What one cannot hear?
Quantum graphs which sound the same

Rami Band, Ori Parzanchevski, Gilad Ben-Shach
‘Can one hear the shape of a drum?’

- This question was asked by Marc Kac (1966).

- Is it possible to have two different drums with the same spectrum (isospectral drums)?
The spectrum of a drum

- A **Drum** is an elastic membrane which is attached to a solid planar frame.
- The spectrum is the set of the Laplacian’s eigenvalues, \( \{ \lambda_n \}_{n=1}^{\infty} \), (usually with Dirichlet boundary conditions):

\[
- \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f = \lambda f \quad f \big|_{\text{boundary}} = 0
\]

- A few eigenfunctions of the Sinai ‘drum’:
Isospectral drums


‘One cannot hear the shape of a drum’

Using Sunada’s construction (1985)
‘Can one hear the shape of the universe?’

- How do we produce isospectral examples?
- What geometrical / topological properties we can hear?
Metric Graphs - Introduction

- A graph $\Gamma$ consists of a finite set of vertices $V=\{v_i\}$ and a finite set of edges $E=\{e_j\}$.
- A metric graph has a finite length ($L_e > 0$) assigned to each edge.
- A function on the graph is a vector of functions on the edges:
  \[ f = (f_{e_1}, \ldots, f_{e_{|E|}}) \quad f_{e_j} : [0, L_{e_j}] \rightarrow \mathbb{D} \]
Quantum Graphs - Introduction

- A quantum graph is a metric graph equipped with an operator, such as the negative Laplacian:
  \[- \Delta f = (-f''|_{e_1}, \ldots, -f''|_{e_{|E|}})\]

- For each vertex \(v\), we impose vertex conditions, such as
  - **Neumann**
    - Continuity \(\forall e_1, e_2 \in E_v \quad f|_{e_1}(v) = f|_{e_2}(v)\)
  - Zero sum of derivatives \(\sum_{e \in E_v} f'|_e(v) = 0\)
  - **Dirichlet**
    - Zero value at the vertex \(\forall e \in E_v \quad f|_e(v) = 0\)

- A quantum graph is defined by specifying:
  - Metric graph
  - Operator
  - Vertex conditions for each vertex
The Spectrum of Quantum Graphs

We are interested in the eigenvalues of the Laplacian:

\[-\Delta f = \lambda f \implies (-f''|_{e_1}, \ldots, -f''|_{e_{|E|}}) = (\lambda f|_{e_1}, \ldots, \lambda f|_{e_{|E|}})\]

Examples of several eigenfunctions of the Laplacian on the graph:

So...

‘Can one hear the shape of a graph?’
One can hear the shape of a simple graph if the lengths are incommensurare (Gutkin, Smilansky 2001)

Otherwise, we do have isospectral graphs:
- Roth (1984)
- VonBelow (2001)
- Band, Shapira, Smilansky (2006)
- Kurasov, enerback (2010)

There are several methods for construction of isospectrality – the main is due to Sunada (1985).

We present a method based on representation theory arguments which generalizes Sunada’s method.
Theorem (R.B., Ori Parzanchevski, Gilad Ben-Shach)

Let $\Gamma$ be a graph which obeys a symmetry group $G$. Let $H_1, H_2$ be two subgroups of $G$ with representations $R_1, R_2$ that satisfy

$$\text{Ind}_{H_1}^G R_1 \cong \text{Ind}_{H_2}^G R_2$$

then the graphs $\Gamma_{/R_1}, \Gamma_{/R_2}$ are isospectral.
Constructing Quotient Graphs

- Example - A string with Dirichlet vertex conditions.
- It obeys the symmetry group $Z_2 = \{id, r\}$.
- Two representations of $Z_2$ are:
  
  $R_1: \{id \rightarrow (1), r \rightarrow (1)\}$

  $R_2: \{id \rightarrow (1), r \rightarrow (-1)\}$

We may encode these functions by the following *quotient graphs*:

\[ \frac{\Gamma}{R_1} \quad \frac{\Gamma}{R_2} \]
Example: The Dihedral group – the symmetry group of the square

\[ G = \{ \text{id} , a , a^2 , a^3 , r_x , r_y , r_u , r_v \} \]

How does the Dihedral group act on a square?

Two subgroups of the Dihedral group:

\[ H_1 = \{ \text{id} , a^2 , r_x , r_y \} \]
\[ H_2 = \{ \text{id} , a^2 , r_u , r_v \} \]
Groups - Representations

- **Representation** – Given a group $G$, a representation $R$ is an assignment of a matrix $\rho_R(g)$ to each group element $g \in G$, such that: $\forall g_1, g_2 \in G \quad \rho_R(g_1) \cdot \rho_R(g_2) = \rho_R(g_1 g_2)$.

- **Example 1** - $G$ has the following 1-dimensional representation

  $\text{id} \rightarrow (1) \quad a \rightarrow (-1) \quad a^2 \rightarrow (1) \quad a^3 \rightarrow (-1) \quad r_x \rightarrow (-1) \quad r_y \rightarrow (-1) \quad r_u \rightarrow (1) \quad r_v \rightarrow (1)$

- **Example 2** - $G$ has the following 2-dimensional representation

  \[
  \begin{align*}
  \text{id} & \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
a & \rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
a^2 & \rightarrow \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\
a^3 & \rightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
r_x & \rightarrow \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\
r_y & \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
r_u & \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
r_v & \rightarrow \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}
  \end{align*}
  \]

- **Induction**: take a representation of $H_1$...

  $\text{id} \rightarrow (1) \quad a^2 \rightarrow (1) \quad r_x \rightarrow (-1) \quad r_y \rightarrow (-1)$

  ...And turn it into a representation of $G$ (which we denote $\text{Ind}^G_{H_1} R$)
Isospectral theorem

Theorem (R.B., Ori Parzanchevski, Gilad Ben-Shach)

Let \( \Gamma \) be a graph which obeys a symmetry group \( G \). Let \( H_1, H_2 \) be two subgroups of \( G \) with representations \( R_1, R_2 \) that satisfy \( \text{Ind}_{H_1}^G R_1 \cong \text{Ind}_{H_2}^G R_2 \)

then the graphs \( \Gamma_{R_1}, \Gamma_{R_2} \) are isospectral.

- An application of the theorem with:
  \[
  G = \{\text{id}, a, a^2, a^3, r_x, r_y, r_u, r_v\}
  \]

Two subgroups of \( G \):
\[
H_1 = \{\text{id}, a^2, r_x, r_y\}
\]
\[
H_2 = \{\text{id}, a^2, r_u, r_v\}
\]

We choose representations
\[
R_1 : \{\text{id} \rightarrow (1), a^2 \rightarrow (-1), r_x \rightarrow (-1), r_y \rightarrow (1)\}
\]
\[
R_2 : \{\text{id} \rightarrow (1), a^2 \rightarrow (-1), a_u \rightarrow (1), a_v \rightarrow (-1)\}
\]

such that \( \text{Ind}_{H_1}^G R_1 \cong \text{Ind}_{H_2}^G R_2 \)
Consider the following rep. $R_1$ of the subgroup $H_1$:

$$R_1 : \left\{ \text{id} \rightarrow (1) \quad a^2 \rightarrow (-1) \quad r_x \rightarrow (-1) \quad r_y \rightarrow (1) \right\}$$

We construct $\Gamma / R_1$ by inquiring what do we know about a function $f$ on $\Gamma$ which transforms according to $R_1$.

$$r_x f = -f \quad r_y f = f$$

Dirichlet \quad Neumann

The construction of a *quotient graph* is motivated by an *encoding scheme*. 
Consider the following rep. $R_1$ of the subgroup $H_1$:

$$R_1 : \{ \text{id} \rightarrow (1) \ a^2 \rightarrow (-1) \ r_x \rightarrow (-1) \ r_y \rightarrow (1) \}$$

We construct $\Gamma / R_1$ by inquiring what do we know about a function $f$ on $\Gamma$ which transforms according to $R_1$.

$$r_x f = -f \quad r_y f = f$$

Consider the following rep. $R_2$ of the subgroup $H_2$:

$$R_2 : \{ \text{id} \rightarrow (1) \ a^2 \rightarrow (-1) \ r_u \rightarrow (1) \ r_v \rightarrow (-1) \}$$

We construct $\Gamma / R_2$ by inquiring what do we know about a function $g$ on $\Gamma$ which transforms according to $R_2$.

$$r_u g = g \quad r_v g = -g$$

Neumann \quad Dirichlet
Theorem (R.B., Ori Parzanchevski, Gilad Ben-Shach)

Let $\Gamma$ be a graph which obeys a symmetry group $G$. Let $H_1, H_2$ be two subgroups of $G$ with representations $R_1, R_2$ that satisfy

$$\text{Ind}^G_{H_1} R_1 \cong \text{Ind}^G_{H_2} R_2$$

then the graphs $\Gamma/R_1, \Gamma/R_2$ are isospectral.
Extending the Isospectral pair

Extending our example: \[ \text{Ind}_{H_1}^G R_1 \cong \text{Ind}_{H_2}^G R_2 \cong \text{Ind}_{H_3}^G R_3 \]

\[ H_1 = \{ e, a^2, r_x, r_y \} \quad R_1: \quad e \rightarrow (1) \quad a^2 \rightarrow (-1) \quad r_x \rightarrow (-1) \quad r_y \rightarrow (1) \]

\[ H_2 = \{ e, a^2, r_u, r_v \} \quad R_2: \quad e \rightarrow (1) \quad a^2 \rightarrow (-1) \quad r_u \rightarrow (1) \quad r_v \rightarrow (-1) \]

\[ H_3 = \{ e, a, a^2, a^3 \} \quad R_3: \quad e \rightarrow (1) \quad a \rightarrow (i) \quad a^2 \rightarrow (-1) \quad a^3 \rightarrow (-i) \]

\[ a f = i f \]
Extending the Isospectral pair

Extending our example: \( \text{Ind}^G_{H_1} R_1 \cong \text{Ind}^G_{H_2} R_2 \cong \text{Ind}^G_{H_3} R_3 \)

\( H_1 = \{ e, a^2, r_x, r_y \} \quad R_1: \quad e \rightarrow (1) \quad a^2 \rightarrow (-1) \quad r_x \rightarrow (-1) \quad r_y \rightarrow (1) \)

\( H_2 = \{ e, a^2, r_u, r_v \} \quad R_2: \quad e \rightarrow (1) \quad a^2 \rightarrow (-1) \quad r_u \rightarrow (1) \quad r_v \rightarrow (-1) \)

\( H_3 = \{ e, a, a^2, a^3 \} \quad R_3: \quad e \rightarrow (1) \quad a \rightarrow (i) \quad a^2 \rightarrow (-1) \quad a^3 \rightarrow (-i) \)

\[ a f = i f \]
Extending the Isospectral pair

Extending our example:

\[ \text{Ind}_{H_1}^G R_1 \cong \text{Ind}_{H_2}^G R_2 \cong \text{Ind}_{H_3}^G R_3 \]

\[ H_1 = \{ e, a^2, r_x, r_y \} \quad R_1: \quad e \rightarrow 1 \quad a^2 \rightarrow -1 \quad r_x \rightarrow -1 \quad r_y \rightarrow 1 \]

\[ H_2 = \{ e, a^2, r_u, r_v \} \quad R_2: \quad e \rightarrow 1 \quad a^2 \rightarrow -1 \quad r_u \rightarrow 1 \quad r_v \rightarrow -1 \]

\[ H_3 = \{ e, a, a^2, a^3 \} \quad R_3: \quad e \rightarrow 1 \quad a \rightarrow i \quad a^2 \rightarrow -1 \quad a^3 \rightarrow -i \]

\[ af = if \]
Extending the Isospectral pair

Extending our example: \[ \text{Ind}_{H_1}^G R_1 \cong \text{Ind}_{H_2}^G R_2 \cong \text{Ind}_{H_3}^G R_3 \]

\[
H_1 = \{ e, a^2, r_x, r_y \} \quad R_1: \quad e \rightarrow (1) \quad a^2 \rightarrow (-1) \quad r_x \rightarrow (-1) \quad r_y \rightarrow (1)
\]

\[
H_2 = \{ e, a^2, r_u, r_v \} \quad R_2: \quad e \rightarrow (1) \quad a^2 \rightarrow (-1) \quad r_u \rightarrow (1) \quad r_v \rightarrow (-1)
\]

\[
H_3 = \{ e, a, a^2, a^3 \} \quad R_3: \quad e \rightarrow (1) \quad a \rightarrow (i) \quad a^2 \rightarrow (-1) \quad a^3 \rightarrow (-i)
\]

\[a f = i f\]
Arsenal of isospectral examples

Γ is the Cayley graph of \( G=D_4 \)
(with respect to the generators \( a, r_x \)):

Take the same group and the subgroups:
\( H_1 = \{ e, a^2, r_x, r_y \} \) with the rep. \( R_1 \)
\( H_2 = \{ e, a^2, r_u, r_v \} \) with the rep. \( R_2 \)
\( H_3 = \{ e, a, a^2, a^3 \} \) with the rep. \( R_3 \)

The resulting quotient graphs are:
Arsenal of isospectral examples

\[ G = D_6 = \{ e, a, a^2, a^3, a^4, a^5, r_x, r_y, r_z, r_u, r_v, r_w \} \]
with the subgroups:
\[ H_1 = \{ e, a^2, a^4, r_x, r_y, r_z \} \] with the rep. \( R_1 \)
\[ H_2 = \{ e, a^2, a^4, r_u, r_v, r_w \} \] with the rep. \( R_2 \)
\[ H_3 = \{ e, a, a^2, a^3, a^4, a^5 \} \] with the rep. \( R_3 \)

The resulting quotient graphs are:
$G = S_3 \ (D_3)$ acts on $\Gamma$ with no fixed points.
To construct the quotient graph, we take the same rep. of $G$, but use two different bases for the matrix representation.

The resulting quotient graphs are:
Why quantum graphs? Why not drums?

Following Martin Sieber

However, $\Gamma/R_3$ is not a planar drum:
Arsenal of isospectral examples

Isospectral drums


This isospectral quartet can be obtained when acting with the group $D_4 \times D_4$ on the following torus:
Arsenal of isospectral examples

Isospectral drums

‘One cannot hear the shape of a drum’
Gordon, Webb and Wolpert (1992)

We construct the known isospectral drums of Gordon et al.
but with new boundary conditions:
What one cannot hear?
On drums\graphs which sound the same

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R. Band, O. Parzanchevski and G. Ben-Shach,
"The Isospectral Fruits of Representation Theory: Quantum Graphs and Drums",

O. Parzanchevski and R. Band,
"Linear Representations and Isospectrality with Boundary Conditions",