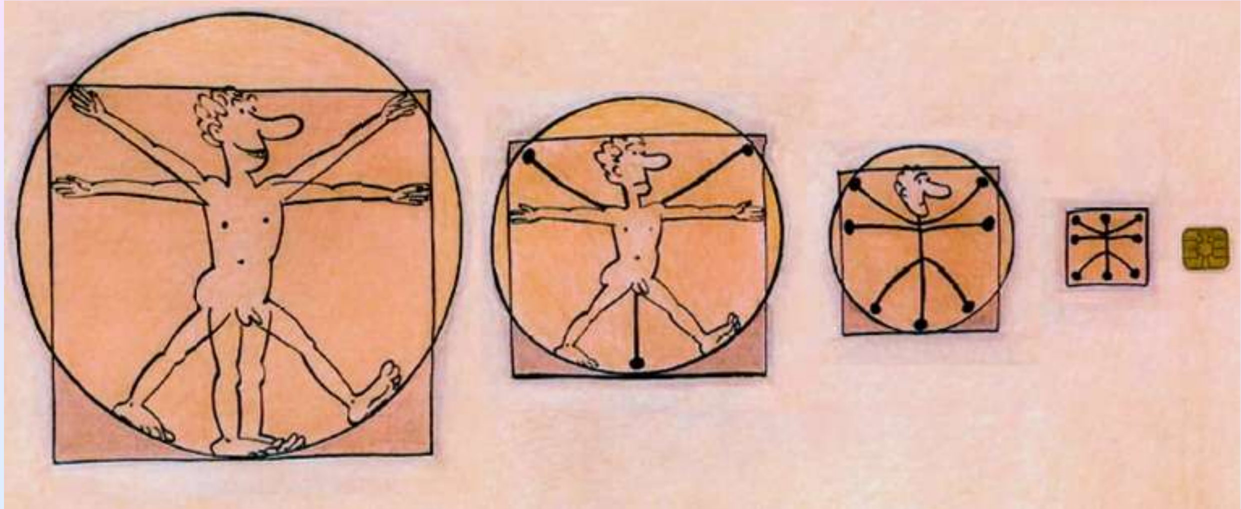
What one cannot hear?
On drums which sound the same

Rami Band, Ori Parzanchevski, Gilad Ben-Shach



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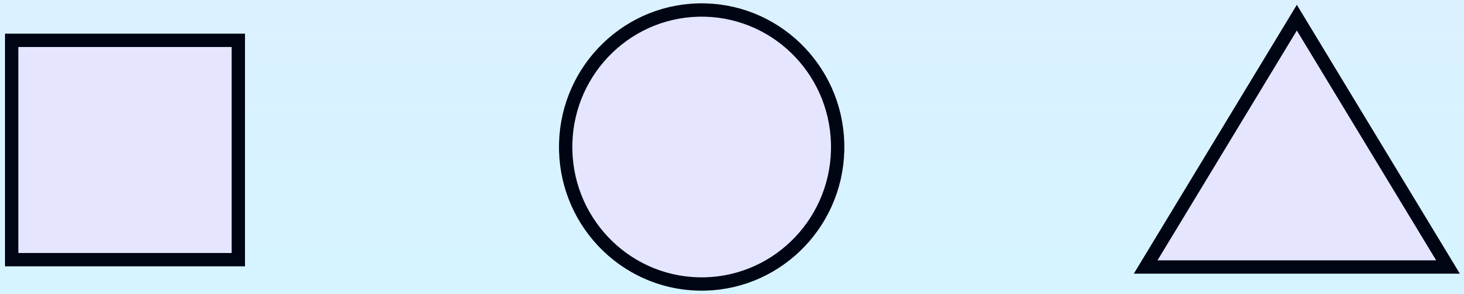
'Can one hear the shape of a drum ?'

- This question was asked by Marc Kac (1966).

Marc Kac (1914-1984)

- Is it possible to have two different drums with the same spectrum (***isospectral drums***) ?

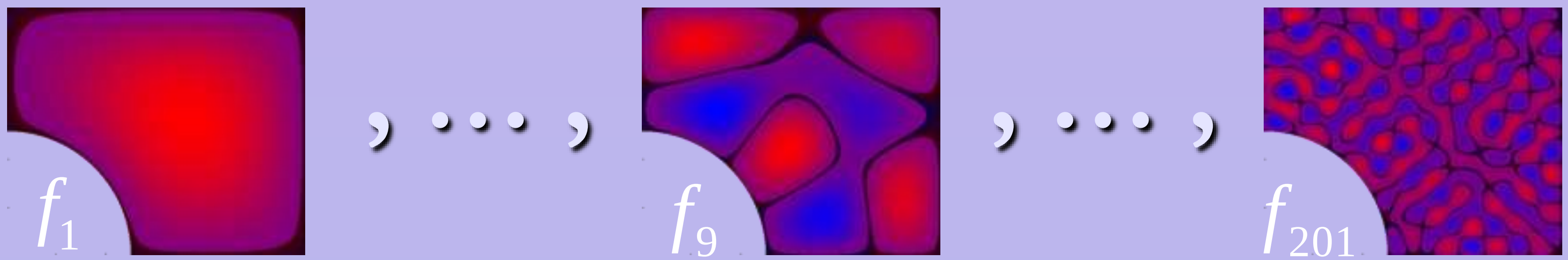


The spectrum of a drum

- A **Drum** is an elastic membrane which is attached to a solid planar frame.
- The spectrum is the set of the Laplacian's eigenvalues, $\{\lambda\}_{n=1}^{\infty}$, (usually with Dirichlet boundary conditions):

$$-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)f=\lambda f \qquad f\big|_{boundary}=0$$

- A few wavefunctions of the Sinai 'drum':

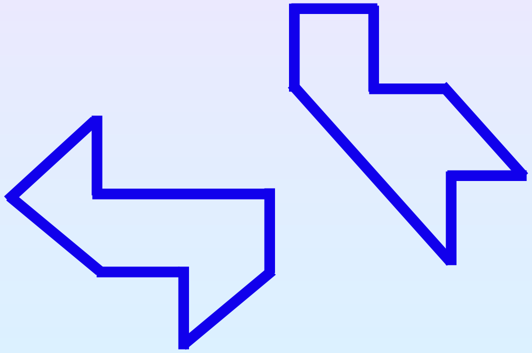


$$f_1, \cdots, f_9, \cdots, f_{201}$$

Isospectral drums

Gordon, Webb and Wolpert (1992):

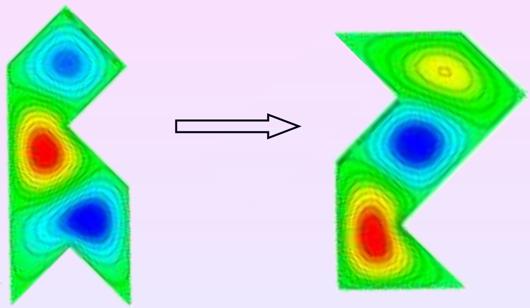
‘One cannot hear the shape of a drum’



Using Sunada’s construction (1985)

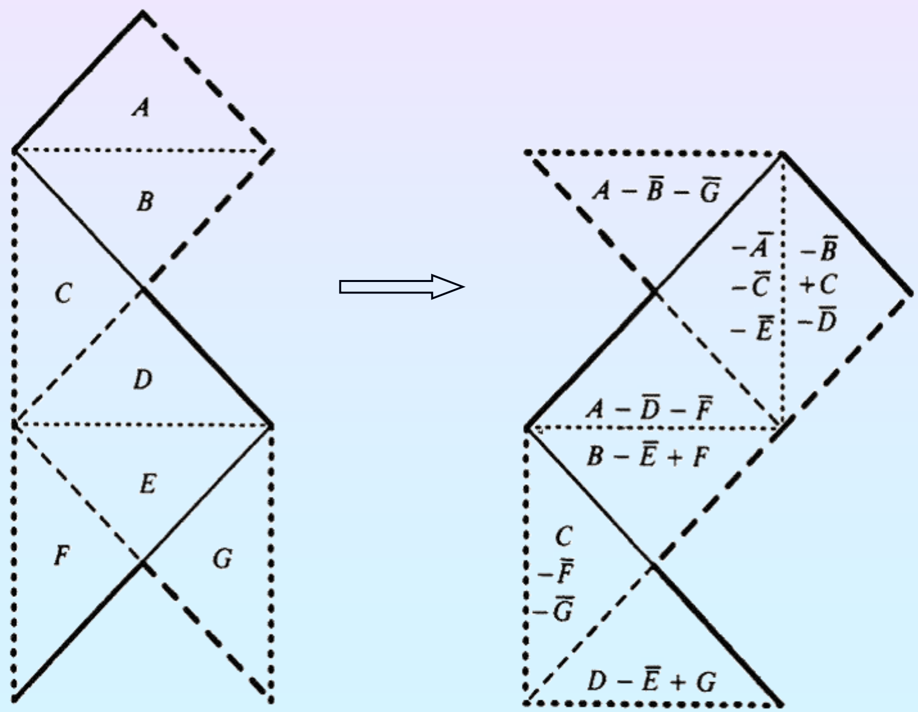
Isospectral drums – A transplantation proof

Given an eigenfunction on drum (a),
create an eigenfunction ***with the same eigenvalue*** on drum (b).



(a) (b)

$$-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) f = \lambda f \qquad f\big|_{boundary} = 0$$

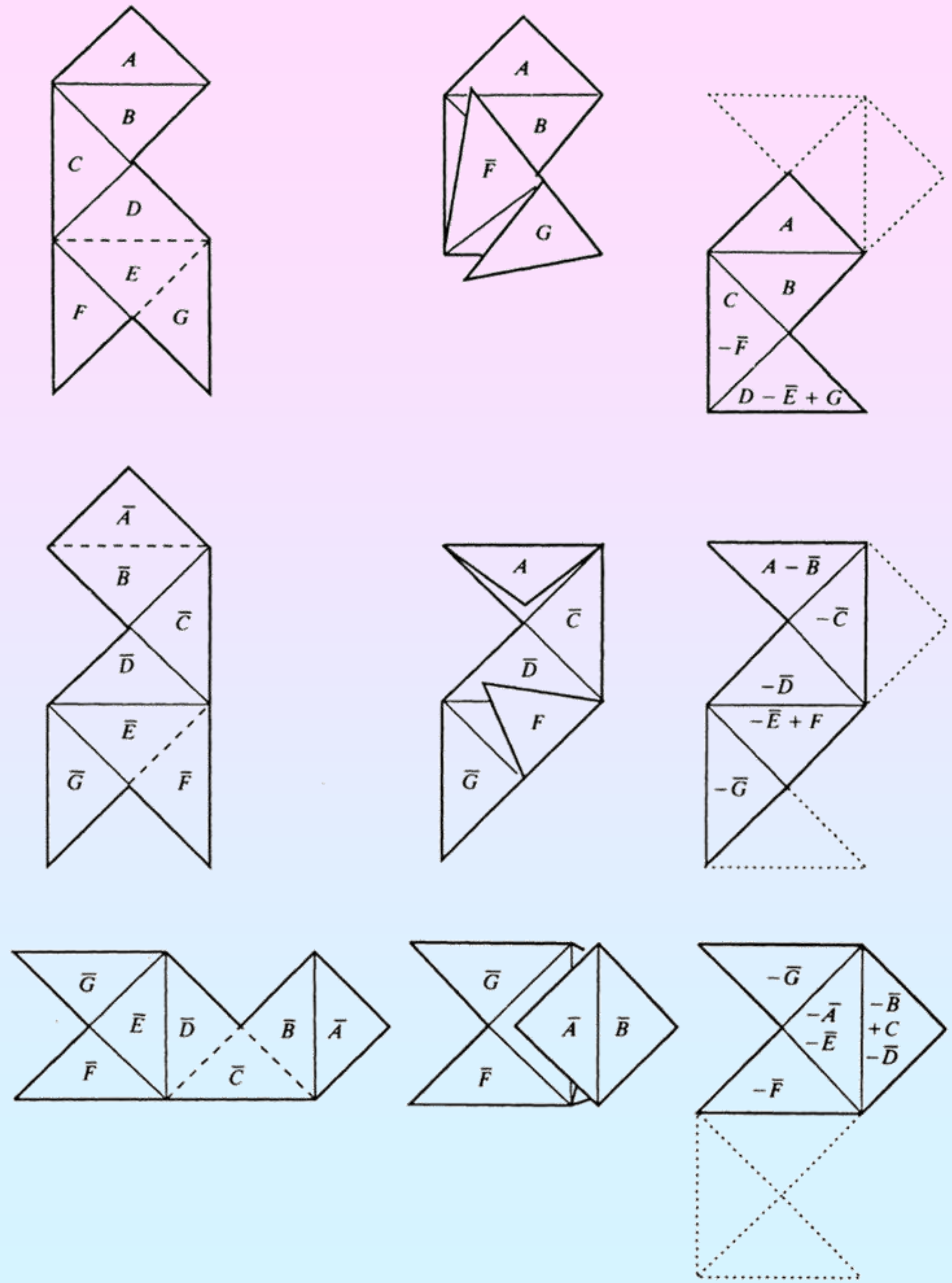


(a) (b)

Isospectral drums

A paper-folding proof

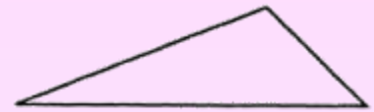
(S.J. Chapman – 2000)

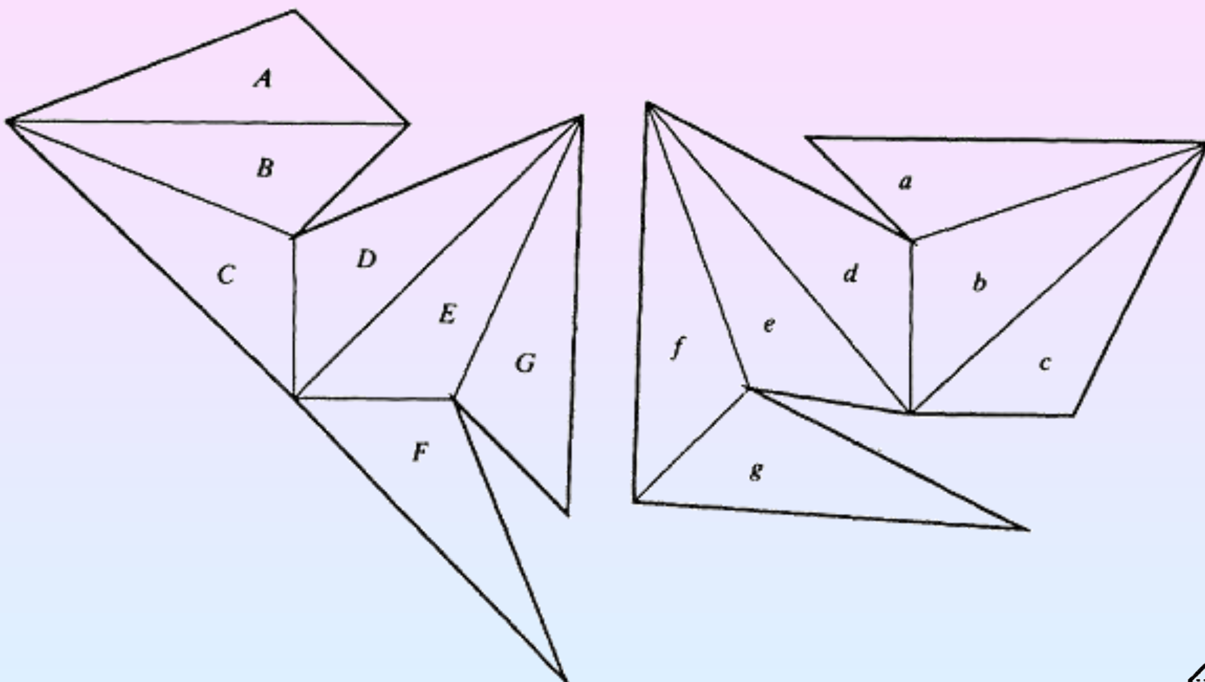


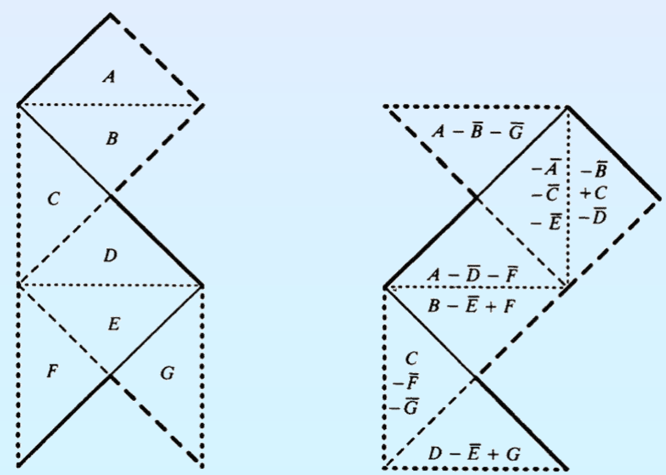
(a) (b) (c)

Isospectral drums – A transplantation proof

We can use another basic building block



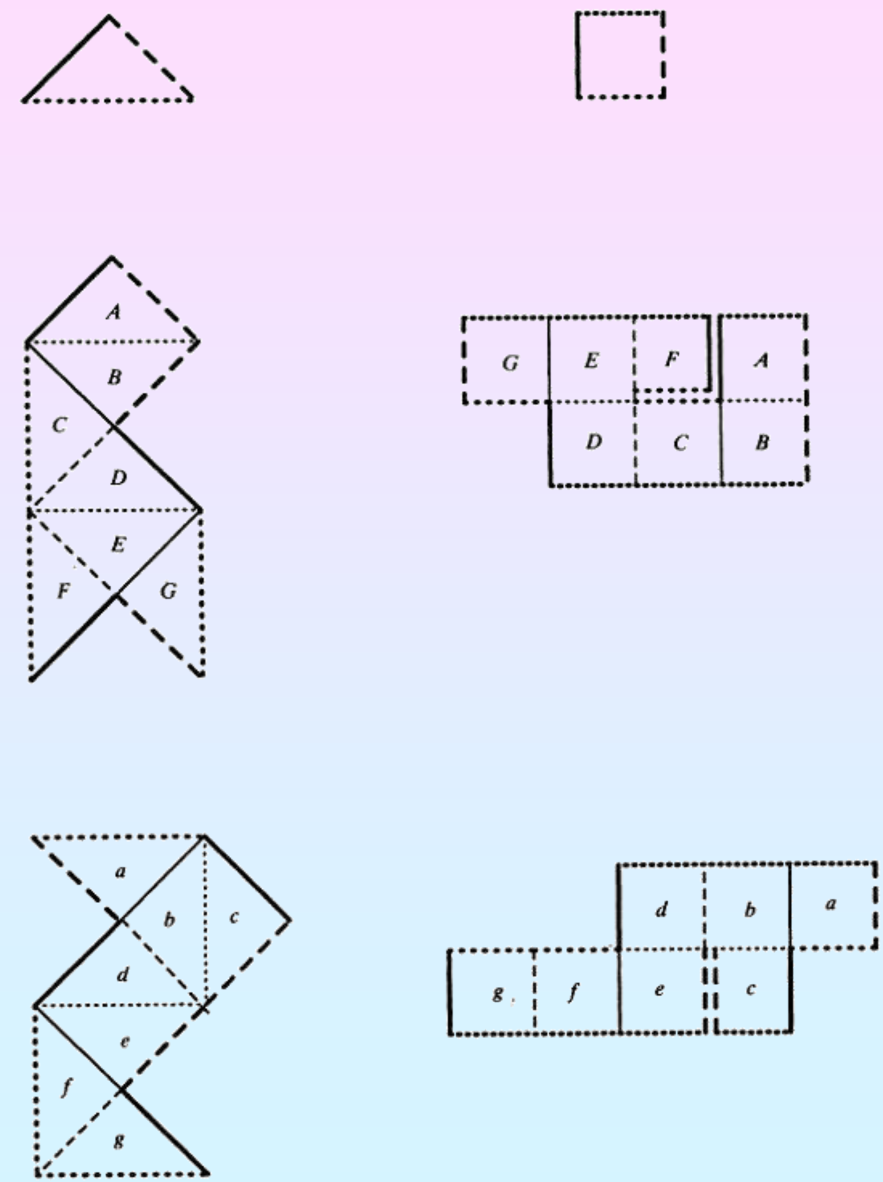




(a) (b)

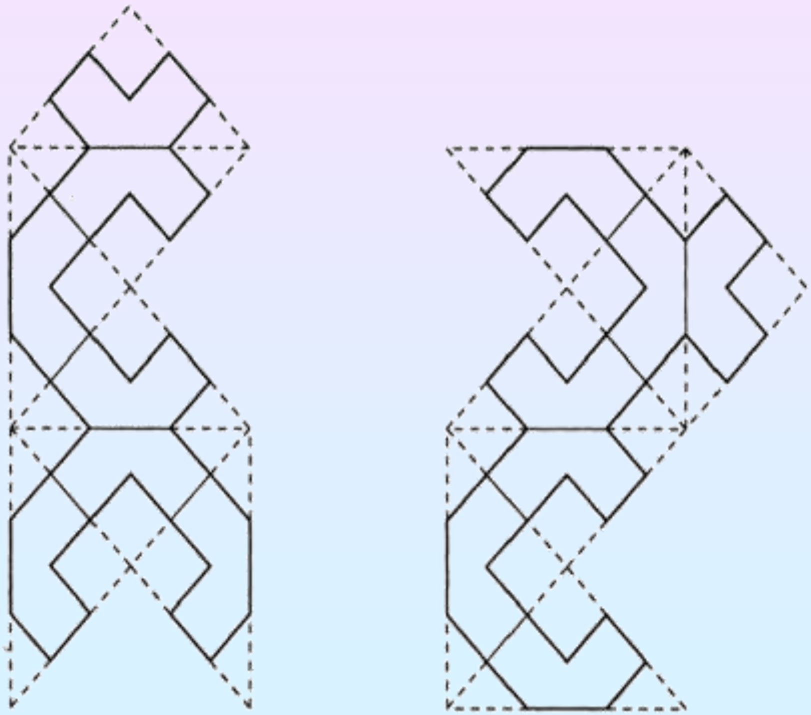
Isospectral drums – A transplantation proof

… or a building block
which is not a triangle …



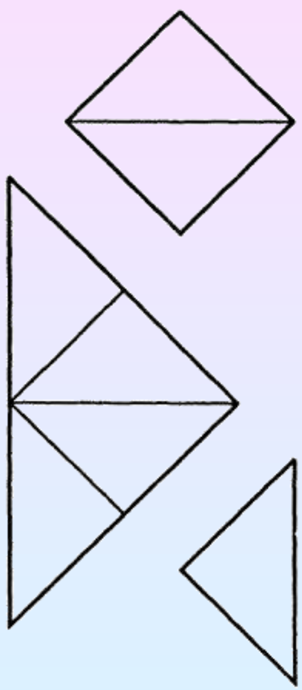
Isospectral drums – A transplantation proof

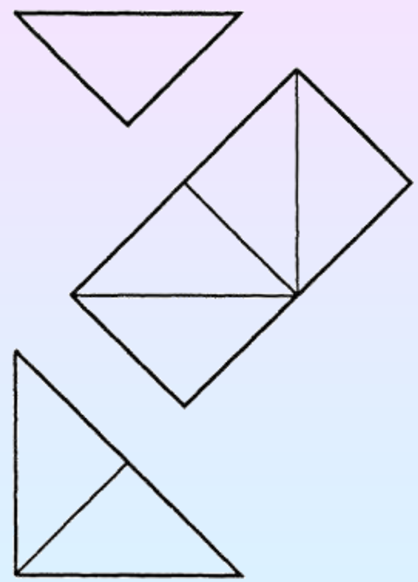
… or even a funny shaped building block …

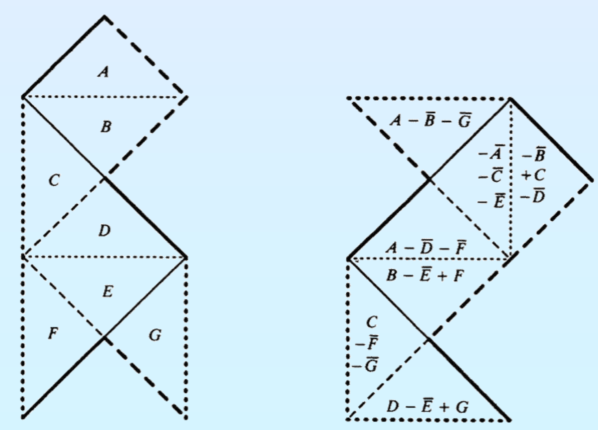


Isospectral drums – A transplantation proof

… or cut it in a nasty way (and ruin the connectivity) …







‘Can one hear the shape of [illegible]?’

- Many examples of isospectral objects (not only drums):
  - Milnor (1964) — *16-dim Tori*
  - Buser (1986) & Berard (1992) — *Transplantation*
  - Gordon, Web, Wolpert (1992) — *Drums*
  - Buser, Conway, Doyle, Semmler (1994) — *More Drums*
  - Brooks (1988,1999) — *Manifolds and Discrete Graphs*
  - Gutkin, Smilansky (2001) — *Quantum Graphs*
  - Gordon, Perry, Schueth (2005) — *Manifolds*

- There are several methods for construction of isospectrality – the main is due to Sunada (1985).

- We present a method based on representation theory arguments which generalizes Sunada’s method.

Isospectral theorem

<u>Theorem</u> (R.B., Ori Parzanchevski, Gilad Ben-Shach)

Let Γ be a drum which obeys a symmetry group G.
Let H_1, H_2 be two subgroups of G with representations R_1, R_2
that satisfy $\mathrm{Ind}_{H_1}^{G} R_1 \cong \mathrm{Ind}_{H_2}^{G} R_2$

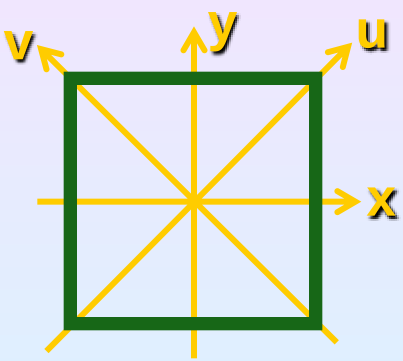
then the drums Γ/R_1 , Γ/R_2 are isospectral.

Groups & Drums

- *Example*: The Dihedral group – the symmetry group of the square.

$$G = \{ id , a , a^2 , a^3 , r_x , r_y , r_u , r_v \}$$

How does the Dihedral group act on the square drum?



- Two subgroups of the Dihedral group:
  $H_1 = \{ id , a^2 , r_x , r_y \}$
  $H_2 = \{ id , a^2 , r_u , r_v \}$

Groups - Representations

- Representation – Given a group G, a representation R is an assignment of a matrix $\rho_R(g)$ to each group element $g \in G$, such that: $\forall\, g_1,g_2 \in G \quad \rho_R(g_1)\cdot\rho_R(g_2) = \rho_R(g_1 g_2)$.

- Example 1 - G has the following 1-dimensional rep. S_1:

$$\mathrm{id} \to (1) \quad a \to (-1) \quad a^2 \to (1) \quad a^3 \to (-1) \quad r_x \to (-1) \quad r_y \to (-1) \quad r_u \to (1) \quad r_v \to (1)$$

- Example 2 - G has the following 2-dimensional rep. S_2:

$$\mathrm{id} \to \begin{pmatrix}1&0\\0&1\end{pmatrix} \quad a \to \begin{pmatrix}0&1\\-1&0\end{pmatrix} \quad a^2 \to \begin{pmatrix}-1&0\\0&-1\end{pmatrix} \quad a^3 \to \begin{pmatrix}0&-1\\1&0\end{pmatrix} \quad r_x \to \begin{pmatrix}-1&0\\0&1\end{pmatrix} \quad r_y \to \begin{pmatrix}1&0\\0&-1\end{pmatrix} \quad r_u \to \begin{pmatrix}0&1\\1&0\end{pmatrix} \quad r_v \to \begin{pmatrix}0&-1\\-1&0\end{pmatrix}$$

- Restriction: $Q = \mathrm{Res}_{H_1}^{G} S_1$ is the following rep. of H_1:

$$\mathrm{id} \to (1) \quad a \to (-1) \quad a^2 \to (1) \quad a^3 \to (-1) \quad r_x \to (-1) \quad r_y \to (-1) \quad r_u \to (1) \quad r_v \to (1)$$

- Induction: $\mathrm{Ind}_{H_1}^{G} Q$ is the following rep. of G :

$$\mathrm{id} \to \begin{pmatrix}1&0\\0&1\end{pmatrix} \quad a \to \begin{pmatrix}-1&0\\0&1\end{pmatrix} \quad a^2 \to \begin{pmatrix}1&0\\0&1\end{pmatrix} \quad a^3 \to \begin{pmatrix}-1&0\\0&1\end{pmatrix} \quad r_x \to \begin{pmatrix}-1&0\\0&-1\end{pmatrix} \quad r_y \to \begin{pmatrix}-1&0\\0&-1\end{pmatrix} \quad r_u \to \begin{pmatrix}1&0\\0&-1\end{pmatrix} \quad r_v \to \begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

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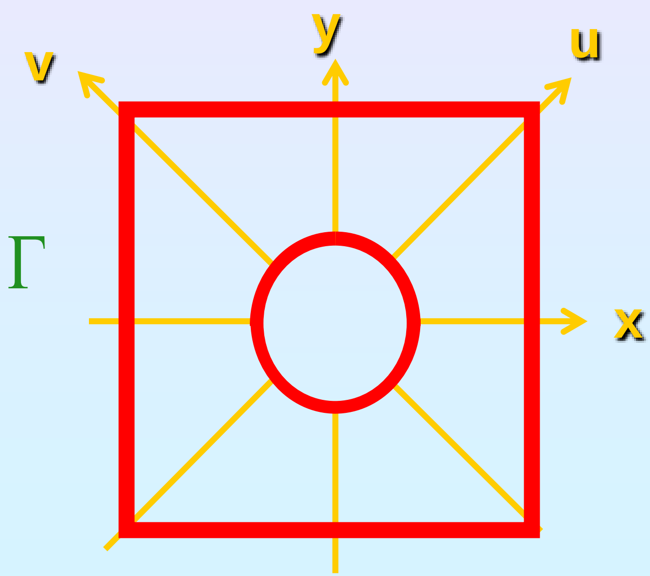
Let H_1, H_2 be two subgroups of G with representations R_1, R_2 that satisfy $\mathrm{Ind}_{H_1}^{G} R_1 \cong \mathrm{Ind}_{H_2}^{G} R_2$

then the drums Γ/R_1 , Γ/R_2 are isospectral.

- An application of the theorem with: $G = \{ \mathrm{id},\ a,\ a^2,\ a^3,\ r_x,\ r_y,\ r_u,\ r_v \}$



Two subgroups of G: $H_1 = \{ \mathrm{id},\ a^2,\ r_x,\ r_y \}$

$H_2 = \{ \mathrm{id},\ a^2,\ r_u,\ r_v \}$

We choose representations

$R_1 : \{ \mathrm{id} \to (1)\quad a^2 \to (-1)\quad r_x \to (-1)\quad r_y \to (1) \}$

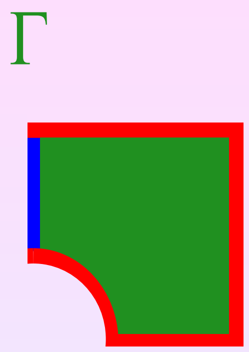
$R_2 : \{ \mathrm{id} \to (1)\quad a^2 \to (-1)\quad r_u \to (1)\quad r_v \to (-1) \}$

such that $\mathrm{Ind}_{H_1}^{G} R_1 \cong \mathrm{Ind}_{H_2}^{G} R_2$

Constructing Quotient Graphs

- Consider the following rep. R_1 of the subgroup H_1:

$$R_1 : \left\{ \text{id} \to (1) \quad a^2 \to (\text{-}1) \quad r_x \to (\text{-}1) \quad r_y \to (1) \right\}$$



We construct Γ/R_1 by inquiring what do we know about a function f on Γ which transforms according to R_1.

$$r_x\, f = -f \qquad r_y\, f = f$$

Dirichlet **Neumann**

The construction of a *quotient drum* is motivated by an *encoding scheme.*

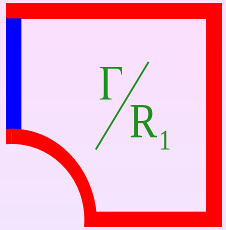
Constructing Quotient Graphs

- Consider the following rep. R_1 of the subgroup H_1:

$$R_1 : \left\{ \text{id} \to (1) \quad a^2 \to (\text{-}1) \quad r_x \to (\text{-}1) \quad r_y \to (1) \right\}$$

We construct $\Gamma/_{R_1}$ by inquiring what do we know about a function f on Γ which transforms according to R_1.

$$r_x f = -f \qquad\qquad r_y f = f$$



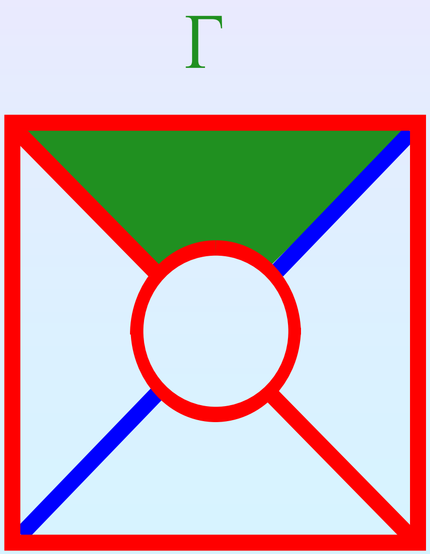
- Consider the following rep. R_2 of the subgroup H_2:

$$R_2 : \left\{ \text{id} \to (1) \quad a^2 \to (\text{-}1) \quad r_u \to (1) \quad r_v \to (\text{-}1) \right\}$$

We construct $\Gamma/_{R_2}$ by inquiring what do we know about a function g on Γ which transforms according to R_2.

$$r_u g = g \qquad\qquad r_v g = -g$$

Neumann **Dirichlet**



Isospectral theorem

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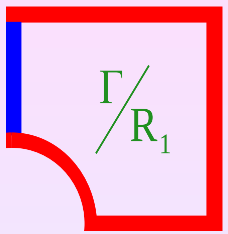
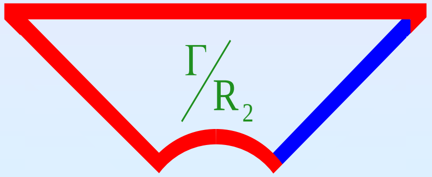




<u>Remarks</u>:

1. *The isospectral theorem is applicable not only for drums, but for general manifolds, graphs, etc.*
2. *The following isospectral example is a courtesy of Martin Sieber.*

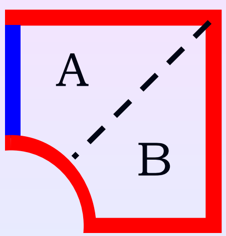
<u>Theorem</u> (R.B., Ori Parzanchevski, Gilad Ben-Shach)

The drums Γ/R_1, Γ/R_2 constructed according to the conditions above possess a transplantation.

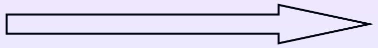
Transplantation

- The transplantation of our example is

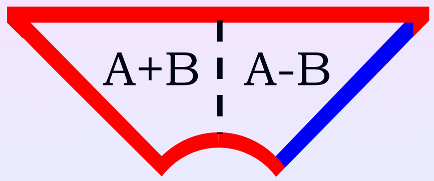
Γ / R_1



$$T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$



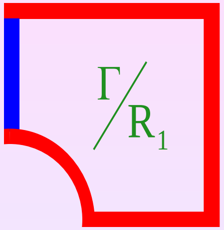
Γ / R_2



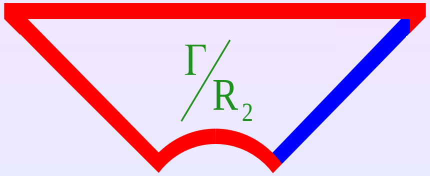
Extending the Isospectral pair

Extending our example: $\mathrm{Ind}_{H_1}^{G} R_1 \cong \mathrm{Ind}_{H_2}^{G} R_2 \cong \mathrm{Ind}_{H_3}^{G} R_3$

$H_1 = \{ e , a^2, r_x , r_y \}$ R_1: $e \to (1)$ $a^2 \to (-1)$ $r_x \to (-1)$ $r_y \to (1)$

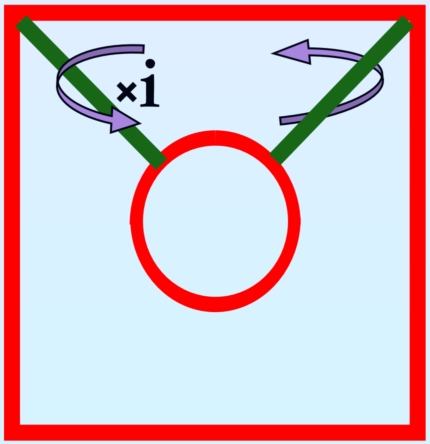


$H_2 = \{ e , a^2, r_u , r_v \}$ R_2: $e \to (1)$ $a^2 \to (-1)$ $r_u \to (1)$ $r_v \to (-1)$



$H_3 = \{ e , a, a^2, a^3 \}$ R_3: $e \to (1)$ $a \to (i)$ $a^2 \to (-1)$ $a^3 \to (-i)$

Γ



$$a f = i f$$

Extending the Isospectral pair

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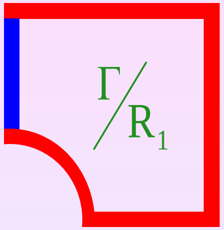
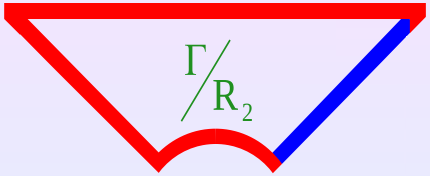
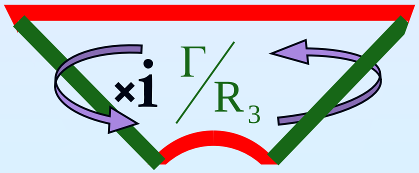
$H_1 = \{e, a^2, r_x, r_y\}$ R_1: $e \to (1)$ $a^2 \to (-1)$ $r_x \to (-1)$ $r_y \to (1)$



$H_2 = \{e, a^2, r_u, r_v\}$ R_2: $e \to (1)$ $a^2 \to (-1)$ $r_u \to (1)$ $r_v \to (-1)$



$H_3 = \{e, a, a^2, a^3\}$ R_3: $e \to (1)$ $a \to (i)$ $a^2 \to (-1)$ $a^3 \to (-i)$

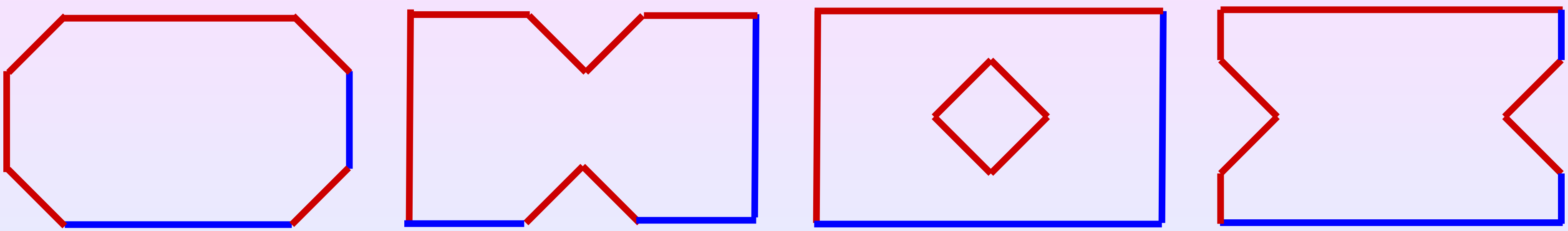


$$a f = i f$$

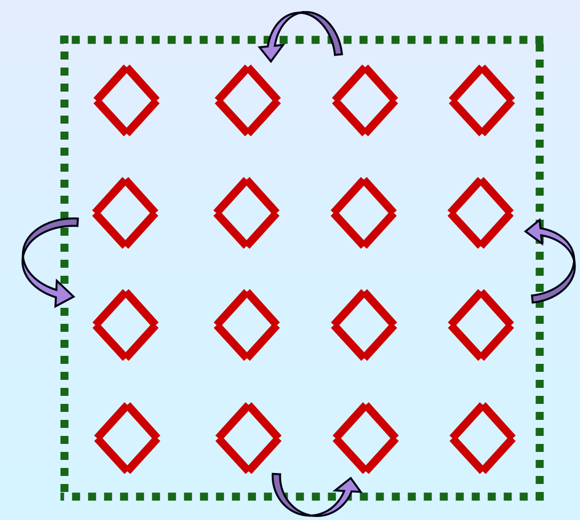
A few isospectral examples

‘Spectral problems with mixed Dirichlet-Neumann boundary conditions: isospectrality and beyond’ D. Jacobson, M. Levitin, N. Nadirashvili, I. Polterovich (2004)

‘Isospectral domains with mixed boundary conditions’ M. Levitin, L. Parnovski, I. Polterovich (2005)

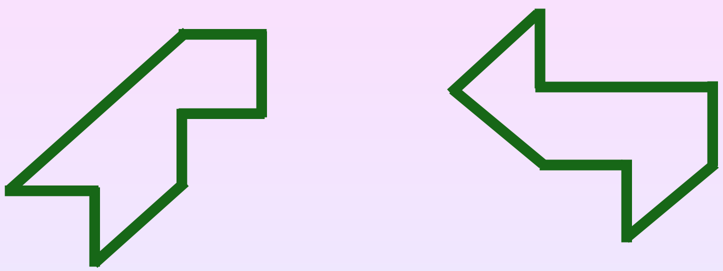


This isospectral quartet can be obtained when acting with the group $D_4 x D_4$ on the following torus:

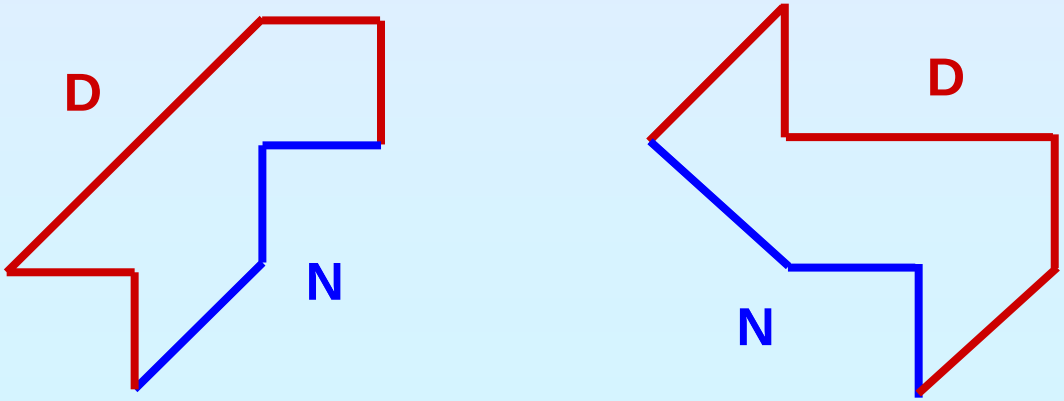


A few isospectral examples

‘One **cannot** hear the shape of a drum’
Gordon, Webb and Wolpert (1992)



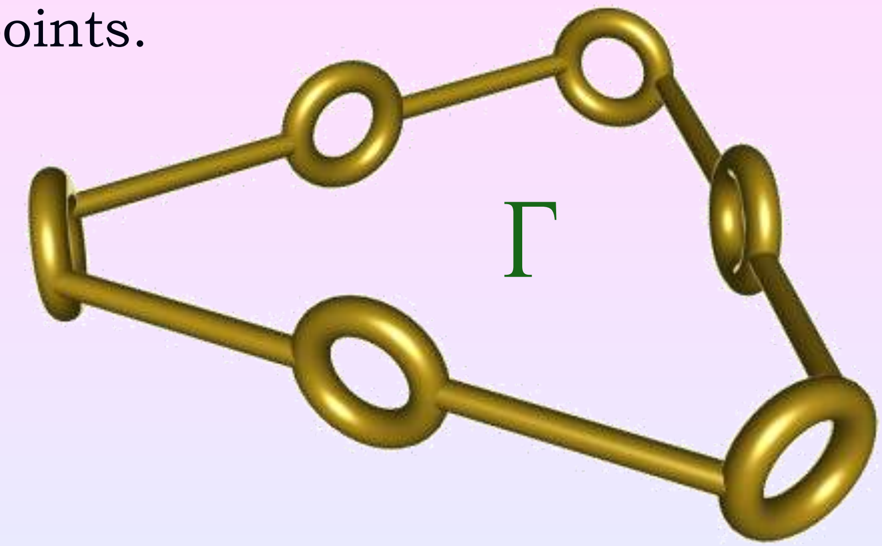
We construct the known isospectral drums of Gordon *et al.* but with new boundary conditions:



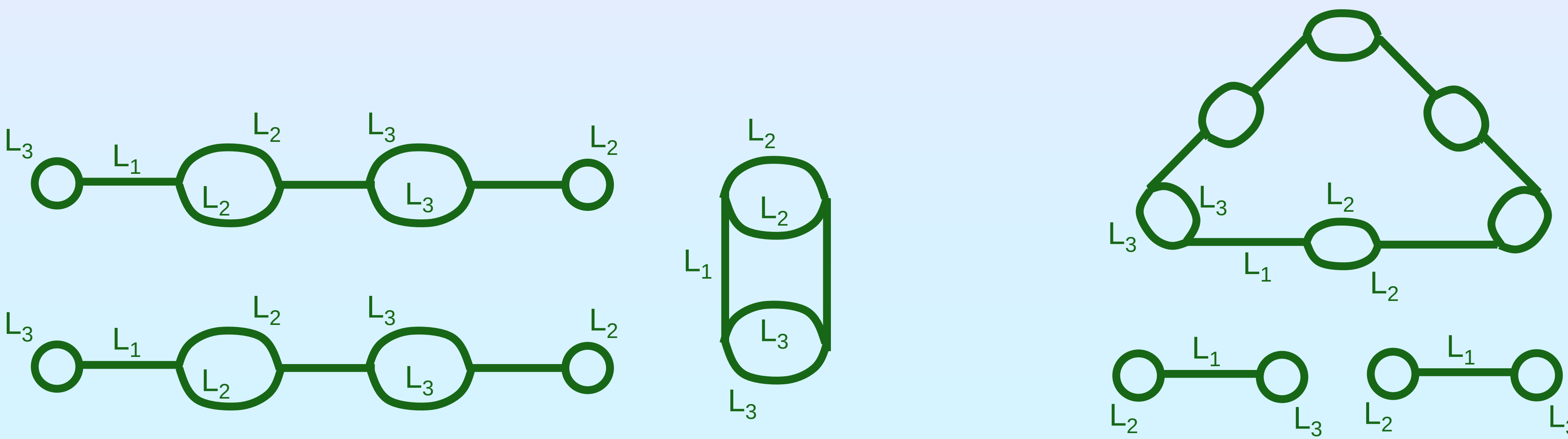
A few isospectral examples – quantum graphs

$G = S_3$ (D_3) acts on Γ with no fixed points.

To construct the quotient graph, we take the same rep. of G, but use two different bases for the matrix representation.



The resulting quotient graphs are:



What one cannot hear?
On drums which sound the same

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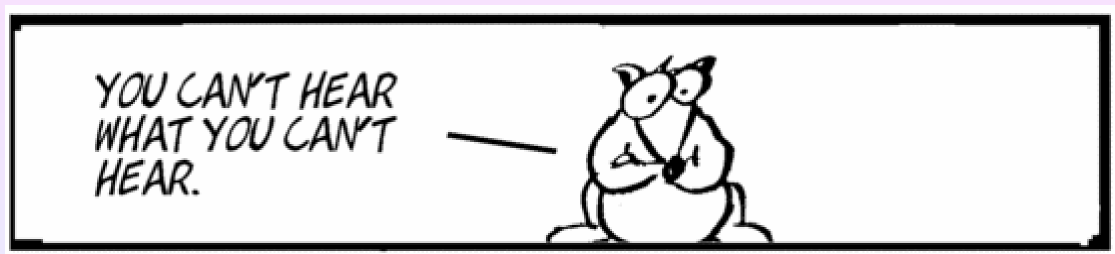

R. Band, O. Parzanchevski and G. Ben-Shach,
"*The Isospectral Fruits of Representation Theory: Quantum Graphs and Drums*",
J. Phys. A (2009).

O. Parzanchevski and R. Band,
"*Linear Representations and Isospectrality with Boundary Conditions*",
Journal of Geometric Analysis (2010).

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