THE DISCOURSE MAPPING TREE AS A TOOL FOR ANALYZING THE POTENTIAL AND IMPLEMENTATION OF LINEAR ALGEBRA TASKS

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We introduce a tool for mapping tasks and their implementation based on the commognitive theory and realization trees. This tool, the Discourse Mapping Tree (DMT), first maps a priori the subdiscourses involved in solving a task, then, a posteriori the discussion of this task in a class. This affords examination of both the mathematical potential of tasks and of how an implementation takes up this potential. We exemplify the DMT on a lesson about linear transformations in a discussion-based linear algebra workshop. The tool highlighted the students' and instructor's role in authoring links between subdiscourses. It also displayed that the instructor was more responsible for meta-level links, while the students made object-level links more readily and easily.

Teaching mathematics includes posing tasks that may offer more or less opportunities for students to engage with mathematical concepts, ideas, and strategies (Sullivan et al., 2015). Examining the potential of such tasks can further our understanding of the disparity between the potential of tasks and their implementation. This is particularly important in the context of cognitively demanding or explorative teaching practices (Smith & Stein, 1998). However, usually the ability to distinguish between the potential of tasks and the take-up in class is difficult. The literature on task design offers a distinction between a priori and a posteriori analysis of tasks (Artigue, 2009). This literature points to the usefulness of detecting through a priori analysis features of tasks that can support certain types of pedagogical goals (e.g. Kieran, 2019).

In line with the task-design literature (e.g. Gravesen et al., 2017), we suggest a tool for mapping the mathematical potential of a task for explorative instruction, made up of two stages: a priori and a posteriori. The a priori stage is independent from the a posteriori analysis thus enabling comparison between the potential and the implementation. Explorative instruction is instruction that affords students maximal opportunities for explorative participation, that is opportunities to author narratives about mathematical objects based on their own reasoning (Weingarden et al., 2019). Encouraging explorative participation, according to Weingarden and colleagues, includes exposing students to multiple realizations of objects and to links between different realizations. These authors suggested a tool, named the Realization Tree Assessment (RTA), to examine the extent to which students were indeed exposed during a whole class discussion to multiple realizations and links between them.

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However, the RTA did not make a clear distinction between a priori and a posteriori analysis of a task, nor did it precisely define the mathematical potential of a task.

In the present work, we build on Weingarden and colleagues' work, and extend it by defining the mathematical potential of a task in correspondence with the socially and historically established mathematical discourse. Inspired by Gee (2015) we distinguish between the mathematical Discourse (with a capital D) - the canonical discourse accepted by the mathematical community, and the classroom discourse (with a small d) - the individualized version of the social Discourse. It is the discourse (with small d) that is seen in discussions around mathematical tasks. Similar to Sfard (2008) we define learning mathematics as becoming a participant in a certain discourse, yet stress that, aligned with Gee, this is a socially and historically established Discourse.

Mathematical Discourses are hierarchical and recursive, where their objects (e.g. $\mathbb{Q}$) build upon previously established objects (e.g. $\mathbb{Z}$) (Sfard, 2008). New mathematical Discourses have historically been created either by several existing Discourses coalescing into one Discourse or by a meta-level Discourse subsuming an older one. Sfard (2008) claims that an individual's adoption of a Discourse, that is learning, often proceeds similarly to how Discourses developed over centuries. Thus, when learners progress from one Discourse to a new subsuming one, the subsuming Discourse includes an isomorphic copy of the old Discourse, as well as new objects and narratives that can only be realized in the new Discourse (Lavie & Sfard, 2019). Adopting new narratives belonging to a familiar Discourse is object-level learning, whereas authoring narratives in the new coalesced Discourse is meta-level learning (Sfard, 2008).

Using this framework, we define the mathematical potential of a task as the potential it affords a student to individualize a Discourse and learning linear algebra as individualizing the Discourse of linear algebra. This includes individualizing all the sub-Discourses of this topic (such as the Discourse of matrices, that of vector spaces, etc.), which we term in this context object-level learning. In addition, it includes linking the realizations from the various sub-Discourses so that the student's individual subsumed discourses coalesce into one discourse, which is considered meta-level learning.

Weingarden and colleagues' (2019) RTA tool constructs a visual representation of realizations of a mathematical object that is at the center of a task, building on Sfard's (2008) definition of a mathematical object being a signifier together with its realization tree. This mapping is constrained, since often multiple mathematical objects are mentioned during a discussion. In the present work, we build on the idea of the RTA to offer a new tool, named the Discourse Mapping Tree (DMT), which enables mapping of a discussion that involves multiple mathematical objects. In addition, we add a clear distinction between the a priori and a posteriori stages of the tool. We conceptualize the building of the a priori DMT as based on an analysis of the mathematical Discourse, whereas the a posteriori stage is based on the classroom discourse that was observed in the lesson.

We apply the Discourse Mapping Tree (DMT) to a single task and to a particular implementation of that task to demonstrate this tool. We ask what the construction of the DMT highlights about the mathematical potential of the task and about the take up in a whole classroom discussion. In addition, we ask what can the DMT tell us about the potential for and the take up of object-level and meta-level learning?

METHOD

Context, Participants and Data

This study is part of a larger project in which linear algebra workshops were offered to students, in addition to the regular lectures and tutorials and held in parallel to them (Wallach, 2022). The first author led the workshops. Overall, 13 workshops were held, and 7 tasks were designed for them. The tasks were designed by the first and third author, instructors of linear algebra with many years of experience.

The study was conducted at a science and engineering university, where all the students have successfully completed advanced level high-school mathematics courses required for entrance. Students take a linear algebra course, a requirement for most science and engineering students, during their first semester. Participation in the workshops was voluntary and the number of students participating in each workshop varied from 7 to 50. The lesson structure of the workshops was an adaptation of the launch, explore and discuss (LED) structure and Smith and Stein's (1998) suggested practices for orchestrating productive mathematics discussions. Data was collected through several video cameras posed at the board and at student groups. The session described in this paper was one academic hour and 7 students participated.

Analysis

Constructing a DMT includes two parts. First, an a priori analysis of the task, as it is presented to the students, directs the construction (see Fig. 1). This is carried out by experts (mathematicians) who represent the canonical Discourse and is supported by textbooks and curricula. This analysis includes determining the root node, which is the object at the center of the task, listing this object's realizations and finally grouping together realizations of similar type. Each type of realization belongs to a certain Discourse as it has its own keywords, its own narratives, and its own routines of manipulation. Each Discourse (or sub-Discourse) is drawn on a separate branch of the DMT. The a posteriori part of constructing a DMT is based on a video recording of a whole class discussion around the task. In this stage, each narrative authored in a discussion is mapped onto the Discourses identified by the a priori DMT (see Fig. 2). This mapping includes drawing the realizations and links mentioned during the discussion in class. Similar to the construction of RTAs (Weingarden et al., 2019), the components of the DTM demarcate if a student authored it or if the instructor did. Dark boxes and solid lines signal narratives that were authored by students and light boxes and broken lines signal narratives that were authored by the instructor. This process is exemplified in detail in the findings section.

FINDINGS

Mapping the potential of a task

This section describes the process used to construct a DMT and the potential, as revealed by the DMT, for the following task.

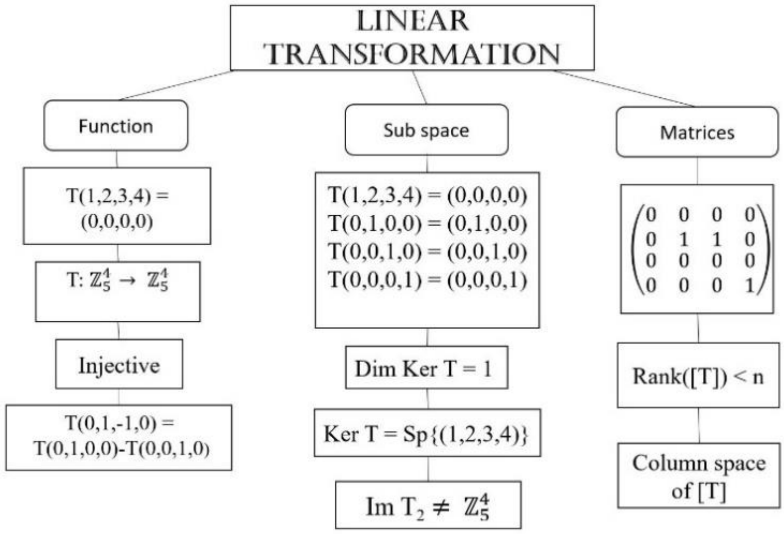
> **The linear transformation task** T: $\mathbb{Z}_5^4 \rightarrow \mathbb{Z}_5^4$ is a linear transformation such that T(1,2,3,4) = (0,0,0,0). For which values of n ϵ $\mathbb{N}$ does there exist such a T so that dim Ker T = n? For those values of n give an example of such a T and find a basis of Ker T.

The first step of constructing a DMT (see Fig. 1) is determining the root node. Solving the linear transformation task includes defining a linear transformation with certain properties, thus we determined the root node to be a "linear transformation".

The next step is listing the object's realizations and classifying them into sub-Discourses. We examined the definitions given in textbooks, solutions, and student discourse in workshops, in tutorials, in exams, in homework sets and in questions posed for realizations. These realizations were classified based on the following discourses. A linear transformation is a type of function, thus it can be realized in the Discourse of functions. This includes narratives about the image of vectors, such as T(1,2,3) = (3,3,3), *the image of a vector (x,y,z) is (x+y,x+y,x+y)* and *the linear transformation is injective*. A linear transformation can also be realized using vector spaces. Within this Discourse, a linear transformation can be realized by its definition on any basis. Narratives within the Discourse of vector spaces include *the linear transformation is uniquely determined by defining it on a basis*. Additionally, a linear transformation can also be realized by a matrix representation. The Discourse of matrices includes narratives such as *the linear transformation is invertible since the matrix is invertible.* Linear transformations can also be realized as elements of Hom(V,W). This notion is not included in the curriculum of the linear algebra course examined in this study, thus is not displayed on the DMT. Thus, our a priori DMT analysis revealed that, in this course, the mathematical object linear transformation could be realized in three sub-Discourses – functions, vector spaces, and matrices. Typical realizations were drawn in boxes on the appropriate branches of the DMT (see Fig. 1).

The a priori DMT offers a clear image of the object that can be exposed through the task and its realizations in three different Discourses. Moreover, the DMT demonstrates the opportunity for object level learning and meta-level learning available in this task. Object level opportunities include authoring a realization within a sub-Discourse or saming between two realizations within the same sub-Discourse. For example, the narrative *the kernel is spanned by a single vector and thus the dimension of the kernel is 1* links between two realizations within the sub-Discourse of subspaces. Practicing routines within a sub-Discourse, such as determining the general element of the kernel from a spanning set, is also object-level learning available in this task. Meta-level opportunities include authoring narratives in the coalesced Discourse connecting between two sub-Discourses. For example, the narrative *the image of (1,2,3,4) is (0,0,0,0) so the dimension of the kernel is not zero* connects between a realization in the functions sub-Discourse (T(1,2,3,4) = (0,0,0,0)) and a realization in the vector space sub-Discourse (dim Ker T ≠ 0). Such a narrative, if authored during the discussion, provides an instance of coalesced discourse which is part of the meta-level learning sought through this task.

Figure 1 - DMT for linear transformation task



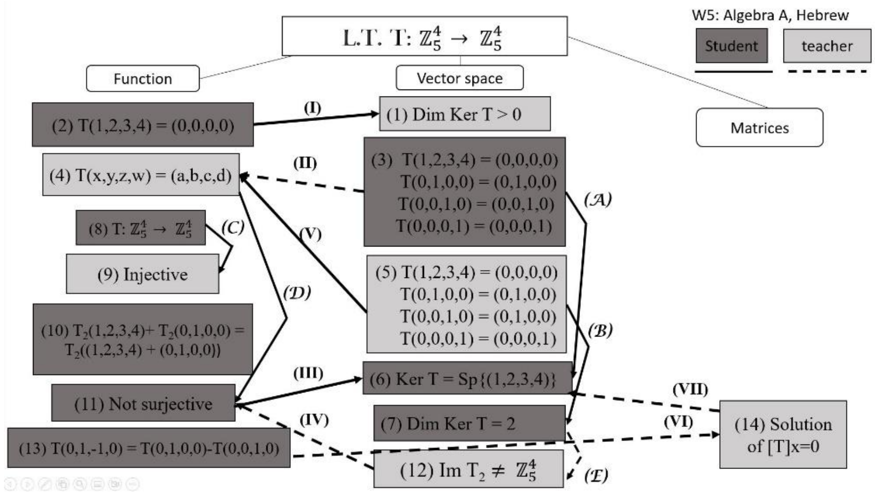
Mapping the implementation of a task

We now exemplify how an implementation of the task was mapped onto the DMT. The workshop on which we focus started with a short reminder of the basic theorems and definitions pertaining to linear transformations. Some of these were in the sub-Discourse of functions and some were in the sub-Discourse of vector spaces. The students were familiar with these narratives from the previous lectures and tutorials. After the launch of the task, the students worked on the task in pairs for 15 minutes. This was followed by a whole class discussion that was 21 minutes long. We use the DMT to map the whole class discussion. There were seven students in the classroom, and they all participated in the discussion. Some talked from their seats, and some came to the board to write out examples or to point to examples already written.

The mapping of the implementation commences by deriving the node and the sub-Discourses from the a priori DMT, described in the previous section. The realizations that were used to determine the sub-Discourses for the DMT are removed. They indicate hypothetical narratives, that may or may not be authored in the class and are only used as examples to map the sub-Discourses and potential links between them. The a posteriori DMT analyzes the discourse in the discussion and maps the narratives authored in class onto the sub-Discourses identified and the connections made between these narratives. Figure 2 presents the DMT mapping of the whole class discussion.

The a posteriori DMT in Figure 2 shows that while there were three available sub-Discourses, the discussion included mostly narratives that belonged to only two of them. The sub-Discourse of matrices, mentioned only briefly by the instructor, was not used by the students at all. In the other two sub-Discourses, the realizations and links were authored both by the instructor and by the students in a mostly balanced manner. Yet there is a noticeable difference between these two sub-Discourses. In the functions sub-Discourse, five out of the seven realizations mentioned were authored by the students, signalling they favored this sub-Discourse, whereas in the subspace sub-Discourse the division is more equal.

Figure 2 - DMT for whole class discussion



We now exemplify the take up of object-level and meta-level learning that occurred during the whole class discussion. During the discussion a student said, “We can define the linear transformation by its behavior on the basis” and wrote this on the board (see Fig. 2, box 3). The instructor agreed and added that the final answer would need to be given for a general vector (4) similar to a function. The instructor outlined how to do so and connected between the realizations (II). The discussion then turned to the kernel of this transformation and a student stated that the kernel is the span of (1,2,3,4) (6, $\mathcal{A}$). The discussion that next ensued about the linear transformation on a general vector (4) within the function subdiscourse sparked a student’s question does such a definition define a function. The student’s explanation of his question and the instructor’s statements elicited from other students that the transformation is not surjective ($\mathcal{D}$,11). The instructor then asked for a connection to the kernel (6), written on the board previously and students authored this (III).

In the excerpt described there was both object-level learning and meta-level learning. The students authored object-level links ($\mathcal{A}$,$\mathcal{D}$) within each sub-Discourse. The instructor, in contrast, stressed the meta-level links (II,III) and put less emphasis on the object-level narratives. More generally, the DMT displays that the students authored more object-level narratives during this discussion. The instructor moved the discussion to include meta-level narratives, as seen by the four meta-level inks authored by the instructor (II, IV, VI, VII), as opposed to a single object-level link ($\mathcal{E}$). Additionally, the links authored by the instructor show that when the discussion was in the functions sub-Discourse, the instructor pushed it to the other ones (II, IV, VI). The potential for object-level learning was taken up by the students authoring narratives mostly within the function sub-Discourse. The potential for meta-level learning was offered mainly by the instructor authoring or supporting the students to author meta-level narratives between sub-Discourses.

DISCUSSION

This study examined what the construction of the DMT highlights about the mathematical potential of the task and about the take up of this potential. The mathematical potential of the task was analyzed based on a priori analysis of mathematical Discourses involved in the task, and an implementation of the task was examined a posteriori based on the discourse that occurred during the classroom discussion. We found that constructing the DMT a priori emphasized the opportunities for both object-level learning and meta-level learning afforded by the task. Additionally, mapping the task afforded a structured look at the various mathematical sub-Discourses involved in solving the task. The a posteriori DMT displayed an image of the whole class discussion that showed how the potential for object-level learning and meta-level learning offered by the task were taken up in an implementation.

The construction of these DMTs highlighted the difference between the expectations from the task and the implementation of the task in terms of the mathematical potential. For example, while the DMT of the task showed that the potential for involving three sub-Discourses in the discussion was available, the implementation only focused on two sub-Discourses. The highlighting of this neglect shows the affordances of a tool which can determine the mathematical potential of a task and the take up of this. The DMT tool's examination of the mathematical potential of a task, independent of the context in which it will be used, allows analysis of tasks before they are implemented in a classroom setting and supports choosing appropriate tasks for use.

The DMT also helps to differentiate between object-level learning and meta-level learning. In the workshop discussion we analyzed, the a posteriori DMT showed that the instructor was more responsible for authoring meta-level links. The students easily and more readily authored object-level links, with little or no support from the instructor. This aligns with Nachlieli and Elbaum-Cohen's (2021) suggestion that student-centered instruction might support meta-level learning when strongly guided by an instructor who can explicate the rules of the new discourse and stress the limitations of the old, familiar discourse.

In this paper, we applied the DMT to one lesson, mainly to illustrate its construction. Applying the DMT to multiple lessons affords comparison of various aspects of the whole class discussion, similarly to was been achieved with the RTA (Weingarden et al., 2019). In the larger study, constructing DMTs for multiple workshops strengthened the impressions from the present analysis that the links between sub-Discourses were dependent on the students' familiarity with the narratives within the sub-Discourses and that there was usually a dominant Discourse which is either more familiar to the students or which includes familiar procedures (e.g. the function subdiscourse in this study) (Wallach, 2022). Constructing DMTs can thus be productive in examining various aspects of whole class discussions, especially in explorative and learner-centered forms of instruction.

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