# Explorative potential of linear algebra tasks

Miriam N. Wallach, Einat Heyd-Metzuyanim, Ram Band

# Explorative potential of linear algebra tasks

Miriam N. Wallach[1], Einat Heyd-Metzuyanim[1] and Ram Band[2]

[1]Technion – Israel Institute of Technology, Faculty of Education, Israel; miriamw@campus.technion.ac.il and einathm@technion.ac.il

[2]Technion – Israel Institute of Technology, Faculty of Mathematics, Israel; ramband@technion.ac.il

*Learner centered instruction, although becoming more prevalent, has been less explored in university settings. In this study, we examine tasks that were aimed at supporting student-centered learning in a linear algebra course. First, we constructed "realization trees", based on the commognitive theory, to examine the opportunities for exploring mathematical objects afforded by these tasks. This analysis revealed that all the tasks held the potential for transitioning between multiple discourses. Next, we used commognitive analysis of possible solution paths to examine the transitions between discourses within a solution. This analysis revealed that the task encouraged such transitions by including impasses, where the students had no available routines to continue within the discourse. The contribution of this study comes from presenting a commognitive tool for examining the potential of tasks for affording explorative engagement.*

*Keywords: Commognition, linear algebra, explorative potential of task.*

## Introduction

Learner centered instruction, where students are engaged and involved in meaningful, explorative learning activities, has proven productive in the K-12 setting (e.g. Schoenfeld, 2014). Mathematical learning in this type of setting requires careful task design (Cooper & Lavie, 2021). Various considerations are mentioned in choosing tasks included in tertiary classrooms promoting inquiry-oriented teaching, including real world problems (Chang, 2011), conceptual inclusiveness (Stewart & Thomas, 2009) and the opportunity for student engagement in disciplinary practices (Zandieh et al., 2017). These do not provide operational characteristics of the task itself. We turned to K-12 tasks, which have been more broadly studied. There, one of the main characteristics is a high level of cognitive demand, which is operationally difficult to determine (Weingarden et al., 2019). We use commognitive tools to determine in a well-defined manner what are the characteristics of tasks that provide opportunities for productive mathematical discussions and afford meaningful, rich learning.

## Theoretical Background

Mathematical tasks may offer more or less opportunities for students to engage with mathematical concepts, ideas, and strategies (Sullivan et al., 2015). Many inquiry-based tasks are aimed at constructing cognitive conflicts, commognitive conflicts or boundary objects (Sfard, 2021) to elicit from the students the need for new mathematical objects, new rules or new discourses or to amend familiar ones. Once the students are motivated to adopt new narratives, new objects and new rules, they still need to actually adopt them and engage with them. Therefore, tasks geared towards this are also necessary.

In this study, we rely on the commognitive framework (Sfard, 2008) to assist in operationalizing the notion of a task that encourages the construction of intricate connections between the objects and routines involved in linear algebra and to examine what are the characteristics of tasks that provide opportunities for productive mathematical discussions.

The commognitive framework defines learning as changing one's discourse, as part of becoming a participant in a certain community. Discourse is defined within this framework as "set apart by its objects, the kinds of mediators used, and the rules followed by participants and thus defining different communities of communicating actors" (Sfard, 2008, p. 93). Learning mathematics thus involves familiarizing oneself with discursive objects that only exist in one's discourse. Forming narratives about such objects, when one has no familiarity with them and has not *objectified* the different signs and procedures involved in the discourse, can thus be done at first only *ritually*, that is, by imitation of more knowledgeable experts (Sfard, 2008). Objectification happens when students come to communicate about mathematical symbols (e.g. $\sqrt[8]{2}$) as representing objects in the world (e.g. "the roots of 2 of order 8 divide a circle into equal parts"). This is an important part of entering the discourse. *Saming* the different realizations of an object (e.g. $1 + i$ and $\sqrt{2}\mathrm{cis}\frac{\pi}{4}$) is an essential step towards such objectification.

In commognitive terms, the learning of linear algebra can be conceptualized as the process of becoming familiar with different realizations of algebraic objects (such as matrices, systems of linear equations, vector spaces, etc.) while saming the equivalent realizations and treating them as objects "existing" in the world. A "rich discussion" in commognitive terms includes saming between realizations (Weingarden et al., 2019). Thus, the potential of a task can be thought of as the capacity to provoke a discussion including the possible links and realizations and the possibility of compelling students to construct these links.

According to commognition (Sfard, 2008), learning can proceed in two fashions: the object level and meta-level. In object level learning, students gradually produce (or endorse) an increasing number of narratives about familiar mathematical objects (Sfard, 2008). This involves, for example, producing narratives about different matrices by using routines tailored for matrix manipulation (e.g. reducing a matrix to Echelon form). Yet, the learning of linear algebra is also rich with demands for meta-level learning, which involves becoming familiar with new objects, with new rules (meta-rules) for how narratives can be produced about these objects and with the connections of these with familiar objects and rules. For example, familiarizing oneself with the routines of manipulating vectors, which are in certain ways "isomorophically equivalent" (Sfard, 2008; p. 175) to manipulating matrices. Previous studies (not within the commognitive framework) have stressed the importance of making students aware of the equivalence of the various representations being studied (e.g. Selinski & Rasmussen, 2014). This shows that meta-level learning, i.e. adopting new discourses, is probably ubiquitous in linear algebra classrooms.

Mathematical discourses are hierarchical and recursive, where their objects (e.g. rationale numbers in $\mathbb{Q}$) are built upon previously established objects (e.g. whole numbers in $\mathbb{Z}$) (Sfard, 2008). Historically, new mathematical discourses were created either by several familiar discourses coalescing into one discourse or by a meta-level discourse *subsuming* an older one. This historical process may be reconstructed in the development of students' individualized discourse (Lavie & Sfard, 2019). When learners progress from one discourse to a subsuming one, the subsuming discourse includes an isomorphic copy of the old ones, as well as new objects and narratives that can only be realized in the new discourse.

Previous commognitive studies (Sfard, 2008) hinted that the coalescing of discourses, which involves meta-level learning, may be problematic in student-centered classrooms. This, since students usually cannot coalesce discourses (nor invent meta-rules) autonomously. The rules stating that the discourses are isomorphic have often been laboriously and slowly invented over hundreds of years, and cannot simply be "discovered" by students. Recently, Nachlieli and Elbaum-Cohen (2021) showed how student-centered instruction might support meta-level learning if strongly guided by an instructor who can explicate the new rules of the subsuming discourse and stress the limitations of the old, familiar discourse.

In linear algebra there are multiple discourses that first have to be adopted, and then coalesced into one single subsuming one. We exemplify this on the mathematical notion of systems of linear equations (SLE). Historically, there are several different domains that represent SLEs, as described by Andrews-Larson (2015). Originally, systems of constraints on everyday problems were described verbally. Next, linear systems and their solutions were described by Chinese mathematicians in 200 BC and by Gauss (early 19th century) without matrix notation. Significant advances in notation, including matrices and determinants, led to SLEs being described as mathematical objects, and not merely as a process to a solution. This allowed SLEs to be represented by their properties. The modern, formal, axiomatic definitions of relations and operations on vectors utilizes vector spaces and transformations to describe SLEs. Guided by the historical development of the representations of SLEs, we can divide the various realizations of SLEs into five main domains - the solution set (constraints), a list of equations, matrix notation, properties of SLEs and vector spaces and transformations. Learning linear algebra includes entering into each of these discourses and producing narratives within them via object-level learning and eventually coalescing these separate domains, or discourses, into one unified discourse of SLEs via meta-level learning.

Choosing appropriate tasks for supporting all of these learning processes is a complex project with multiple facets. One of them is that the tasks should afford the students opportunities for saming and objectification of the mathematical objects embedded in them. In addition, one should consider if object level-learning or meta-level learning is involved, and how each type of learning is supported. Based on this theorizing, we examined the tasks of the linear algebra workshops with the following research questions (RQs):

(1) What are the objects that can be exposed through the tasks, their different realizations and the opportunities for saming that can be afforded by these tasks? (2) Can the tasks support the coalescing of different discourses, and if so, how?

## Method

We designed workshops to encourage and support student-centered learning and academically productive mathematical discussions in a linear algebra course. These workshops used discussion-based teaching methods and included small group and whole classroom discussions around a given task. Our data consisted of seven tasks designed for these workshops. All tasks were tried at least in one workshop and some of them were refined through several iterations of design and implementation.

To answer the first RQ, regarding the objects exposed in the task, we made a novel use of an existing tool named the Realization Tree Assessment tool or RTA (Weingarden et al., 2019). Originally, this tool was used to map the engagement with mathematical objects in discussion-based lessons by mapping which realizations were mentioned during a discussion and which links were constructed. However, in this study, we used the tool in a different way. That is, we constructed empty RTAs as a way of examining the *potential* of a task.

An RTA uses the notion of an object being a "signifier together with its realization tree" (Sfard, 2008). It is a visual representation of realizations of a mathematical object and the hierarchy between them. The first step in constructing an RTA is determining the root node. This is not necessarily straightforward, since, theoretically, all realizations of an object are equivalent and thus any realization can be the root node. For convenience, we chose as the root node the title given to the object in textbooks (for example: "Complex number" or "Diagonalizable matrix"). The next step of constructing an RTA is listing the object's realizations, determining the possible relations between them, and grouping together realizations of similar type. Each type of realization usually belongs to a certain discourse; thus, it has its own keywords, its own narratives and its own routines of manipulation. Each type of realizations was placed in a separate branch of the RTA. This is illustrated in the findings section (Figure 1).

The process of constructing RTAs gave a general view of the richness embedded in each task. However, it did not allow us to see the extent to which a task demanded the use of more than one (object-level) branch. For this, we moved to a more micro-level analysis. The micro analysis involved coming up with possible solution paths (approved by mathematical experts) for each task. We next characterized each step of these solution paths (or routines) by what discourses they tapped, and which realizations of the object they utilized. Finally, we examined whether the task could be solved based on a single discourse by examining the application of familiar routines (taught and rehearsed in the course) to the task at hand, or whether it necessitated traversing multiple discourses.

# Findings

## RTAs for Linear Algebra Tasks

We start our findings with an exemplification of how one of the RTAs was constructed so that readers may gain a sense of how these RTAs can be interpreted. This RTA was built around the task: Give a system of linear equations (SLE) whose solution is the set $\{(x, 2x, 3x) \mid x \in \mathbb{R}\}$.

Solving this task includes, mainly, the exploration of the system of linear equations (SLE) object. Thus, this object was the node of the RTA. The separate branches of the RTA included the different types of realizations of the SLE mathematical object, as described above. Only examples of realizations that were part of the course's curriculum are represented in this RTA. The fifth discourse, the domain of vector spaces and transformations, was not yet introduced to the students at this point in the course.

The first form, in which an SLE can be realized, the one most familiar to secondary school students, is as a list of linear equations with variables. For example: $\begin{cases} 2x - y = 0 \\ 3x - z = 0 \end{cases}$ or $\begin{cases} 2x = y \\ 5x = y + z \end{cases}$

The next type of realization is introduced during the beginning of a linear algebra course, that of an augmented matrix. The notion of an SLE was introduced in the course as a list of equations, immediately followed by the augmented matrix notation. This notation is almost exclusively used by the instructors and students due to its efficiency. Specifically, for the SLE considered in this task, the matrix could look like this: $\left(\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 3 & 0 & -1 & 0 \end{array}\right)$. This is the domain of matrix notation.

A third way by which an SLE can be realized is by its *properties*. For example: the system consists of 2 equations with 3 variables, it is a homogenous system, and it has a system rank of 2. There are also properties of the SLE's solution set, such as: the zero vector is a solution, the number of degrees of freedom in the solution, and the number of parameters in the solution set. This type of realizations are of a family of SLEs, and not a unique one. This type of realizations aligns with the domain of the SLE as a mathematical object with properties, not as a process.

A fourth way that an SLE can be realized is simply by its solution set, in this example $\{(x, 2x, 3x) \mid x \in \mathbb{R}\}$. Although there are infinitely many SLEs with this solution set, they are equivalent in the sense that the Gaussian matrix representing these systems will have the same row space. This can be realized as a general element of a set: $\{(x,2x,3x) \mid x \in \mathbb{R}\}$, the linear span of a finite set: Span{(1,2,3), (4,8,12)}, or the kernel of a linear transformation Ker (T(x,y,z) = (y-2x,z-3x)). This is the domain of the constraints on the solution set.

The RTA in Figure 1 depicts the realizations and the domains described above. Notice it is constructed around the specific SLE searched for by the task, but other SLEs could be described similarly. Using the process described above, we constructed RTAs for six other tasks that were used in our linear algebra discussion-based workshops. A detailed view and discussion of each of these RTAs is beyond the scope of this paper. These RTAs demonstrate that, similar to the task represented in Figure 1, embedded in each task there were at least four different realizations available to the students (as seen in the major branches stemming from the root object). This shows us that there were multiple opportunities for students to same the various realizations of the mathematical objects. The RTAs thus provided a clear picture of the objects whose exploration was supported by the tasks, including their various realizations (RQ I).

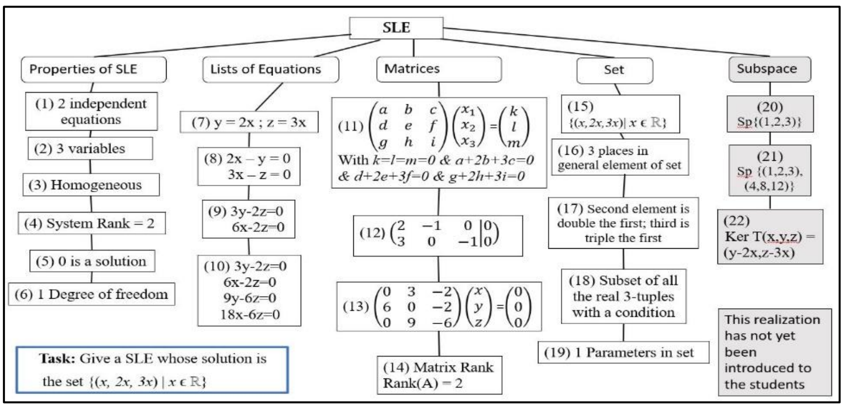


**Figure 1: RTA for SLE task**

## Micro analysis: Mapping a possible routine for solving the SLE task

In general, we found that all the tasks supported solution paths that traversed different realizations. These potential transitions appear when one follows a particular routine for solving the task. Here we demonstrate our analysis on one such routine for solving the SLE task.

We present the routine in Table 1 together with an analysis of the discourses traversed in each sub-routine. The nodes on the RTA (Figure 1) that represent each realization are marked in parentheses.

**Table 1: A possible routine for solving: Give a SLE whose solution is the set {(x, 2x, 3x) | x ϵ ℝ}**

| Routine sub-step | Discourse traversed |
|---|---|
| a) There are 3 places in the general element of the given set (16) that solves the SLE so there are 3 variables in the expected SLE (2). | Sets (general element) → Properties of SLE (3 variables) |
| (b) The general element of the set that would solve the expected SLE can be expressed using a single parameter (19), which is equivalent to stating that there is one degree of freedom in the expected SLE (6). | Sets (general element) → Properties of SLE (degree of freedom) |
| (c) The degrees of freedom of a SLE (6) is the number of variables (2) less the rank of the representative matrix (14), thus the rank of the system is 2 (4). That is, there are two independent equations in the expected SLE (1). | Properties of SLE (degree of freedom, number of variables) → Matrix discourse (rank of matrix) → SLE (two independent equations) |
| (d) The elements of the set that solves the SLE are the 3-tuples whose second element is double the first element and the third element is triple the first element (17), thus the conditions on the set can be expressed as y = 2x & z = 3x & x, y, z ϵ ℝ (hybrid between 7 & 18) or as 2x – y = 0 & 3x – z = 0 & x, y, z ϵ ℝ (hybrid between 8 & 18). | Sets (elements of the set, 3-tuples, etc.) → System of equations (e.g. y=2x) |
| (e) The solution to the task is the SLE which is $\begin{cases} 2x - y = 0 \\ 3x - z = 0 \end{cases}$ (8) or $\left(\begin{array}{ccc\|c} 2 & -1 & 0 & 0 \\ 3 & 0 & -1 & 0 \end{array}\right)$ (12). | Systems of equations, matrices, SLEs. |

As can be seen in Table 1, this routine tapped four different discourses – Properties of SLEs, Matrices, List of Equations and Sets. In fact, the wording of the task by itself includes transitions between discourses. Thus, the beginning of the task - "Give a SLE whose solution is…" - belongs to the List of Equations discourse involving "equations" and "solutions" of these equations. However, if a student stayed only in the List of Equations discourse, by employing, for example, the routines of solving sets of equations or by Gaussian elimination, they would reach an impasse. Those routines are appropriate for finding a solution of a given system. However, in this task a student must first construct a system. The wording in the last part of the task, namely "the *solution* is the *set* {(*x*, *2x*, *3x*) | *x* ϵ ℝ}", belongs to the Sets discourse. As the expected answer is a list of equations or a matrix representation, the Sets discourse is also not sufficient for solving the task. Thus, any solution necessitates tapping multiple discourses and linking between them.

It is worth noting that there might be students whose individual routines for lists of equations is still finding a single solution. The familiar use of these lists, as is presented in secondary school, tends to be finding the intersection of lines. Unless they have successfully samed the different realizations of the SLE discourse, these students are still likely to rely on the familiar routine of looking for single points as realizations of SLEs. Thus, for these students this task supports the meta-level shift that systems of equation can have solutions which are sets of points rather than a single point.

The assumption underlying this task is that students *have already been* introduced to the various subsumed discourses underlying the SLE (full linear algebra) discourse, as well as to the equivalence of the various realizations. This task is not supposed to *introduce* students to new meta-rules but rather to support them in enacting and rehearsing the saming actions critical for the objectification of SLEs. Given the difficulty of meta-level learning, it is conceivable that the instructor would have an important role in supporting students' struggle with such tasks. Thus, when students reach an impasse, the instructor should guide them to search in other discourses for a possible routine.

To summarize, the micro-analysis of one possible solution path indicated that such a solution path included several transitions between discourses. Moreover, examining sub-routines along the way revealed that most probably, none of the familiar routines at the object-level (in one subsumed discourse) would have sufficed to solve the task. The task thus afforded, in fact demanded, both object-level and meta-level learning.

## Discussion

Our goal in this study was to examine whether tasks we posed afforded meaningful, rich opportunities for saming different realizations of linear algebra objects, and traversing the discourses involved in this domain. We found that RTAs provide a clear picture of the objects for which the tasks support exploration, including their various realizations. Our findings further revealed that the task afforded both the construction of object-level narratives, as well as traversing between different discourses. Increasingly, tertiary classrooms have been including inquiry-based practices to encourage student engagement and meaningful participation (e.g. Zandieh et al., 2017). Our work may advance these attempts by providing tools to characterize the tasks underlying such practices.

This study is limited to the focus on seven tasks from one course. Nevertheless, it has several contributions. Methodologically it makes a novel use of the RTA tool as a tool for examining the potential of tasks. This may link to works on cognitive demand of tasks and further elucidate what tasks that support rich discussion entail (Tekkumru-Kisa et al., 2020). The second contribution is related to our study of the feasibility and design principles of discussion-based linear algebra lessons. Mapping the potential of tasks can serve as the first step towards understanding the ways by which linear algebra instruction can become more exploration-requiring, despite the multiple demands for meta-level learning (Nachlieli & Elbaum-Cohen, 2021; Nachlieli & Tabach, 2019). Thus, this study may be a first step towards an operational method of evaluating tasks by mapping objects, links between realizations, and the relevant discourses to which they belong.

**Acknowledgement:** We thank Merav Weingarden for her assistance with the RTA's construction.

## References

Andrews-Larson, C. (2015). Roots of linear algebra: An historical exploration of linear systems. *Primus*, *25*(6), 507–528. https://doi.org/10.1080/10511970.2015.1027975

Chang, J. M. (2011). A practical approach to inquiry-based learning in linear algebra. *International Journal of Mathematical Education in Science and Technology*, *42*(2), 245–259. https://doi.org/10.1080/0020739X.2010.519795

Cooper, J., & Lavie, I. (2021). Bridging incommensurable discourses – A commognitive look at instructional design in the zone of proximal development. *Journal of Mathematical Behavior*, *61*(October 2020), 1–9. https://doi.org/10.1016/j.jmathb.2020.100822

Lavie, I., & Sfard, A. (2019). How children individualize numerical routines: Elements of a discursive theory in making. *Journal of the Learning Sciences*, *28*(4–5), 419–461. https://doi.org/10.1080/10508406.2019.1646650

Nachlieli, T., & Elbaum-Cohen, A. (2021). Teaching practices aimed at promoting meta-level learning: The case of complex numbers. *Journal of Mathematical Behavior*, *62*(February), 1–13. https://doi.org/10.1016/j.jmathb.2021.100872

Nachlieli, T., & Tabach, M. (2019). Ritual-enabling opportunities-to-learn in mathematics classrooms. *Educational Studies in Mathematics*, *101*(2), 253–271. https://doi.org/10.1007/s10649-018-9848-x

Schoenfeld, A. H. (2014). What makes for powerful classrooms, and how can we support teachers in creating them? A story of research and practice, productively intertwined. *Educational Researcher*, *43*(8), 404–412.

Selinski, N., & Rasmussen, C. (2014). A method for using adjacency matrices to analyze the connections students make within and between concepts: The case of linear algebra. *Journal for Research in Mathematics Education*, *45*(5), 550–583.

Sfard, A. (2008). *Thinking as Communicating: Human development, the growth of discourses, and mathematizing*. Cambridge University Press.

Sfard, A. (2021). Taming fantastic beasts of mathematics: Struggling with incommensurability. *International Journal of Research in Undergraduate Mathematics Education*, *1–33*. https://doi.org/10.1007/s40753-021-00156-7

Stewart, S., & Thomas, M. O. J. (2009). A framework for mathematical thinking: The case of linear algebra. *International Journal of Mathematical Education in Science and Technology*, *40*(7), 951–961. https://doi.org/10.1080/00207390903200984

Sullivan, P., Knott, L., & Yang, Y. (2015). The relationships between task design, anticipated pedagogies, and student learning. In A. Watson & M. Ohtani (Eds.), *Task Design in Mathematics Education, New ICMI Study Series* (pp. 83–114). https://doi.org/10.1007/978-3-319-09629-2_3

Tekkumru-Kisa, M., Stein, M. K., & Doyle, W. (2020). Theory and research on tasks revisited: Task as a context for students' thinking in the era of ambitious reforms in mathematics and science. *Educational Researcher*, *49*(8), 606–617. https://doi.org/10.3102/0013189X20932480

Weingarden, M., Heyd-Metzuyanim, E., & Nachlieli, T. (2019). The realization tree assessment tool – examining explorative participation in mathematics lessons. *Journal of Mathematical Behavior*, *56*(May), 1–13. https://doi.org/10.1016/j.jmathb.2019.100717

Zandieh, M., Wawro, M., & Rasmussen, C. (2017). *An Example of Inquiry in Linear Algebra : The Roles of Symbolizing and Brokering An Example of Inquiry in Linear Algebra : The Roles of Symbolizing and Brokering*. *1970*. https://doi.org/10.1080/10511970.2016.1199618