldsymbol{c},m,n]}) \in \mathbb{R}^{q_3}$ such that*

$$H_{[\boldsymbol{c},m,n],V}\left(\theta_{[\boldsymbol{c},m,n]}\right) - H^{\oplus}_{[\boldsymbol{c},m,n]}(\theta_{[\boldsymbol{c},m]}, \theta_{\boldsymbol{c}}) = xx^t - yy^t$$

is a symmetric rank two perturbation with trace zero. Furthermore, set

$$d_1 := \begin{cases} q_1 & \text{if } k \equiv 0 \mod 2, \\ nq_2 & \text{if } k \equiv 1 \mod 2, \end{cases} \quad \text{and} \quad d_2 := \begin{cases} nq_2 & \text{if } k \equiv 0 \mod 2, \\ q_1 & \text{if } k \equiv 1 \mod 2. \end{cases}$$

(a) If $w = (w_1, \ldots w_{d_1}, 0, \ldots, 0)^t \in \mathbb{R}^{q_3}$ is orthogonal to x and to y then $w_1 = w_{d_1} = 0$.
(b) If $w = (0, \ldots, 0, w_1, \ldots w_{d_2})^t \in \mathbb{R}^{q_3}$ is orthogonal to x and to y then $w_1 = w_{d_2} = 0$.

Proof. Let $e_1, \ldots, e_{q_3}$ be the standard orthonormal basis of $\mathbb{R}^{q_3}$, namely e_i is the i-th unit vector in $\mathbb{R}^{q_3}$. Recall that for the continued fraction expansion, we have the identity $q_3 = nq_2 + q_1 = d_1 + d_2$, see e.g. Lemma I.1. The statement of the lemma follows by straightforward calculations invoking Lemma I.1, so we just explicitly write here the expressions for the $x, y \in \mathbb{R}^{q_3}$ in the statement of the lemma. Let $k \in \mathbb{N}$ be even. If $(\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}) = (0, 0, 0)$, then

$$x = \frac{1}{\sqrt{2}}\big(e_1 - e_{nq_2} - e_{nq_2+1} + e_{q_3}\big), \quad y = \frac{1}{\sqrt{2}}\big(-e_1 - e_{nq_2} + e_{nq_2+1} + e_{q_3}\big).$$

If $(\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}) = (\pi, \pi, 0)$, then

$$x = \frac{1}{\sqrt{2}}\big(e_1 + e_{nq_2} + e_{nq_2+1} + e_{q_3}\big), \quad y = \frac{1}{\sqrt{2}}\big(-e_1 + e_{nq_2} - e_{nq_2+1} + e_{q_3}\big).$$

If $(\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}) = (\pi, 0, \pi)$, then

$$x = \frac{1}{\sqrt{2}}\big(-e_1 + e_{nq_2} + e_{nq_2+1} + e_{q_3}\big), \quad y = \frac{1}{\sqrt{2}}\big(e_1 + e_{nq_2} - e_{nq_2+1} + e_{q_3}\big).$$

If $(\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}) = (0, \pi, \pi)$, then

$$x = \frac{1}{\sqrt{2}}\big(e_1 + e_{nq_2} + e_{nq_2+1} - e_{q_3}\big), \quad y = \frac{1}{\sqrt{2}}\big(-e_1 + e_{nq_2} - e_{nq_2+1} - e_{q_3}\big).$$

The case when $n \in \mathbb{N}$ is odd is treated similarly. □

Remark III.4. Statements (a) and (b) in Lemma III.3 are used to conclude strict inequalities in Theorem 3.4. Towards this, we use that if two consecutive (up to cyclic permutation) entries of a solution (see (a) and (b)) vanish, then the whole solution vanishes, since we have a nearest neighbor interaction.

Now we have all tools at hand to prove Theorem 3.4.

Proof of Theorem 3.4. Recall the statement of the theorem. Let $V > 0$, $m, n \in \mathbb{N}$ and $\mathbf{c} = [0, c_0, \ldots, c_k] \in \mathcal{C}$ be such that $\varphi(\mathbf{c}) \notin \{-1, \infty\}$ and $[\mathbf{c}, m] \in \mathcal{C}$. Thus, $c_k \in \mathbb{N}$ if $k \geq 1$. Furthermore, $\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]} \in \{0, \pi\}$ are admissible, i.e., they satisfy $\theta_{\mathbf{c}} + \theta_{[\mathbf{c},m]} + \theta_{[\mathbf{c},m,n]} \in \{0, 2\pi\}$. Consider $Y = H_{[\mathbf{c},m,n],V}(\theta_{[\mathbf{c},m,n]})$ and $X = H^{\times n}_{[\mathbf{c},m],V}(\theta_{[\mathbf{c},m]}) \oplus H_{\mathbf{c},V}(\theta_{\mathbf{c}})$. We need to prove that

$$\lambda_{j-1}(Y) \leq \lambda_j(X) \leq \lambda_{j+1}(Y)$$

and that the inequalities are strict whenever $\lambda_j(X)$ is a simple eigenvalue of X. First, observe that by construction X and $Z := H^{\oplus}_{[\mathbf{c},m,n]}(\theta_{[\mathbf{c},m]}, \theta_{\mathbf{c}})$ share the same eigenvalues (with same multiplicities). Thus, the claimed inequalities follow directly from Corollary III.2 and Lemma III.3. It is left to prove that those inequalities are strict if $\lambda_j(X) = \lambda_j(Z)$ is simple.

Let $n \in \mathbb{N}$ and borrow the notation of Lemma III.3 for $q_1, q_2, q_3 \in \mathbb{N}$ and $d_1, d_2 \in \mathbb{N}$. Following Lemma III.3, there are $x := x(\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]})$, $y := y(\theta_{\mathbf{c}}, \theta_{[\mathbf{c},m]}, \theta_{[\mathbf{c},m,n]}) \in \mathbb{R}^{q_3}$ such that $Y - Z = xx^t - yy^t$. Moreover, x and y satisfy the assertions (a) and (b) in Lemma III.3. Set

$$Z(x) := Z + xx^t \qquad \text{and} \qquad Z(y) := Z - yy^t.$$

Then we have

$$Y = Z + xx^t - yy^t = Z(x) - yy^t = Z(y) + xx^t.$$

Recall that $q_3 = nq_2 + q_1 = d_1 + d_2$ (Lemma I.1) and $H^{\times n}_{[\mathbf{c},m],V}(\theta_{[\mathbf{c},m]})$ is an $nq_2 \times nq_2$ matrix while $H_{\mathbf{c},V}(\theta_{\mathbf{c}})$ is an $q_1 \times q_1$.

(a) We prove $\lambda_j(Z) < \lambda_{j+1}(Y)$. Assume by contradiction that $\lambda_j(Z) = \lambda_{j+1}(Y)$ holds and $\lambda_j(Z)$ is a simple eigenvalue of Z. Due to Proposition III.1 (using that xx^t and yy^t are positive semidefinite), the previous identities lead to

$$\lambda_j(Z) \leq \lambda_{j+1}(Z(y)) \leq \lambda_{j+1}(Y) \quad \text{and} \quad \lambda_j(Z) \leq \lambda_j(Z(x)) \leq \lambda_{j+1}(Y).$$

Thus,

$$\lambda := \lambda_j(Z) = \lambda_j(Z(x)) = \lambda_{j+1}(Z(y)) = \lambda_{j+1}(Y).$$

follows by our assumption.

Let $w \in \mathbb{R}^{q_3} \setminus \{0\}$ be an eigenvector of Z corresponding to the eigenvalue λ. Since λ is a simple eigenvalue of Z, then either (1) λ is an eigenvalue of $H^{\times n}_{[\mathbf{c},m],V}(\theta_{[\mathbf{c},m]})$ or (2) λ is an eigenvalue of $H_{\mathbf{c},V}(\theta_{\mathbf{c}})$, but not both. These two cases can be treated similarly using Lemma III.3. We only prove here case (1).

Since λ is an eigenvalue of $H^{\times n}_{[\mathbf{c},m],V}(\theta_{[\mathbf{c},m]})$ but not of $H_{\mathbf{c},V}(\theta_{\mathbf{c}})$, we conclude that the corresponding eigenvector w of z is of the form $w = (w_1, \ldots, w_{nq_2}, 0, \ldots, 0)^t \in \mathbb{R}^{q_3}$ if $k \in \mathbb{N}$ is odd (where k is determined by the length of the tuple $\mathbf{c}$) and $w = (0, \ldots, 0, w_1, \ldots, w_{nq_2})^t \in \mathbb{R}^{q_3}$ if $k \in \mathbb{N}$ is even. Set $u := (w_1, \ldots, w_{nq_2})^t$.

We claim that $\langle x, w \rangle = 0 = \langle w, y \rangle$ holds. Before proving this identity, let us show how these equalities finish our proof. If $\langle x, w \rangle = 0 = \langle w, y \rangle$, then $w_1 = w_{nq_2} = 0$ follow from Lemma III.3 (a) if k is odd and from Lemma III.3 (b) if k is even. Since $H^{\times n}_{[\mathbf{c},m],V}(\theta_{[\mathbf{c},m]})u = \lambda u$ and each equation in the system involves three consecutive (going cyclically) of the entries of $u \in \mathbb{R}^{nq_2}$, we derive $u = 0$ and so $w = 0$. This is a contradiction as $w \neq 0$ is an eigenvector of Z for the eigenvalue λ.

Now let us prove the claim $\langle x, w\rangle = 0 = \langle w, y\rangle$. Since $\lambda = \lambda_j(Z) = \lambda_j(Z(x))$, there is an eigenvector v of $Z + xx^t$ with eigenvalue λ. Using that $Z + xx^t$ is hermitian and $x^t w = \langle x, w\rangle$, we conclude

$$\lambda\langle v, w\rangle = \langle v, (Z + xx^t)w\rangle = \langle v, Zw\rangle + \langle x, w\rangle\langle v, x\rangle = \lambda\langle v, w\rangle + \langle x, w\rangle\langle v, x\rangle$$

implying $\langle x, w\rangle\langle v, x\rangle = 0$. If $\langle v, x\rangle \neq 0$, we immediately derive $\langle x, w\rangle = 0$ as desired. If $\langle v, x\rangle = 0$, then

$$\lambda v = (Z + xx^t)v = Zv + \langle x, v\rangle x = Zv$$

follows. Thus, $v = Cw$ holds for some $C \in \mathbb{R} \setminus \{0\}$ as λ is a simple eigenvalue of Z with eigenvector w. Hence, $\langle v, x\rangle = 0$ leads to $\langle w, x\rangle = 0$ as claimed.

Similarly, we conclude $\langle w, y\rangle = 0$ using that λ is an eigenvalue of $Z(y)$.

(b) Similarly to case (a), we can prove $\lambda_{j-1}(Y) < \lambda_j(Z)$. Assume by contradiction that $\lambda_{j-1}(Y) = \lambda_j(Z)$ holds and $\lambda_j(Z)$ is a simple eigenvalue of Z. Then Proposition III.1 leads to

$$\lambda_{j-1}(Y) \leq \lambda_{j-1}(Z(x)) \leq \lambda_j(Z) \quad \text{and} \quad \lambda_{j-1}(Y) \leq \lambda_j(Z(y)) \leq \lambda_j(Z).$$

Thus, our assumption yields

$$\lambda := \lambda_j(Z) = \lambda_{j-1}(Z(x)) = \lambda_j(Z(y)) = \lambda_{j-1}(Y).$$

Next, let $w \in \mathbb{R}^{q_3} \setminus \{0\}$ be an eigenvector of Z for the eigenvalue λ. By simplicity of the eigenvalue $\lambda = \lambda_j(Z)$, λ is either an eigenvalue of $H^{\times n}_{[\mathbf{c},m],V}(\theta_{[\mathbf{c},m]})$ or of $H_{\mathbf{c},V}(\theta_{\mathbf{c}})$. Thus, w has either the form $w = (w_1, \ldots, w_{d_1}, 0, \ldots, 0)^t \in \mathbb{R}^{q_3}$ or $w = (0, \ldots, 0, w_1, \ldots, w_{d_2})^t \in \mathbb{R}^{q_3}$. As before one can show that in both cases $\langle x, w\rangle = 0 = \langle w, y\rangle$ holds. Then Lemma III.3 yields $w_1 = w_{d_1} = 0$ (respectively $w_1 = w_{d_2} = 0$) and so $w = 0$ follows, a contradiction. □

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