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Courant-sharp eigenvalues of Neumann 2-rep-tiles

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Abstract We find the Courant-sharp Neumann eigenvalues of the Laplacian on some 2-rep-tile domains. In $\mathbb{R}^2$, the domains we consider are the isosceles right triangle and the rectangle with edge ratio $\sqrt{2}$ (also known as the A4 paper). In $\mathbb{R}^n$, the domains are boxes which generalize the mentioned planar rectangle. The symmetries of those domains reveal a special structure of their eigenfunctions, which we call folding\unfolding. This structure affects the nodal set of the eigenfunctions, which, in turn, allows to derive necessary conditions for Courant-sharpness. In addition, the eigenvalues of these domains are arranged as a lattice which allows for a comparison between the nodal count and the spectral position. The Courant-sharpness of most eigenvalues is ruled out using those methods. In addition, this analysis allows to estimate the nodal deficiency—the difference between the spectral position and the nodal count.

Keywords Nodal domains · Courant nodal theorem · Nodal set

Mathematics Subject Classification 35B05 · 58C40 · 58J50

1 Introduction

It is nearly a century, since Courant proved his famous nodal result, stating that the nth Laplacian eigenfunction cannot have more than n nodal domains [16]. Eigenfunctions which achieve this upper bound are called Courant-sharp and it was Pleijel who showed

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that there are only finitely many of them [37]. Much more recently, Polterovich proved a similar result for domains with Neumann boundary conditions [38]. In his paper, Pleijel also pointed out the Courant-sharp eigenfunctions of the square with Dirichlet boundary conditions. What started in this early work of Pleijel is recently revived as a systematic search of Courant-sharp eigenfunctions of various domains. Part of this analysis owes to the broad interest in the general subject of nodal domains, but this line of research in particular stems from the latest works on nodal partitions. The search for Courant-sharpness was pioneered by Helffer, Hoffmann-Ostenhof, and Terracini, who found that minimization of some energy functional over a set of domain partitions is connected to nodal patterns of eigenfunctions [21]. In particular, they have shown that if the minimum of this functional over domain partitions of k subdomains is equal to the kth eigenvalue, then the kth eigenvalue is Courant-sharp. In addition, the nodal partition of the corresponding Courant-sharp eigenfunction is a minimizing partition.[1] This led to a particularized search for Courant-sharp eigenfunctions of various domains over just the last couple of years. Among the domains that were treated are the square, the disk, the annulus, irrational rectangles, the torus, and some triangles, where the analysis in those cases is specialized to the considered boundary conditions (either Dirichlet or Neumann). Most of those investigations are done by Helffer collaborating with Hoffmann-Ostenhof and Terracini [23–25], with Bérard [7,8], and with Sundqvist [26,27]. Additional results for various tori are proved by Léna [31,32]. For further details and references, we refer the reader to the recent reviews by Bonnaillie-Noël and Helffer [14] and by Laugesen and Siudeja [30]. While these reviews came out and afterwards, three additional results, which for the first time concern high-dimensional domains, were proven. Helffer and Kiwan determined the Courant-sharp eigenfunctions of the cube [22], Léna found them for the three-dimensional square torus [31] and Helffer with Sundqvist solved the problem for Euclidean balls in any dimension [26]. Finally, in another direction, Helffer and Bérard and also van den Berg and Gittins provided bounds on the largest Courant-sharp Dirichlet eigenvalue and on the total number of them for a general domain [6,9].

In the present work, we determine the Courant-sharp eigenfunctions of certain 2-rep-tile domains with Neumann boundary conditions. A domain is said to be rep-tile (replicating figure, a name coined by Golomb [19]) if it can be decomposed into k isometric domains, each of which is similar to the original domain. According to the number of its subdomains (k), the domain is called rep-k or a k-rep-tile. The convex polygonal 2-rep-tiles in the plane are known to be the isosceles right triangle and parallelograms with edge length ratio $\sqrt{2}$ [35,36]. In this paper, we find the Courant-sharp eigenfunctions of such Neumann triangle (Theorem 1.1) and Neumann rectangle, together with all the 2-rep-tile high-dimensional boxes, which generalize this rectangle (Theorem 1.2). In addition to being 2-rep-tiles, they all have the special property that the cut which separates them into the mentioned two subdomains serves

[1] It is worthwhile to mention that a similar variational approach was recently developed by Berkolaiko, Kuchment, and Smilansky. Their results also characterize the nodal sets of non-Courant-sharp eigenfunctions [11].

also as a symmetry axis[2] (or hyperplane for the boxes). Thus, for an eigenfunction which is even with respect to the symmetry axis, its restriction to the subdomain, when rescaled, yields again an eigenfunction of the same eigenvalue problem. This allows us to identify a special structure, ordering all of the eigenfunctions, which we call the folding (unfolding) structure. Using this classification, we prove that all eigenfunctions within a certain class vanish on the same subset.[3] This property allows to rule out Courant-sharpness of eigenfunctions without using the Faber–Krahn inequality [18, 29] or similar isoperimetric inequalities. Such isoperimetric inequalities form the first step in ruling out Courant-sharpness in most of the works mentioned above (with the exception of irrational rectangles, the disk, and Euclidean balls, where properties of minimal partitions were used for that purpose). We present our result for the rectangle in a general form (Theorem 1.2), valid for all n-dimensional ($n \geq 2$) boxes which are 2-rep-tiles and symmetric with respect to their hyperplane cut (see Fig. 8). It is interesting to note that the case of the rectangle goes beyond the irrational rectangles which were explored so far, as its square of edge ratio is rational (it equals two). Its spectrum is, therefore, not simple as in the case of the irrational rectangles (treated in [21]). Yet, the folding structure mentioned above allows to quickly rule out all of its multiple eigenvalues, and the same goes for all high-dimensional boxes.

The outline of the paper is as follows. This section continues by providing useful notations and exact statements of our main results. We then present the so-called folding structure for eigenfunctions of the isosceles right triangle in Sect. 2. In Sect. 3, we complete the investigation of the triangle's eigenfunctions and prove Theorem 1.1. In Sect. 4, we present the folding structure for the box eigenfunctions and prove Theorem 1.2. In Appendix A, we present some identities concerning eigenvalue multiplicities for the boxes, connecting those to the two-dimensional problem of the rectangle. Finally, in Appendix B, we go beyond Courant-sharpness by describing some results on the nodal deficiency, and in Appendix C, we point out how some of our methods apply for the same domains, but with Dirichlet boundary conditions.

1.1 Notations and preliminaries

We consider the Laplacian eigenvalue problem on a bounded domain Ω with Neumann boundary conditions

$$-\Delta\varphi = \lambda\varphi, \quad \left.\frac{\partial\varphi}{\partial\vec{n}}\right|_{\partial\Omega} = 0. \tag{1.1}$$

We denote the corresponding spectrum by $\sigma(\Omega)$ and note that it can be described by an increasing sequence of eigenvalues

$$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \nearrow \infty.$$

[2] Using involution symmetry for studying nodal counts may be found already in the early studies of Leydolod [33,34].

[3] This connects nicely to the recent works [1,15], though those do not concern Courant-sharpness.

We define the following spectral counting functions:

$$\overline{N}(\lambda) := \left|\left\{j \mid \lambda_j \leq \lambda\right\}\right|, \tag{1.2}$$

$$\underline{N}(\lambda) := \left|\left\{j \mid \lambda_j < \lambda\right\}\right|, \tag{1.3}$$

$$N(\lambda) := \begin{cases} \underline{N}(\lambda) & \lambda \notin \sigma(\Omega) \\ \underline{N}(\lambda) + 1 & \lambda \in \sigma(\Omega) \end{cases} \tag{1.4}$$

and denote the multiplicity of an eigenvalue by

$$\mathrm{d}(\lambda) := \overline{N}(\lambda) - \underline{N}(\lambda). \tag{1.5}$$

For an eigenfunction φ on Ω, we denote by $\nu(\varphi)$ the number of connected components of $\Omega \backslash \varphi^{-1}(0)$, also known as nodal domains.

In terms of those definitions, the celebrated Courant bound reads $\nu(\varphi) \leq N(\lambda)$, where φ is an eigenfunction of the eigenvalue λ [16].
Let φ be an eigenfunction on Ω with eigenvalue λ. We say that φ is a Courant-sharp eigenfunction if $\nu(\varphi) = N(\lambda)$. In this case, we also say that λ is a Courant-sharp eigenvalue.

1.2 Main results

Theorem 1.1 *The Courant-sharp eigenvalues of the Neumann Laplacian on the isosceles right triangle are* $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_6$.

Theorem 1.2 *Let* $n \in \mathbb{N},\ \ n \geq 2$, *and let* $\mathcal{B}^{(n)}$ *be an n-dimensional box of measures* $l_1 \times l_2 \times \cdots \times l_n$, *where the ratios of edge lengths are given by* $\frac{l_j}{l_{j+1}} = 2^{\frac{1}{n}}$ $(1 \leq j \leq n-1)$. *The Courant-sharp eigenvalues of the Neumann Laplacian on* $\mathcal{B}^{(n)}$ *are* $\lambda_1, \lambda_2, \lambda_4, \lambda_6$ *for* $n = 2$ *and* λ_1, λ_2 *for* $n \geq 3$.

2 Eigenfunction folding structure of the triangle

We consider the following scaling for the isosceles right triangle:

$$\mathcal{D} = \{(x, y) \in [0, \pi] \times [0, \pi] \,\big|\, y \leq x\}.$$

For geometric convenience to be exploited later, $\mathcal{D}$ denotes the closed domain and the Laplacian is defined on its interior $\Omega = \mathcal{D}^{\circ}$.
Denote $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and define the set

$$\mathcal{Q} := \{(m, n) \in \mathbb{N}_0 \times \mathbb{N}_0 | \, m \geq n\}, \tag{2.1}$$

which we call the set of quantum numbers. A complete orthogonal basis of eigenfunctions is given by

$$\varphi_{m,n}(x, y) = \cos(mx)\cos(ny) + \cos(my)\cos(nx)\ ;\ (m, n) \in \mathcal{Q}, \tag{2.2}$$

and the spectrum is given by

$$\sigma(\mathcal{D}) = \left\{\lambda_{m,n} = \|(m, n)\|^2 \;\middle|\; (m, n) \in \mathcal{Q}\right\},$$

where

$$\|(m, n)\|^2 = m^2 + n^2.$$

It is useful to define

$$\mathcal{Q}(\lambda) := \left\{(m, n) \in \mathcal{Q} \;\middle|\; \|(m, n)\|^2 < \lambda\right\}$$

and observe that

$$\underline{N}(\lambda) = |\mathcal{Q}(\lambda)|.$$

The isosceles right triangle $\mathcal{D}$ is symmetric with respect to the median to the hypotenuse

$$\mathrm{L} = \{(x, y) \in \mathcal{D} |\, x + y = \pi\}$$

and the symmetry is expressed by

$$R(x, y) = (\pi - y, \pi - x).$$

We describe, in the following, a special feature of eigenfunctions on the triangle, which is based on the symmetry above.

Lemma 2.1 *Let* $\lambda \in \sigma(\mathcal{D})$*, then its corresponding eigenfunctions are odd (even) with respect to* L if and only if λ *is odd (even).*

Proof Let $\lambda_{m,n} \in \sigma(\mathcal{D})$, we get

$$\begin{aligned}\varphi_{m,n}(R(x, y)) &= \varphi_{m,n}(\pi - y, \pi - x)\\ &= \cos(m\pi - my)\cdot\cos(n\pi - nx)\\ &\quad + \cos(m\pi - mx)\cdot\cos(n\pi - ny)\\ &= (-1)^{m+n}[\cos(mx)\cos(ny) + \cos(my)\cos(nx)]\\ &= (-1)^{m+n}\varphi_{m,n}(x, y),\end{aligned}$$

so that $\varphi_{m,n}$ is odd if and only if $m \neq n \pmod 2$ and even if and only if $m = n \pmod 2$. As $\lambda_{m,n} = m^2 + n^2$, we get that $\varphi_{m,n}$ is odd if and only if $\lambda_{m,n}$ is odd and even if and only if $\lambda_{m,n}$ is even. The lemma now follows, since the elements of $\left\{\varphi_{m,n}\right\}_{m^2+n^2=\lambda}$ form a basis for the eigenspace. □

Lemma 2.1 motivates the following definition.

Definition 2.2 We define the subsets of $\mathcal{Q}$ that correspond to the odd and even eigenvalues

$$\begin{aligned} \mathcal{O} &:= \{(m,n) \in \mathcal{Q} \mid m \neq n \pmod 2\} \\ \mathcal{E} &:= \{(m,n) \in \mathcal{Q} \mid m = n \pmod 2\} \end{aligned} \tag{2.3}$$

and we denote the corresponding sets of eigenvalues by

$$\sigma_{\text{odd}}(\mathcal{D}) := \left\{\lambda_{m,n} \middle| (m,n) \in \mathcal{O}\right\}$$

$$\sigma_{\text{even}}(\mathcal{D}) := \left\{\lambda_{m,n} \middle| (m,n) \in \mathcal{E}\right\},$$

where in those sets, each eigenvalue appears as many times as its multiplicity.

Denote

$$\frac{1}{2}\mathcal{D} := \left\{(x,y) \in \mathcal{D} \middle| (x+y, x-y) \in [0,\pi]^2\right\},$$

and observe that L partitions $\mathcal{D}$ into the two isometric triangles $\frac{1}{2}\mathcal{D}$ and $\left(\mathcal{D}\backslash\frac{1}{2}\mathcal{D}\right) \cup \mathrm{L}$, each is a scaled version of $\mathcal{D}$ by a factor $\sqrt{2}$. The following defines the transformations which describe the similarity relation between $\mathcal{D}$ and $\frac{1}{2}\mathcal{D}$.

Definition 2.3 We define the *coordinate folding transformation*

$$\begin{aligned} F &: \tfrac{1}{2}\mathcal{D} \to \mathcal{D} \\ F(x,y) &:= (x+y, x-y) \end{aligned} \tag{2.4}$$

and the *coordinate unfolding transformation* as the inverse of F by

$$\begin{aligned} U &: \mathcal{D} \to \tfrac{1}{2}\mathcal{D} \\ U(u,v) &= \left(\tfrac{u+v}{2}, \tfrac{u-v}{2}\right). \end{aligned} \tag{2.5}$$

Remark The mappings F and U are, indeed, similarity transformations between $\mathcal{D}$ and $\frac{1}{2}\mathcal{D}$ as

$$F(x,y) = \underbrace{\sqrt{2}}_{\text{scaling}} \underbrace{\begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\text{isometry}} \begin{pmatrix} x \\ y \end{pmatrix}$$

and

$$U(x,y) = \underbrace{\frac{1}{\sqrt{2}}}_{\text{scaling}} \underbrace{\begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\text{isometry}} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The next definition introduces the notion of folding and unfolding of an eigenfunction.

Definition 2.4 Let φ be an eigenfunction corresponding to $\lambda \in \sigma(\mathcal{D})$,

(1) Assume $\lambda \in \sigma_{\text{even}}(\mathcal{D})$, we define the folded function, $\mathbf{F}\varphi : \mathcal{D} \to \mathbb{R}$, as

$$\mathbf{F}\varphi(x,y) = \varphi \circ U(x,y), \ (x,y) \in \mathcal{D}. \tag{2.6}$$

(2) We define the unfolded function, $\mathbf{U}\varphi : \mathcal{D} \to \mathbb{R}$, as

$$\mathbf{U}\varphi(x,y) = \begin{cases} \varphi \circ F(x,y) & (x,y) \in \frac{1}{2}\mathcal{D} \\ (\varphi \circ F) \circ R(x,y) & (x,y) \in \mathcal{D} \backslash \frac{1}{2}\mathcal{D} \end{cases}. \tag{2.7}$$

Note that only folding of an even eigenfunction gives a new function whose normal derivative vanishes on $\partial\mathcal{D}$. Therefore, it follows that only folding of an even eigenfunction results with another eigenfunction. Unfolding of any eigenfunction always results with another eigenfunction. Hence, we consider the foldings for the even eigenfunctions and the unfoldings for all eigenfunctions. We also remark that the folded (unfolded) eigenfunction is of an eigenvalue which is twice as small (large), since the coordinate folding (unfolding) transformation is a similarity transformation with a scaling factor of $\sqrt{2}$ $(1/\sqrt{2})$. Those results are stated and proved below.

Lemma 2.5 *Let*

$$\varphi = \sum_{k^2+l^2=\lambda} \alpha_{k,l}\varphi_{k,l}$$

be an eigenfunction corresponding to the eigenvalue λ, then the following holds:

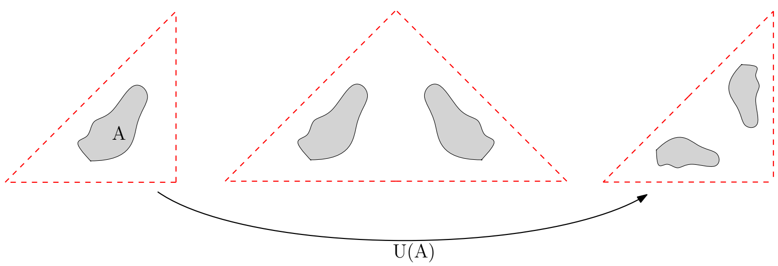


Fig. 1 Unfolding of a set

(1) *If* $\lambda \in \sigma_{even}(\mathcal{D})$, *then the folded function* $\mathbf{F}\varphi$ *is an eigenfunction corresponding to the eigenvalue* $\frac{\lambda}{2}$ *and is given by*

$$\mathbf{F}\varphi = \sum_{k^2+l^2=\lambda} \alpha_{k,l}\varphi_{F_{\mathcal{Q}}(k,l)}, \quad \textit{where} \quad F_{\mathcal{Q}}(k,l) = \left(\tfrac{k+l}{2}, \tfrac{k-l}{2}\right). \tag{2.8}$$

(2) *The unfolded function* $\mathbf{U}\varphi$ *is an eigenfunction corresponding to the eigenvalue* 2λ *and is given by*

$$\mathbf{U}\varphi = \sum_{k^2+l^2=\lambda} \alpha_{k,l}\varphi_{U_{\mathcal{Q}}(k,l)}, \quad \textit{where} \quad U_{\mathcal{Q}}(k,l) = (k+l, k-l). \tag{2.9}$$

Proof First, consider the case that $\varphi = \varphi_{k,l}$. A simple calculation of $\mathbf{F}\varphi_{k,l}$ and $\mathbf{U}\varphi_{k,l}$ involving the trigonometric identity

$$2\cos(\alpha)\cos(\beta) = \cos(\alpha+\beta) + \cos(\alpha-\beta),$$

yields

$$\mathbf{F}\varphi_{k,l} = \varphi_{F_{\mathcal{Q}}(k,l)}$$

and

$$\mathbf{U}\varphi_{k,l} = \varphi_{U_{\mathcal{Q}}(k,l)}.$$

To conclude that $\mathbf{F}\varphi_{k,l}$ and $\mathbf{U}\varphi_{k,l}$ are eigenfunctions, we need to verify that $F_{\mathcal{Q}}(k,l),\ U_{\mathcal{Q}}(k,l) \in \mathcal{Q}$. This is obvious for $U_{\mathcal{Q}}(k,l)$, and as for $F_{\mathcal{Q}}(k,l)$, we use that $\lambda_{k,l} \in \sigma_{\text{even}}(\mathcal{D})$ implies $(k,l) \in \mathcal{E}$, and thus, $\left(\frac{k+l}{2}, \frac{k-l}{2}\right) \in \mathcal{Q}$. The last part of the claim is that $\mathbf{F}\varphi_{k,l}$ and $\mathbf{U}\varphi_{k,l}$ correspond to eigenvalues $\frac{1}{2}\lambda_{k,l}$ and $2\lambda_{k,l}$. Indeed, we have

$$\left\| F_{\mathcal{Q}}(k,l)\right\|^2 = \frac{1}{2}\left(k^2+l^2\right) = \frac{1}{2}\lambda_{k,l},$$

and

$$\left\| U_{\mathcal{Q}}(k,l)\right\|^2 = 2\left(k^2+l^2\right) = 2\lambda_{k,l}.$$

Finally, using the linearity of $\mathbf{F}$ and $\mathbf{U}$, we conclude that the claim holds for

$$\varphi = \sum_{k^2+l^2=\lambda} \alpha_{k,l}\varphi_{k,l}.$$

□

The last lemma allows for a useful characterization of all eigenvalues.

Corollary 2.6

(1) *Let* $0 \neq \lambda \in \sigma(\mathcal{D})$. *Then, there exist unique* $\lambda^{(0)} \in \sigma_{odd}(\mathcal{D})$ *and* $k \in \mathbb{N}_0$, *such that* $\lambda = 2^k\lambda^{(0)}$. *Furthermore,* $\mathrm{d}(\lambda) = \mathrm{d}\left(\lambda^{(0)}\right)$.

(2) *Let* $\lambda^{(0)} \in \sigma_{odd}(\mathcal{D})$ *and* $k \in \mathbb{N}_0$. *Then,* $2^k\lambda^{(0)} \in \sigma(\mathcal{D})$.

Proof

(1) Let $0 \neq \lambda \in \sigma(\mathcal{D})$. As $\sigma(\mathcal{D}) \subseteq \mathbb{N}_0$, we can write uniquely

$$\lambda = 2^k\lambda^{(0)} \quad \text{with} \quad k \in \mathbb{N}_0, \quad \lambda^{(0)} \in \mathbb{N}_0 \quad \text{odd}. \tag{2.10}$$

To show $\lambda^{(0)} \in \sigma_{\text{odd}}(\mathcal{D})$, consider φ to be an eigenfunction of $\lambda = 2^k\lambda^{(0)}$. By Lemma 2.5, it follows that $\mathbf{F}^k\varphi$ is an eigenfunction and its corresponding eigenvalue equals $2^{-k}\lambda = \lambda^{(0)}$. Hence, $\lambda^{(0)} \in \sigma(\mathcal{D})$, but as it is odd, we further have $\lambda^{(0)} \in \sigma_{\text{odd}}(\mathcal{D})$. The equality of multiplicities of λ and $\lambda^{(0)}$ arises as $\mathbf{F}^k$ is a linear isomorphism (its inverse is $\mathbf{U}^k$) from the eigenspace of λ to the eigenspace of $\lambda^{(0)}$.

(2) Let $\lambda^{(0)} \in \sigma_{\text{odd}}(\mathcal{D})$ and $k \in \mathbb{N}_0$. By Lemma 2.5, $\mathbf{U}^k$ maps an eigenfunction of $\lambda^{(0)}$ to an eigenfunction of $2^k\lambda^{(0)}$. □

The last corollary implies that the spectrum has the following hierarchical structure:

$$\sigma(\mathcal{D}) = \bigsqcup_{k=0}^{\infty}\left(\bigcup_{\lambda^{(0)}\in\sigma_{\text{odd}}(\mathcal{D})}\left\{2^k\lambda^{(0)}\right\}\right) = \bigsqcup_{k=0}^{\infty}\left(\bigcup_{(m,n)\in\mathcal{O}}\left\{\lambda_{U_Q^k(m,n)}\right\}\right), \tag{2.11}$$

where the second equality follows, since

$$\begin{aligned} \lambda^{(0)} \in \sigma_{\text{odd}}(\mathcal{D}) \quad &\Longleftrightarrow \quad \lambda^{(0)} = \lambda_{m,n} \text{ s.t. } (m,n) \in \mathcal{O} \\ &\Longleftrightarrow \quad 2^k\lambda^{(0)} = \lambda_{U_Q^k(m,n)} \text{ s.t. } (m,n) \in \mathcal{O}. \end{aligned}$$

We will use this structure to divide the spectrum into three subsets and rule out separately the Courant-sharpness of the eigenvalues in each of those subsets (the three parts of Proposition 3.1).

In the following, we will show for a given $k \in \mathbb{N}_0$ and any $\lambda^{(0)} \in \sigma_{\text{odd}}(\mathcal{D})$ that the eigenfunctions corresponding to the eigenvalue $2^k\lambda^{(0)}$ all vanish on a specific (k-dependent) subset of $\mathcal{D}$.

Definition 2.7 Let $A \subseteq \mathcal{D}$, we define the unfold of A as (Fig. 1)

$$U(A) = \left\{(x,y) \in \frac{1}{2}\mathcal{D} \,\middle|\, F(x,y) \in A\right\} \cup \left\{(x,y) \in \mathcal{D}\backslash\frac{1}{2}\mathcal{D} \,\middle|\, F \circ R(x,y) \in A\right\}. \tag{2.12}$$

For $k \geq 0$, we define the k-frame as (see Fig. 2)

$$S^{(0)} = \mathrm{L} \quad \text{and} \quad \forall k \geq 1, \quad S^{(k)} = U^k(\mathrm{L}). \tag{2.13}$$

Proposition 2.8 *Let* $k \in \mathbb{N}_0$ *and* $\lambda^{(0)} \in \sigma_{odd}(\mathcal{D})$, *then any eigenfunction corresponding to* $2^k\lambda^{(0)}$ *vanishes on the k-frame.*

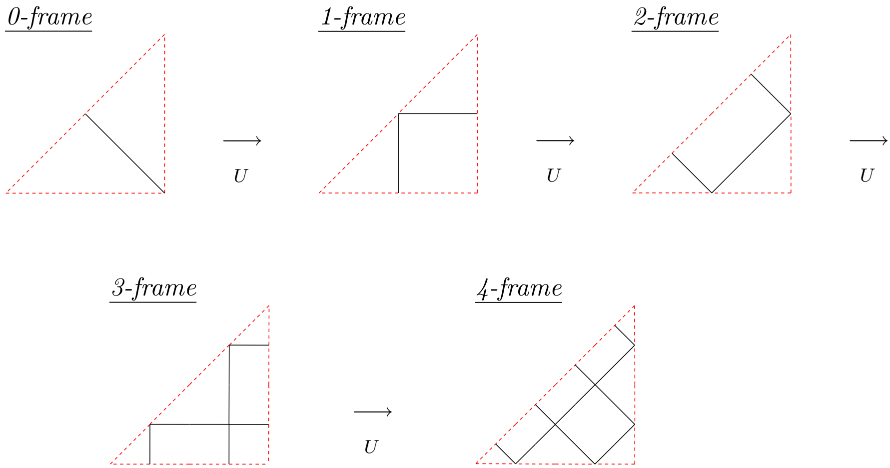


Fig. 2 The first five k-frames indicated by *black solid lines*

Proof Let $\lambda^{(0)} \in \sigma_{\text{odd}}(\mathcal{D})$, we shall prove the claim for $2^k\lambda^{(0)}$ by induction on k. For $k = 0$, by Lemma 2.1, we get that any eigenfunction φ of $\lambda^{(0)}$ is anti-symmetric with respect to L, and therefore

$$\varphi|_{S^{(0)}} = 0.$$

Next assume that the claim holds for $k-1$, and let φ be an eigenfunction corresponding to $2^k\lambda^{(0)}$. By Lemma 2.5, we have that $F\varphi$ is an eigenfunction corresponding to the eigenvalue $2^{k-1}\lambda^{(0)}$, and hence

$$\mathbf{F}\varphi|_{S^{(k-1)}} = 0.$$

Now, let $(x, y) \in S^{(k)}$. Since $S^{(k)} = U\left(S^{(k-1)}\right)$, we get that $F(x, y) \in S^{(k-1)}$ or $F \circ R(x, y) \in S^{(k-1)}$ [see (2.12)]. If $F(x, y) \in S^{(k-1)}$, then

$$\varphi(x, y) = \varphi \circ U \circ F(x, y) = \mathbf{F}\varphi(F(x, y)) = 0.$$

If $F \circ R(x, y) \in S^{(k-1)}$, we note that φ is symmetric as $2^k\lambda^{(0)}$ is even and hence

$$\varphi(x, y) = \varphi(R(x, y)) = \varphi \circ U \circ F(R(x, y)) = \mathbf{F}\varphi(F \circ R(x, y)) = 0.$$

Therefore, $\varphi|_{S^{(k)}} = 0$. $\square$

3 Proof of Theorem 1.1

Using the definitions of Sect. 2, the set of Courant-sharp eigenvalues according to Theorem 1.1 can be written as

$$\mathcal{C} = \{0\} \cup \left\{\lambda_{U_Q^k(1,0)}\right\}_{k=0}^{3}. \tag{3.1}$$

We divide the remaining eigenvalues, $\sigma(\mathcal{D})\backslash\mathcal{C}$, into three subsets and rule out their Courant-sharpness by the following proposition.

Proposition 3.1 *The eigenvalues of each of the following sets are not Courant-sharp:*

(1)

$$\Lambda^{(1)} := \bigcup_{(m,n)\,\in\,\mathcal{O}\backslash\,\{(1,0)\}} \left\{\lambda_{m,n}\right\}$$

(2)

$$\Lambda^{(2)} := \bigsqcup_{k=1}^{\infty}\left(\bigcup_{(m,n)\in\mathcal{O}:\, n\neq 0}\left\{\lambda_{U_{\mathcal{Q}}^{k}(m,n)}\right\}\right)$$

(3)

$$\Lambda^{(3)} := \bigsqcup_{k=1}^{\infty}\left(\bigcup_{(m,0)\in\mathcal{O}\backslash\{(1,0)\}}\left\{\lambda_{U_{\mathcal{Q}}^{k}(m,0)}\right\}\right)\bigcup\left\{\lambda_{U_{\mathcal{Q}}^{k}(1,0)}\right\}_{k=4}^{\infty}.$$

The proofs of the three parts of the proposition are essentially different and each appears in a designated subsection.

3.1 Proving Proposition 3.1(1)

We start by providing some additional constructions, needed for the proof.

Definition 3.2 We define the following subsets of the lattice $\mathcal{Q}$:

(1) For $0 \leq \lambda \in \mathbb{R}$, we define

$$\begin{aligned}\mathcal{O}(\lambda) &= \mathcal{Q}(\lambda) \cap \mathcal{O},\\ \mathcal{E}(\lambda) &= \mathcal{Q}(\lambda) \cap \mathcal{E}.\end{aligned} \tag{3.2}$$

(2) Let $A \subseteq \mathcal{Q}$, define $\underset{\rightarrow}{\partial} A$ to be the set of points in A, such that their right neighbor is outside A, that is

$$\underset{\rightarrow}{\partial} A = \{(m,n) \in A \mid (m+1,n) \notin A\}. \tag{3.3}$$

We consider the following auxiliary eigenvalue problem on $\frac{1}{2}\mathcal{D}$ with mixed Dirichlet–Neumann boundary conditions:

$$-\Delta\varphi = \lambda\varphi, \quad \varphi\Big|_{\mathrm{L}} = 0, \quad \frac{\partial\varphi}{\partial\vec{n}}\Big|_{\partial\left(\frac{1}{2}\mathcal{D}\right)\backslash L} = 0. \tag{3.4}$$

Denote the corresponding spectrum by $\sigma\left(\frac{1}{2}\mathcal{D}\right)$ and the spectral counting functions of (3.4) by $\underline{\tilde{N}}(\lambda)$, $\tilde{N}(\lambda)$ as in (1.3)–(1.4).

Lemma 3.3 *Let* $0 \le \lambda \in \mathbb{R}$, *then*

$$\underline{\tilde{N}}(\lambda) = |\mathcal{O}(\lambda)|\,. \tag{3.5}$$

Proof Note that if $\lambda_{p,q} \in \sigma_{\text{odd}}(\mathcal{D})$ and $\tilde{\varphi}$ is any of its eigenfunctions, then by Lemma 2.1, it follows that $\tilde{\varphi}\big|_{\frac{1}{2}\mathcal{D}}$ is an eigenfunction of (3.4) which means that $\lambda_{p,q} \in \sigma\left(\frac{1}{2}\mathcal{D}\right)$. This motivates to consider the mapping

$$\Phi : \mathcal{O}(\lambda) \to \left\{\tilde{\lambda} \in \sigma\left(\frac{1}{2}\mathcal{D}\right) \middle|\ \tilde{\lambda} < \lambda\right\}$$

$$\Phi : (p, q) \longmapsto \lambda_{p,q},$$

where we note that the set $\left\{\tilde{\lambda} \in \sigma\left(\frac{1}{2}\mathcal{D}\right) \middle|\ \tilde{\lambda} < \lambda\right\}$ contains each eigenvalue as many times as its multiplicity in $\sigma\left(\frac{1}{2}\mathcal{D}\right)$. Showing that Φ is a bijection proves the lemma. To show that Φ is onto, let $\tilde{\lambda} \in \left\{\tilde{\lambda} \in \sigma\left(\frac{1}{2}\mathcal{D}\right) \middle|\ \tilde{\lambda} < \lambda\right\}$ and extend one of its corresponding eigenfunctions $\tilde{\varphi}$ anti-symmetrically along L, i.e., consider

$$\varphi(x, y) = \begin{cases} \tilde{\varphi}(x, y) & (x, y) \in \frac{1}{2}\mathcal{D} \\ -\tilde{\varphi} \circ R(x, y) & (x, y) \in \mathcal{D}\backslash\frac{1}{2}\mathcal{D} \end{cases}.$$

By the reflection principle (see [28] for example), φ is an odd eigenfunction of (1.1). By Lemma 2.1, we deduce $\tilde{\lambda} \in \sigma_{\text{odd}}(\mathcal{D})$, and so there exists $(p, q) \in \mathcal{O}(\lambda)$, such that $\lambda_{p,q} = \tilde{\lambda}$. To show that Φ is an injection, take $(p_1, q_1) \neq (p_2, q_2)$, and observe that the eigenfunctions $\varphi_{p_1,q_1}, \varphi_{p_2,q_2}$ are linearly independent and anti-symmetric, and hence, $\varphi_{p_1,q_1}\big|_{\frac{1}{2}\mathcal{D}}, \varphi_{p_2,q_2}\big|_{\frac{1}{2}\mathcal{D}}$ are linearly independent and are eigenfunctions of (3.4), which means that $\Phi(p_1, q_1) \neq \Phi(p_2, q_2)$. □

We are now able to prove Proposition 3.1(1).

Proof of Proposition 3.1(1) Let $\lambda_{m,n}$ be such that $(m, n) \in \mathcal{O}\backslash\{(1, 0)\}$ and φ be any eigenfunction corresponding to $\lambda_{m,n}$, Lemma 2.1 gives

$$\nu(\varphi) = 2 \cdot \nu\left(\varphi|_{\frac{1}{2}\mathcal{D}}\right). \tag{3.6}$$

Since $\varphi|_{\frac{1}{2}\mathcal{D}}$ is an eigenfunction of (3.4) with an eigenvalue $\lambda_{m,n}$, we get by Courant's nodal theorem [16] that

$$\nu\left(\varphi|_{\frac{1}{2}\mathcal{D}}\right) \le \tilde{N}\left(\lambda_{m,n}\right), \tag{3.7}$$

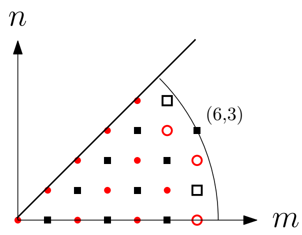


Fig. 3 *Red disks* correspond to $\mathcal{E}\left(\lambda_{6,3}\right)$ and the *black squares* correspond to $\mathcal{O}\left(\lambda_{6,3}\right)$. The *empty disks* and *squares* correspond to $\underset{\rightarrow}{\partial}\mathcal{Q}\left(\lambda_{6,3}\right)$ (color figure online)

so that

$$\nu(\varphi) \leq 2 \cdot \tilde{N}\left(\lambda_{m,n}\right)$$

and by (3.5), we obtain

$$\nu(\varphi) \leq 2 \cdot \left(\mathcal{O}\left(\lambda_{m,n}\right) + 1\right). \tag{3.8}$$

Next, observe that the following mapping:

$$\begin{gathered} B: \ \mathcal{O}\left(\lambda_{m,n}\right) \to \mathcal{E}\left(\lambda_{m,n}\right) \setminus \left(\underset{\rightarrow}{\partial}\mathcal{Q}\left(\lambda_{m,n}\right) \cap \mathcal{E}\right) \\ B(p,q) = (p-1,q), \end{gathered} \tag{3.9}$$

is a bijection (see Fig. 3), and thus, we obtain

$$\left|\mathcal{O}\left(\lambda_{m,n}\right)\right| = \left|\mathcal{E}\left(\lambda_{m,n}\right)\right| - \left|\underset{\rightarrow}{\partial}\mathcal{Q}\left(\lambda_{m,n}\right) \cap \mathcal{E}\right|. \tag{3.10}$$

We now have

$$\begin{aligned} \nu(\varphi) \underbrace{\leq}_{(3.8)} 2\left(\left|\mathcal{O}\left(\lambda_{m,n}\right)\right| + 1\right) \underbrace{=}_{(3.10)} & \left|\mathcal{O}\left(\lambda_{m,n}\right)\right| + \left|\mathcal{E}\left(\lambda_{m,n}\right)\right| + 1 \\ & + \left(1 - \left|\underset{\rightarrow}{\partial}\mathcal{Q}\left(\lambda_{m,n}\right) \cap \mathcal{E}\right|\right). \end{aligned} \tag{3.11}$$

Note that

$$\mathcal{Q}\left(\lambda_{m,n}\right) = \mathcal{E}\left(\lambda_{m,n}\right) \bigsqcup \mathcal{O}\left(\lambda_{m,n}\right),$$

hence

$$\underline{N}\left(\lambda_{m,n}\right) = \left|\mathcal{O}\left(\lambda_{m,n}\right)\right| + \left|\mathcal{E}\left(\lambda_{m,n}\right)\right|. \tag{3.12}$$

Combining (3.11) with (3.12), we get

$$\nu(\varphi) \leq N\left(\lambda_{m,n}\right) + \left(1 - \left|\underset{\rightarrow}{\partial}\mathcal{Q}\left(\lambda_{m,n}\right) \cap \mathcal{E}\right|\right). \tag{3.13}$$

Therefore, to rule out the Courant-sharpness of $\lambda_{m,n}$, we only require that

$$\left| \underset{\rightarrow}{\partial} \mathcal{Q}\left(\lambda_{m,n}\right) \cap \mathcal{E} \right| > 1. \tag{3.14}$$

Indeed, since $(m,n) \in \mathcal{O}$, we get $(m-1,n) \in \underset{\rightarrow}{\partial} \mathcal{Q}\left(\lambda_{m,n}\right) \cap \mathcal{E}$ and we are left with finding one more point $(p,q) \in \underset{\rightarrow}{\partial} \mathcal{Q}\left(\lambda_{m,n}\right) \cap \mathcal{E}$. As we consider $(m,n) \in \mathcal{O}\backslash\{(1,0)\}$, a simple calculation shows that if $n \geq 1$, then $(m, n-1) \in \underset{\rightarrow}{\partial} \mathcal{Q}\left(\lambda_{m,n}\right) \cap \mathcal{E}$, and if $n = 0$, we have $(m-1,2) \in \underset{\rightarrow}{\partial} \mathcal{Q}\left(\lambda_{m,0}\right) \cap \mathcal{E}$. □

Remark It is easy to see that the last argument does not work for $(m,n) = (1,0)$. Indeed, we show later that this is a Courant-sharp eigenvalue (Lemma 3.9).

3.2 Proving Proposition 3.1(2)

The k-frame structure divides the triangle into k-dependent number of subdomains. This is defined below and is used in the proofs of the current subsection.

Definition 3.4 Define the k-frame partition as

$$\mathcal{P}^{(k)} := \mathcal{D}^{\circ}\backslash S^{(k)} = \bigsqcup_{i=1}^{M(k)} \mathcal{D}_i^{(k)},$$

where $\left\{\mathcal{D}_i^{(k)}\right\}_{i=1}^{M(k)}$ denote the subdomains of this partition and $M(k)$ is their number. Consider the following eigenvalue problems with the boundary conditions induced by the k-frame:

$$-\Delta\varphi = \lambda\varphi, \quad \varphi\Big|_{S^{(k)} \bigcap \partial\mathcal{D}_i^{(k)}} = 0, \quad \frac{\partial\varphi}{\partial\vec{n}}\Big|_{\partial\mathcal{D} \bigcap \partial\mathcal{D}_i^{(k)}} = 0. \tag{3.15}$$

We denote the corresponding spectra by $\sigma\left(\mathcal{D}_i^{(k)}\right)$ and define the corresponding spectral counting functions, and multiplicities

$$\overline{N}_i^{(k)}(\lambda), \ \underline{N}_i^{(k)}(\lambda), \ N_i^{(k)}(\lambda), \ \mathrm{d}_i^{(k)}(\lambda),$$

as in (1.2)–(1.5).

Next, we bring two lemmata, the second of which provides necessary conditions for an eigenvalue to be Courant-sharp.

Lemma 3.5 *Let $k \in \mathbb{N}_0$ and $\lambda^{(0)} \in \sigma_{odd}(\mathcal{D})$. We have that*

$$\mathrm{d}\left(2^k\lambda^{(0)}\right) \leq \mathrm{d}_i^{(k)}\left(2^k\lambda^{(0)}\right) \quad i \in \{1,\ldots,M(k)\}. \tag{3.16}$$

Proof Let $\lambda = 2^k \lambda^{(0)}$ and let

$$\mathcal{B} = \left\{\varphi_1, \ldots, \varphi_{\mathrm{d}(\lambda)}\right\}$$

be a basis for the eigenspace of λ. Let $i \in \{1, \ldots, M(k)\}$, by Proposition 2.8, we have that $\mathcal{B}' = \left\{ \varphi_1 \Big|_{\mathcal{D}_i^{(k)}}, \ldots, \varphi_{\mathrm{d}(\lambda)}\Big|_{\mathcal{D}_i^{(k)}} \right\}$ are eigenfunctions of (3.15) on the domain $\mathcal{D}_i^{(k)}$. Assume by contradiction that the set $\mathcal{B}'$ turns out to be linearly dependent, then we have scalars $\alpha_l \in \mathbb{R}$ not all zero, such that

$$\sum_l \alpha_l \varphi_l|_{\mathcal{D}_i^{(k)}} \equiv 0.$$

However, then the eigenfunction $\sum_l \alpha_l \varphi_l$ of (1.1) vanishes on the open subset $\mathcal{D}_i^{(k)}$, and by the unique continuation property [4], we obtain that

$$\sum_l \alpha_l \varphi_l \equiv 0,$$

contradicting the linear independence of $\mathcal{B}$. Thus, it follows that the dimension of the eigenspace that corresponds to $\lambda \in \sigma\left(\mathcal{D}_i^{(k)}\right)$ is at least $\mathrm{d}(\lambda)$. □

Remark The strict inequality in (3.16) may, indeed, occur. This can be demonstrated by applying Corollary 2.6(1) and Lemma 3.25 to some simple eigenvalue $\lambda_{m,n} \in \sigma_{\mathrm{odd}}(\mathcal{D})$, such that $n \neq 0$.

Lemma 3.6 *Let $k \in \mathbb{N}_0$ and $\lambda^{(0)} \in \sigma_{odd}(\mathcal{D})$. If $2^k \lambda^{(0)}$ is a Courant-sharp eigenvalue of $\mathcal{D}$, then both of the following hold:*

(1) The eigenvalue $2^k \lambda^{(0)}$ is a simple eigenvalue in $\sigma\left(\mathcal{D}_i^{(k)}\right)$ for all $i \in \{1, \ldots, M(k)\}$.
(2) The eigenvalue $2^k \lambda^{(0)}$ is a simple eigenvalue in $\sigma(\mathcal{D})$.

Proof Let φ be a Courant-sharp eigenfunction of $2^k \lambda^{(0)} \in \sigma(\mathcal{D})$, then

$$N\left(2^k \lambda^{(0)}\right) = \nu(\varphi). \tag{3.17}$$

By Proposition 2.8, we have

$$\nu(\varphi) = \sum_{i=1}^{M(k)} \nu\left(\varphi|_{\mathcal{D}_i^{(k)}}\right). \tag{3.18}$$

For all $i \in \{1, \ldots, M(k)\}$, we have that $\varphi|_{\mathcal{D}_i^{(k)}}$ is an eigenfunction of the eigenvalue problem (3.15), and therefore, by Courant's nodal theorem, we have

$$\sum_{i=1}^{M(k)} \nu\left(\varphi|_{\mathcal{D}_i^{(k)}}\right) \leq \sum_{i=1}^{M(k)} N_i^{(k)}\left(2^k \lambda^{(0)}\right). \tag{3.19}$$

Rewriting the right-hand side of (3.19) and combining (3.17) and (3.18), we arrive at

$$N\left(2^k\lambda^{(0)}\right) \leq \sum_{i=1}^{M(k)} \overline{N}_i^{(k)}\left(2^k\lambda^{(0)}\right) + \sum_{i=1}^{M(k)}\left[N_i^{(k)}\left(2^k\lambda^{(0)}\right) - \overline{N}_i^{(k)}\left(2^k\lambda^{(0)}\right)\right].$$

If we consider the eigenvalue problem on $\bigcup_i \mathcal{D}_i^{(k)}$ and use the variational principle to compare with the eigenvalue problem on $\mathcal{D}$, we obtain

$$\sum_{i=1}^{M(k)} \overline{N}_i^{(k)}\left(2^k\lambda^{(0)}\right) \leq \overline{N}\left(2^k\lambda^{(0)}\right). \tag{3.20}$$

The conclusion above appears, for example, in [17], page 408, Theorem 2 for Dirichlet boundary conditions. Having Neumann boundary conditions, as in our case, brings to the same conclusion.

It follows that

$$\sum_{i=1}^{M(k)}\left[\overline{N}_i^{(k)}\left(2^k\lambda^{(0)}\right) - N_i^{(k)}\left(2^k\lambda^{(0)}\right)\right] \leq \overline{N}\left(2^k\lambda^{(0)}\right) - N\left(2^k\lambda^{(0)}\right). \tag{3.21}$$

By Lemma 3.5, we have

$$\overline{N}\left(2^k\lambda^{(0)}\right) - N\left(2^k\lambda^{(0)}\right) \leq \overline{N}_i^{(k)}\left(2^k\lambda^{(0)}\right) - N_i^{(k)}\left(2^k\lambda^{(0)}\right) \quad \forall i \in \{1, \ldots, M(k)\}. \tag{3.22}$$

Plugging this in (3.21), we get

$$\begin{aligned}&\sum_{l=1}^{M(k)}\left[\overline{N}_l^{(k)}\left(2^k\lambda^{(0)}\right) - N_l^{(k)}\left(2^k\lambda^{(0)}\right)\right]\\ &\quad\leq \overline{N}_i^{(k)}\left(2^k\lambda^{(0)}\right) - N_i^{(k)}\left(2^k\lambda^{(0)}\right) \quad \forall i \in \{1, \ldots, M(k)\},\end{aligned}$$

and as $M(k) \geq 2$, it has to be that

$$\overline{N}_i^{(k)}\left(2^k\lambda^{(0)}\right) - N_i^{(k)}\left(2^k\lambda^{(0)}\right) = 0 \quad \forall i \in \{1, \ldots, M(k)\},$$

which proves the first part of the lemma. The second part follows immediately from a combination of the first part with Lemma 3.5. □

Remark The first claim of Lemma 3.6 is not restricted to the domains dealt with so far and, indeed, appears in a more general form in Corollary 3.5(ii) of [2], where it is proven for domains with Dirichlet boundary conditions. Yet, the second claim of Lemma 3.6 does not hold for arbitrary domains, as it is based on the inequality (3.16) which is not satisfied, in general.

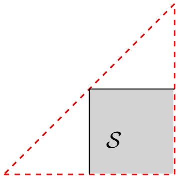


Fig. 4 Subdomain $\mathcal{S}$ as it appears in the 1-frame partition, $\mathcal{P}_1$

Remark Lemma 3.6(2) may be also obtained as a direct corollary of Lemma B.1 (appears in Appendix B) and Corollary 2.6(1). In fact, Lemma B.1 is a generalization of Lemma 3.6(2).

With Lemma 3.6 at hand, it is now possible to rule out the Courant-sharpness of many more eigenvalues. This is done by applying the lemma to the following particular subdomains of the triangle (see Figs. 4, 5).

Definition 3.7 We define the following subdomains of the k-frame partition:

(1) A square subdomain $\mathcal{S} \in \mathcal{P}^{(1)}$ expressed by

$$\mathcal{S} = \left(\frac{\pi}{2}, \pi\right) \times \left(0, \frac{\pi}{2}\right). \tag{3.23}$$

(2) Rectangular subdomains $\mathcal{R}^{(k)} \in \mathcal{P}^{(k)}$, $\forall k \geq 2$, expressed recursively by

$$\mathcal{R}^{(2)} = \left(U\left(\overline{\mathcal{S}}\right)\right)^{\circ} \quad \text{and} \quad \mathcal{R}^{(k)} = \left(\left\{(x, y) \in \mathcal{D} \middle| \, F(x, y) \in \overline{\mathcal{R}^{(k-1)}}\right\}\right)^{\circ}. \tag{3.24}$$

Lemma 3.8 *Let* $(m, n) \in \mathcal{O}$, $n \neq 0$*. Then, the following holds:*

(1) *Consider the eigenvalue problem* (3.15) *on* $\mathcal{S}$*. Then,* $\lambda_{U_{\mathcal{Q}}(m,n)} \in \sigma(\mathcal{S})$ *is a non-simple eigenvalue.*
(2) *Let* $k > 1$ *and consider the eigenvalue problem* (3.15) *on* $\mathcal{R}^{(k)}$*. Then,* $\lambda_{U^k_{\mathcal{Q}}(m,n)} \in \sigma\left(\mathcal{R}^{(k)}\right)$ *is a non-simple eigenvalue.*

Proof We start by giving explicit expressions for the eigenvalues and the eigenfunctions of (3.15) on the domains $\mathcal{S}$ and $\mathcal{R}^{(k)}$. To do that, we choose the following convenient parametrizations for the domains. First, we consider $\mathcal{S}$, which we write as

$$\mathcal{S} = \left(0, \frac{\pi}{2}\right) \times \left(0, \frac{\pi}{2}\right).$$

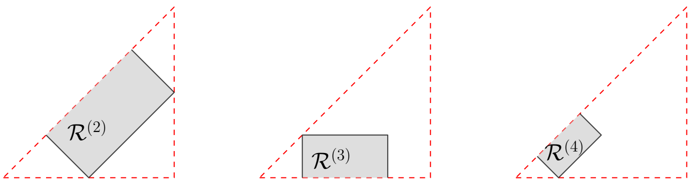


Fig. 5 Subdomain $\mathcal{R}^{(k)}$ as it appears in the k-frame partition, $\mathcal{P}^{(k)}$, for $k = 2, 3, 4$

The boundary conditions induced by the 1-frame are expressed by

$$\hat{\varphi}\big|_{\{x=0 \text{ or } y=0\}} \equiv 0\ ; \quad \frac{\partial\hat{\varphi}}{\partial n}\Big|_{\partial\mathcal{S}\setminus\{x=0 \text{ or } y=0\}} \equiv 0.$$

The eigenvalues are

$$\hat{\lambda}_{p,q} = (2p+1)^2 + (2q+1)^2 \quad (p,q) \in \mathbb{N}_0 \times \mathbb{N}_0 \tag{3.25}$$

and the orthogonal set of eigenfunctions is given by

$$\hat{\varphi}_{p,q}(x,y) = \sin((2p+1)x)\sin((2q+1)y)\ ; \ \ (p,q) \in \mathbb{N}_0 \times \mathbb{N}_0. \tag{3.26}$$

We proceed with $\mathcal{R}^{(k)}$. The edges have ratio $1:2$ and the longest one is of length $l = \frac{\pi}{2^{\frac{k-1}{2}}}$, thus we may write

$$\mathcal{R}^{(k)} = (0,l) \times \left(0, \frac{l}{2}\right).$$

The boundary conditions induced by the k-frame are expressed by

$$\hat{\varphi}\big|_{\partial\mathcal{R}^{(k)}\setminus\left\{y=\frac{l}{2}\right\}} \equiv 0\ ; \quad \frac{\partial\hat{\varphi}}{\partial n}\Big|_{\left\{y=\frac{l}{2}\right\}} \equiv 0.$$

The eigenvalues are given to be

$$\hat{\lambda}_{p,q} = 2^{k-1}\left(p^2+q^2\right)\ ; \ (p,q) \in \mathbb{N} \times [2\mathbb{N}_0+1] \tag{3.27}$$

and the orthogonal set of eigenfunctions is given by

$$\hat{\varphi}_{p,q}(x,y) = \sin\left(\frac{\pi\cdot p\cdot x}{l}\right)\sin\left(\frac{\pi\cdot q\cdot y}{l}\right) \ \ (p,q) \in \mathbb{N} \times [2\mathbb{N}_0+1]. \tag{3.28}$$

We proceed to prove both parts of the lemma by pointing out on two linearly independent eigenfunctions which correspond to the relevant eigenvalue. Recall that we consider $(m,n) \in \mathcal{O}$, with $n \neq 0$.

(1) Define

$$(p_1,q_1) = \left(\frac{m+n-1}{2}, \frac{m-n-1}{2}\right),$$
$$(p_2,q_2) = (q_1,p_1).$$

As $m \neq n \pmod 2$, we get (p_1,q_1), $(p_2,q_2) \in \mathbb{N}_0 \times \mathbb{N}_0$, and since $n \neq 0$, we get $(p_1,q_1) \neq (p_2,q_2)$. By (3.26), we get that $\hat{\varphi}_{p_1,q_1}$ and $\hat{\varphi}_{p_2,q_2}$ are linearly independent and by (3.27), we obtain

$$2\left(m^2+n^2\right)=\lambda_{U_{\mathcal{Q}}(m,n)}=\hat{\lambda}_{p_1,q_1}=\hat{\lambda}_{p_2,q_2}.$$

Thus, the eigenvalue $\lambda_{U_{\mathcal{Q}}(m,n)}\in\sigma\left(\mathcal{S}\right)$ is non-simple.

(2) Define

$$(p_1,q_1)=(m+n,m-n),$$
$$(p_2,q_2)=(q_1,p_1).$$

As $m\neq n \pmod 2$, we get $(p_1,q_1),\ (p_2,q_2)\in\mathbb{N}\times[2\mathbb{N}_0+1]$, and since $n\neq 0$, we get $(p_1,q_1)\neq(p_2,q_2)$. By (3.28), we get that $\hat{\varphi}_{p_1,q_1}$ and $\hat{\varphi}_{p_2,q_2}$ are linearly independent, and by (3.25), we obtain

$$2^k\left(m^2+n^2\right)=\lambda_{U^k_{\mathcal{Q}}(m,n)}=\hat{\lambda}_{p_1,q_1}=\hat{\lambda}_{p_2,q_2}.$$

Thus, the eigenvalue $\lambda_{U^k_{\mathcal{Q}}(m,n)}\in\sigma\left(\mathcal{R}^{(k)}\right)$ is non-simple.

Proposition 3.1(2) follows immediately by combining Lemma 3.6(1) with Lemma 3.25.

3.3 Proving Proposition 3.1(3)

By Lemma 3.6(2), it follows that we only need to rule out the Courant-sharpness of the simple eigenvalues of $\Lambda^{(3)}$.

Proof of Proposition 3.1(3) To show that the simple eigenvalues of $\Lambda^{(3)}$ are not Courant-sharp, we find, in the following, a subset $\mathcal{T}_{\mathcal{Q}}\left(\lambda_{m,n}\right)\subsetneq\mathcal{Q}\left(\lambda_{m,n}\right)\cup\{(m,n)\}$, such that $\left|\mathcal{T}_{\mathcal{Q}}\left(\lambda_{m,n}\right)\right|=\nu\left(\varphi_{m,n}\right)$. This will rule out Courant-sharpness of a simple eigenvalue, since then

$$N\left(\lambda_{m,n}\right)=\left|\mathcal{Q}\left(\lambda_{m,n}\right)\cup\{(m,n)\}\right|>\left|\mathcal{T}_{\mathcal{Q}}\left(\lambda_{m,n}\right)\right|=\nu\left(\varphi_{m,n}\right).$$

Before proceeding, we rewrite $\Lambda^{(3)}$ as an expression that is more adjusted to the following arguments:

$$\Lambda^{(3)}=\left\{\lambda_{2m,0}\right\}_{m=3}^{\infty}\cup\left\{\lambda_{m,m}\right\}_{m=3}^{\infty}.$$

First, we treat the case of a simple eigenvalue $\lambda_{m,m}$ for $m\geq 3$. Its nodal set is

$$\varphi^{-1}_{m,m}\{0\}=\left\{(x,y)\in\mathcal{D}\,\middle|\,\cos(mx)\cos(my)=0\right\}.$$

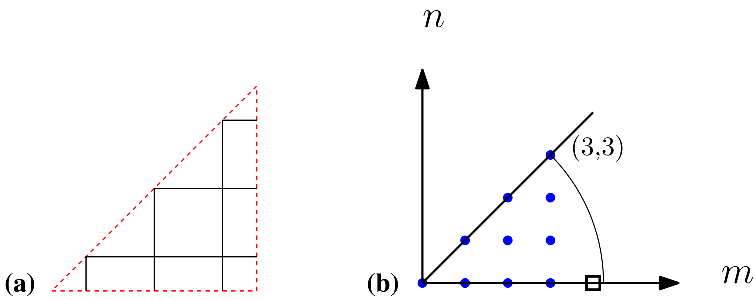


Fig. 6 **a** Nodal set of $\varphi_{3,3}$. **b** *Blue points* correspond to $\mathcal{T}_{\mathcal{Q}}\left(\lambda_{3,3}\right)$ and the *empty square* corresponds to the point (4, 0) (color figure online)

Thus, we deduce that there are m nodal lines inside $\mathcal{D}$ parallel to the x-axis and m that are parallel to the y-axis (Fig. 6a). The number of nodal domains is, therefore,

$$\nu\left(\varphi_{m,m}\right) = \sum_{i=0}^{m}(i+1). \tag{3.29}$$

Denote

$$\mathcal{T}_{\mathcal{Q}}\left(\lambda_{m,m}\right) := \left\{(i,j)\ \middle|\ 0 \leq j \leq i \leq m\right\},$$

and observe that

$$\left|\mathcal{T}_{\mathcal{Q}}\left(\lambda_{m,m}\right)\right| = \sum_{i=0}^{m}(i+1) = \nu\left(\varphi_{m,m}\right). \tag{3.30}$$

A simple calculation shows that (see Fig. 6b)

$$\mathcal{T}_{\mathcal{Q}}\left(\lambda_{m,m}\right) \subseteq \mathcal{Q}\left(\lambda_{m,m}\right) \cup \{(m,m)\}.$$

In addition, for $m \geq 3$, we have $(m+1,0) \in \mathcal{Q}\left(\lambda_{m,m}\right) \backslash \mathcal{T}_{\mathcal{Q}}\left(\lambda_{m,m}\right)$, since

$$\|(m+1,0)\|^2 = m^2 + 2m + 1 < 2m^2 = \lambda_{m,m}.$$

Thus, we showed

$$\mathcal{T}_{\mathcal{Q}}\left(\lambda_{m,m}\right) \subsetneq \mathcal{Q}\left(\lambda_{m,m}\right) \cup \{(m,m)\}.$$

Next, we treat the case of a simple eigenvalue $\lambda_{2m,0} \in \sigma(\mathcal{D})$ for $m \geq 3$. This eigenvalue is the unfolding of $\lambda_{m,m}$ which we treated above (and, therefore, their multiplicity is equal). Its nodal set is, therefore, determined easily (Fig. 7a) and the nodal count is given by

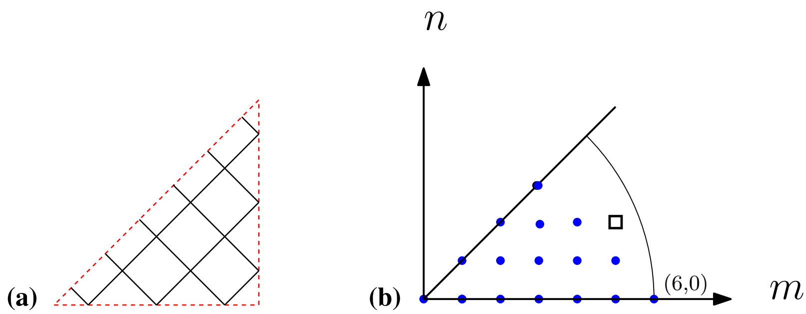


Fig. 7 **a** Nodal set of $\varphi_{6,0}$. **b** *Blue points* correspond to $\mathcal{T}_{\mathcal{Q}}\left(\lambda_{6,0}\right)$ and the *empty square* corresponds to the point (5, 2) (color figure online)

$$\nu\left(\varphi_{2m,0}\right)=2\nu\left(\varphi_{m,m}\right)-(m+1)=\sum_{i=0}^{m}(2i+1)\,. \tag{3.31}$$

Denote

$$\mathcal{T}_{\mathcal{Q}}\left(\lambda_{2m,0}\right)=\left\{(m+j,m-i)\;\middle|\;0\leq i\leq m\;,\;-i\leq j\leq i\right\}.$$

Observe that

$$\left|\mathcal{T}_{\mathcal{Q}}\left(\lambda_{2m,0}\right)\right|=\sum_{i=0}^{m}\sum_{j=-i}^{i}1=\sum_{i=0}^{m}(2i+1)=\nu\left(\varphi_{2m,0}\right), \tag{3.32}$$

and a simple calculation shows that (see Fig. 7b)

$$\mathcal{T}_{\mathcal{Q}}\left(\lambda_{2m,0}\right)\subseteq\mathcal{Q}\left(\lambda_{2m,0}\right)\cup\{(2m,0)\}\,.$$

Observe that for $m\geq 3$, we have $(2m-1,2)\in\mathcal{Q}\left(\lambda_{2m,0}\right)\backslash\mathcal{T}_{\mathcal{Q}}\left(\lambda_{2m,0}\right)$, since

$$\|(2m-1,2)\|^2=4m^2-4m+5<4m^2=\lambda_{2m,0}.$$

Thus, we showed

$$\mathcal{T}_{\mathcal{Q}}\left(\lambda_{2m,0}\right)\subsetneq\mathcal{Q}\left(\lambda_{2m,0}\right)\cup\{(2m,0)\}\,.$$

$\square$

3.4 Concluding the Proof of Theorem 1.1

Finally, Theorem 1.1 is proved once we show that the eigenvalues we have not ruled out are, indeed, Courant-sharp.

Lemma 3.9 *The eigenvalues of* $\mathcal{C}=\{0\}\cup\left\{\lambda_{U_{\mathcal{Q}}^k(1,0)}\right\}_{k=0}^{3}$ *are Courant-sharp*

Proof By Courant's bound and orthogonality, the first two eigenvalues, $\lambda_{0,0} = 0$ and $\lambda_{1,0} = 1$, are Courant-sharp. Next, note that the eigenvalue $\lambda_{1,0}$ is simple, and therefore, all of the eigenvalues in $\left\{\lambda_{U_Q^k(1,0)}\right\}_{k=1}^{3}$ are simple as well. It is now straightforward to find the number of nodal domains of the eigenfunctions in the set $\left\{\varphi_{U_Q^k(1,0)}\right\}_{k=1}^{3}$ (see (3.29), (3.31)) and verify that those three eigenvalues are Courant-sharp as well. □

We end by noting that the nodal sets of the non-constant Courant-sharp eigenfunctions are exactly the first four k-frames (see Fig. 2).

4 Proof of Theorem 1.2

We start by developing the eigenfunction folding structure of an n-dimensional box, $\mathcal{B}^{(n)}$, whose edge length ratio is given by $\frac{l_j}{l_{j+1}} = \gamma_n := 2^{\frac{1}{n}}$ $(1 \le j \le n-1)$. For convenience, we choose a scaling according to which $l_1 = \pi$. We start by following the construction from Sect. 2 and present the folding structure of the $\mathcal{B}^{(n)}$ eigenfunctions.

The set of quantum numbers in this case is

$$\mathcal{Q} := \left\{\vec{m} \in \mathbb{N}_0^n\right\}. \tag{4.1}$$

The orthogonal basis of eigenfunctions is

$$\varphi_{\vec{m}}(\vec{x}) = \prod_{j=1}^{n} \cos(\gamma_n^{j-1} m_j x_j) \; ; \; \vec{m} \in \mathcal{Q} \tag{4.2}$$

and the corresponding eigenvalues are

$$\lambda_{\vec{m}} = \sum_{j=1}^{n} \left(\gamma_n^{j-1} m_j\right)^2 \; ; \; \vec{m} \in \mathcal{Q}. \tag{4.3}$$

We use the notations

$$\sigma\left(\mathcal{B}^{(n)}\right) := \left\{\lambda_{\vec{m}} \,\middle|\, \vec{m} \in \mathcal{Q}\right\},$$
$$\mathcal{Q}(\lambda) := \left\{\vec{m} \in \mathcal{Q} \,\middle|\, \lambda_{\vec{m}} < \lambda\right\}$$

and have as before that

$$\underline{N}(\lambda) = |\mathcal{Q}(\lambda)|.$$

The box $\mathcal{B}^{(n)}$ is symmetric with respect to the following hyperplane:

$$\mathrm{L} = \left\{\vec{x} \in \mathcal{B}^{(n)} \,\middle|\, x_1 = \frac{\pi}{2}\right\}, \tag{4.4}$$

and the reflection transformation is

$$R(\vec{x}) = (\pi - x_1,\ x_2, \ldots, x_n). \tag{4.5}$$

As opposed to the case of the triangle, the eigenvalues of $\mathcal{B}^{(n)}$ are not integers (with the exception of the case $n = 2$, where they are), but rather belong to $\mathbb{Z}\left[\gamma_n^2\right]$, a finite ring extension of $\mathbb{Z}$. We consider $\mathbb{Z}\left[\gamma_n^2\right]$ as a free module with the following basis:

$$\mathcal{G}^{(n)} = \begin{cases} \left\{\gamma_n^j\right\}_{j=0}^{n-1} & n \text{ is odd} \\ \left\{\gamma_n^{2j}\right\}_{j=0}^{\frac{n}{2}-1} & n \text{ is even.} \end{cases} \tag{4.6}$$

Furthermore, in the unique representation of $\lambda \in \sigma\left(\mathcal{B}^{(n)}\right)$ as a linear combination of this basis, the coefficients are taken from $\mathbb{N}_0$. This is used to define the parity of an eigenvalue.

Definition 4.1 Denoting by $\mathfrak{p}(\lambda)$ the coefficient multiplying $\gamma_n^0 = 1$ when spanning $\lambda \in \sigma\left(\mathcal{B}^{(n)}\right)$ by $\mathcal{G}^{(n)}$, we call λ an *odd* (*even*) *eigenvalue* if $\mathfrak{p}(\lambda)$ is odd (even).

Hence, we adopt here the dichotomy to even and odd eigenvalues, similarly to the one we had in Sect. 2. The parity of an eigenvalue dictates the parity of all of its eigenfunctions with respect to the reflection across L, which is proved in the following (analogously to Lemma 2.1).

Lemma 4.2 *Let* $\lambda \in \sigma\left(\mathcal{B}^{(n)}\right)$*, then any eigenfunction of* λ *is odd (even) with respect to* L if and only if λ *is an odd (even) eigenvalue.*

Proof Let $\vec{m} \in \mathcal{Q}$, such that $\varphi_{\vec{m}}$ is an eigenfunction corresponding to λ. Writing λ as $\sum_{j=1}^{n}\left(\gamma_n^{j-1} m_j\right)^2$ and using that $\mathcal{G}^{(n)}$ is a basis, we have that

$$\mathfrak{p}(\lambda) = \begin{cases} m_1^2 & n \text{ is odd} \\ m_1^2 + 2m_{\frac{n}{2}+1}^2 & n \text{ is even,} \end{cases} \tag{4.7}$$

and in both cases, the parity of λ equals the parity of m_1. From the explicit expression of the eigenfunction, (4.2), we see that $\varphi_{\vec{m}}$ is odd (even) with respect to L if and only if m_1 is odd (even). If λ is a multiple eigenvalue and it is odd (even), the argument above gives that the basis, $\left\{\varphi_{\vec{m}} \mid \lambda_{\vec{m}} = \lambda\right\}$, of its eigenspace consists of odd (even) eigenfunctions, and therefore, so is any eigenfunction of λ. □

As in Sect. 2, Lemma 4.2 motivates the following definition (compare with Definition 2.2).

Definition 4.3 We define the subsets of $\mathcal{Q}$ that correspond to the odd and even eigenvalues

$$\begin{aligned} \mathcal{O} &:= \{\vec{m} \in \mathcal{Q} \mid m_1 = 1\,(mod\ 2)\}, \\ \mathcal{E} &:= \{\vec{m} \in \mathcal{Q} \mid m_1 = 0\,(mod\ 2)\} \end{aligned} \tag{4.8}$$

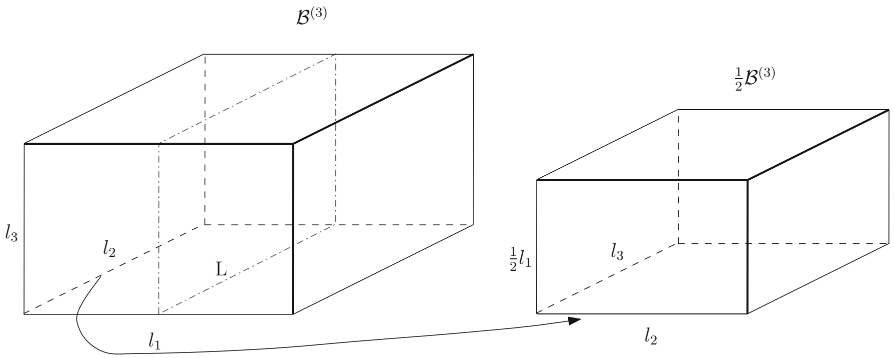


Fig. 8 Illustration of $\mathcal{B}^{(3)}$ decomposed into two similar *boxes*, one of which is $\frac{1}{2}\mathcal{B}^{(3)}$ (up to rotation)

and we denote the corresponding sets of eigenvalues by

$$\sigma_{\text{odd}}\left(\mathcal{B}^{(n)}\right) := \left\{\lambda_{\vec{m}}\middle|\vec{m}\in\mathcal{O}\right\},$$
$$\sigma_{\text{even}}\left(\mathcal{B}^{(n)}\right) := \left\{\lambda_{\vec{m}}\middle|\vec{m}\in\mathcal{E}\right\}.$$

Denote

$$\frac{1}{2}\mathcal{B}^{(n)} = \left\{\vec{x}\in\mathcal{D}:\ x_1\leq\frac{\pi}{2}\right\}.$$

Observe that L partitions $\mathcal{B}^{(n)}$ into the two isometric boxes $\frac{1}{2}\mathcal{B}^{(n)}$ and $\left(\mathcal{B}^{(n)}\setminus\frac{1}{2}\mathcal{B}^{(n)}\right)\cup$L, each is a scaled version of $\mathcal{B}^{(n)}$ by a factor γ_n. Namely,

$$(l_1,\ldots,l_{n-1},l_n) = \gamma_n\cdot\left(l_2,\ldots,l_n,\ \frac{l_1}{2}\right), \tag{4.9}$$

where the left-hand side gives the edge lengths of $\mathcal{B}^{(n)}$ and the right-hand side gives the edge lengths of $\frac{1}{2}\mathcal{B}^{(n)}$ (see Fig. 8).

Remark Equation (4.9) may be perceived as a generalization of the A series (A_3, A_4, etc.) paper sizes to higher dimensions.

This similarity reveals the folding structure of the $\mathcal{B}^{(n)}$ eigenfunctions. Indeed, the following two definitions and lemma are analogous to Definitions 2.3 and 2.4, and Lemma 2.5 of the triangle case.

Definition 4.4 The *coordinate folding transformation* is

$$\begin{aligned} F:\ &\tfrac{1}{2}\mathcal{B}^{(n)}\to\mathcal{B}^{(n)}\\ F\left(x_1,x_2,\ldots,x_n\right) &:= \gamma_n\cdot\left(x_2,\ x_3,\ldots,x_n,\ x_1\right)\end{aligned} \tag{4.10}$$

and the *coordinate unfolding transformation* U is the inverse of F and is expressed by

$$\begin{gathered} U : \mathcal{B}^{(n)} \to \tfrac{1}{2}\mathcal{B}^{(n)} \\ U(x_1, x_2, \ldots, x_n) = \gamma_n^{-1} \cdot (x_n,\ x_1, \ldots, x_{n-1}) . \end{gathered} \tag{4.11}$$

Definition 4.5 Let φ be an eigenfunction corresponding to the eigenvalue $\lambda \in \sigma\left(\mathcal{B}^{(n)}\right)$.

(1) Assume λ is even. Then, the *folded function* $\mathbf{F}\varphi$ is defined by

$$\mathbf{F}\varphi(\vec{x}) = \varphi \circ U(\vec{x}),\ \vec{x} \in \mathcal{B}^{(n)}. \tag{4.12}$$

(2) The *unfolded function*, $\mathbf{U}\varphi$, is

$$\mathbf{U}\varphi(\vec{x}) = \begin{cases} \varphi \circ F(\vec{x}) & \vec{x} \in \frac{1}{2}\mathcal{B}^{(n)} \\ (\varphi \circ F) \circ R(\vec{x}) & \vec{x} \in \mathcal{B}^{(n)} \backslash \frac{1}{2}\mathcal{B}^{(n)} \end{cases}. \tag{4.13}$$

Lemma 4.6 *Let $\varphi = \sum_{\vec{m};\ \lambda_{\vec{m}}=\lambda} \alpha_{\vec{m}} \cdot \varphi_{\vec{m}}$ be an eigenfunction corresponding to the eigenvalue $\lambda \in \sigma\left(\mathcal{B}^{(n)}\right)$.*

(1) *If λ is even, then the folded function is $\mathbf{F}\varphi = \sum_{\vec{m};\ \lambda_{\vec{m}}=\lambda} \alpha_{\vec{m}} \cdot \varphi_{F_{\mathcal{Q}}(\vec{m})}$ and corresponds to the eigenvalue $\lambda_{F_{\mathcal{Q}}(\vec{m})} = \gamma_n^{-2}\lambda_{\vec{m}}$, with*

$$F_{\mathcal{Q}}(\vec{m}) := \left(m_2, m_3, \ldots, m_n, \tfrac{m_1}{2}\right). \tag{4.14}$$

(2) *The unfolded function is $\mathbf{U}\varphi = \sum_{\vec{m};\ \lambda_{\vec{m}}=\lambda} \alpha_{\vec{m}} \cdot \varphi_{U_{\mathcal{Q}}(\vec{m})}$ and corresponds to the eigenvalue $\lambda_{U_{\mathcal{Q}}(\vec{m})} = \gamma_n^2\lambda_{\vec{m}}$, with*

$$U_{\mathcal{Q}}(\vec{m}) := (2m_n,\ m_1,\ m_2, \ldots, m_{n-1}). \tag{4.15}$$

Proof (1) Let λ be an even eigenvalue of $\mathcal{B}^{(n)}$. Let $\vec{m} \in \mathcal{Q}$ be such that $\lambda_{\vec{m}} = \lambda$. As $\mathfrak{p}(\lambda)$ is even, we conclude that m_1 is even as well [see (4.7)], and therefore, $F_{\mathcal{Q}}(\vec{m}) \in \mathcal{Q}$, so that $\varphi_{F_{\mathcal{Q}}(\vec{m})}$ is well defined and it is an eigenfunction of $\mathcal{B}^{(n)}$. Combining the form of the eigenfunction $\varphi_{\vec{m}}$, (4.2), with the definition of its folding, (4.12), we get

$$\begin{aligned} \mathbf{F}\varphi_{\vec{m}}(\vec{x}) &= \varphi_{\vec{m}} \circ U(\vec{x}) \\ &= \cos\left(m_1\gamma_n^{-1}x_n\right) \prod_{j=2}^{n} \cos\left(\gamma_n^{j-1} m_j \gamma_n^{-1} x_{j-1}\right) \\ &= \cos\left(\gamma_n^{n-1}\frac{1}{2}m_1 x_n\right) \prod_{j=1}^{n-1} \cos\left(\gamma_n^{j-1} m_{j+1} x_j\right) \\ &= \varphi_{F_{\mathcal{Q}}(\vec{m})}(\vec{x}). \end{aligned}$$

If λ is a multiple eigenvalue, the calculation above is valid for any eigenfunction of the form $\varphi_{\vec{m}}$, and by linearity, it extends to $\mathbf{F}\varphi = \sum_{\vec{m};\ \lambda_{\vec{m}}=\lambda} \alpha_{\vec{m}} \cdot \varphi_{F_{\mathcal{Q}}(\vec{m})}$. Calculating the eigenvalue corresponding to $\varphi_{F_{\mathcal{Q}}(\vec{m})}$, we get

$$\lambda_{F_{\mathcal{Q}}(\vec{m})} = \gamma_n^{-2} m_1^2 + \sum_{j=1}^{n-1} \left(\gamma_n^{j-1} m_{j+1}\right)^2 = \gamma_n^{-2} \sum_{j=1}^{n} \left(\gamma_n^{j-1} m_j\right)^2 = \gamma_n^{-2} \lambda_{\vec{m}}. \tag{4.16}$$

(2) Let $\lambda \in \sigma\left(\mathcal{B}^{(n)}\right)$ and let $\vec{m} \in \mathcal{Q}$, such that $\lambda_{\vec{m}} = \lambda$. Let $\vec{x} \in \frac{1}{2}\mathcal{B}^{(n)}$.

$$\begin{aligned}
\mathbf{U}\varphi_{\vec{m}}\left(\vec{x}\right) &= \varphi_{\vec{m}} \circ F\left(\vec{x}\right) \\
&= \cos\left(\gamma_n^{n-1} m_n \gamma_n x_1\right) \prod_{j=1}^{n-1} \cos\left(\gamma_n^{j-1} m_j \gamma_n x_{j+1}\right) \\
&= \cos\left(2 m_n x_1\right) \prod_{j=2}^{n} \cos\left(\gamma_n^{j-1} m_{j-1} x_j\right) \\
&= \varphi_{U_{\mathcal{Q}}(\vec{m})}\left(\vec{x}\right).
\end{aligned}$$

For $\vec{x} \in \mathcal{B}^{(n)} \backslash \frac{1}{2}\mathcal{B}^{(n)}$, we have

$$\mathbf{U}\varphi_{\vec{m}}\left(\vec{x}\right) \underbrace{=}_{(4.13)} \mathbf{U}\varphi_{\vec{m}}\left(R\left(\vec{x}\right)\right) \underbrace{=}_{R(\vec{x})\in\frac{1}{2}\mathcal{B}^{(n)}} \varphi_{U_{\mathcal{Q}}(\vec{m})}\left(R\left(\vec{x}\right)\right) \underbrace{=}_{\text{Lemma 4.2}} \varphi_{U_{\mathcal{Q}}(\vec{m})}\left(\vec{x}\right).$$

Just as in the first part of the proof, we may use linearity to extend the relation above to the whole eigenspace of λ. In addition, it is easily verified that $\lambda_{U_{\mathcal{Q}}(\vec{m})} = \gamma_n^2 \lambda_{\vec{m}}$. □

The last lemma allows to show that the eigenvalues inherit the folding structure. This is shown in the following, which is analogous to Corollary 2.6.

Corollary 4.7 (1) *Let* $0 \neq \lambda \in \sigma\left(\mathcal{B}^{(n)}\right)$. *Then, there exist unique* $\lambda^{(0)} \in \sigma_{odd}\left(\mathcal{B}^{(n)}\right)$ *and* $k \in \mathbb{N}_0$, *such that* $\lambda = \gamma_n^{2k}\lambda^{(0)}$. *Furthermore,* $\mathrm{d}\left(\lambda\right) = \mathrm{d}\left(\lambda^{(0)}\right)$.
(2) *Let* $\lambda^{(0)} \in \sigma_{odd}\left(\mathcal{B}^{(n)}\right)$ *and* $k \in \mathbb{N}_0$. *Then,* $\gamma_n^{2k}\lambda^{(0)} \in \sigma\left(\mathcal{B}^{(n)}\right)$.

Proof We start by observing that the second claim may be proven similarly to the second claim of Corollary 2.6—start from any eigenfunction of $\lambda^{(0)}$ and by unfolding it k times get an eigenfunction whose eigenvalue is $\gamma_n^{2k}\lambda^{(0)}$.

Next, we prove the first claim and start by proving the uniqueness of the representation $\lambda = \gamma_n^{2k}\lambda^{(0)}$. Assume by contradiction that $\gamma_n^{2k_1}\lambda_1^{(0)} = \gamma_n^{2k_2}\lambda_2^{(0)}$, with different $k_1, k_2 \in \mathbb{N}_0$ and $\lambda_1^{(0)}, \lambda_2^{(0)} \in \sigma_{\mathrm{odd}}\left(\mathcal{B}^{(n)}\right)$. Without loss of generality, $k_1 > k_2$, and hence, $\lambda_2^{(0)} = \gamma_n^{2(k_1-k_2)}\lambda_1^{(0)}$. Pick an eigenfunction of $\lambda_1^{(0)}$ and unfold it $k_1 - k_2$ times to get an eigenfunction of the eigenvalue $\lambda_2^{(0)}$. By (4.13), this unfolded eigenfunction is even, and by Lemma 4.2, we deduce that its eigenvalue, $\lambda_2^{(0)}$ is also even and arrive at a contradiction.

It remains to show the existence of $k \in \mathbb{N}_0$, $\lambda^{(0)} \in \sigma_{\text{odd}}\left(\mathcal{B}^{(n)}\right)$, such that $\lambda = \gamma_n^{2k}\lambda^{(0)}$. There exists some $\vec{m} \in \mathcal{Q}$, such that $\lambda = \lambda_{\vec{m}}$. If $\lambda \in \sigma_{\text{odd}}\left(\mathcal{B}^{(n)}\right)$, then the statement holds with $k = 0$. Otherwise, by Lemma 4.6, we get that $\lambda_{F_{\mathcal{Q}}(\vec{m})} = \gamma_n^{-2}\lambda_{\vec{m}}$ is an eigenvalue. We keep applying $F_{\mathcal{Q}}$ to $\vec{m}$ until we get that $\lambda_{F_{\mathcal{Q}}^k(\vec{m})} = \gamma_n^{-2k}\lambda_{\vec{m}}$ is an odd eigenvalue. Once we get that the lemma is proved and it only remains to show that this process terminates after a finite (k) number of steps.

To see this, we may present the $\vec{m}$ entries as $m_j = p_j 2^{k_j}$, with k_j being the largest possible (and formally, set $p_j = 0$, $k_j = \infty$ if $m_j = 0$). The subsequent applications of $F_{\mathcal{Q}}$ cyclically shift the vector and divide the first entry by two [see (4.14)]. Eventually, one of the entries would be odd and the process stops (unless $\lambda = 0$).

Finally, the equality of multiplicities of λ and $\lambda^{(0)}$ arises as $\mathbf{F}^k$ is a linear isomorphism (its inverse is $\mathbf{U}^k$) from the eigenspace of λ to the eigenspace of $\lambda^{(0)}$. $\square$

Defining the k-frame exactly as in (2.13) (see also Fig. 9) allows to prove an analogue of Proposition 2.8, namely that for $\lambda^{(0)} \in \sigma_{\text{odd}}\left(\mathcal{B}^{(n)}\right)$, its k-unfolded eigenvalue, $\lambda = \gamma_n^{2k}\lambda^{(0)}$ vanishes on the k-frame. This, in turn, shows that Lemmata 3.5 and 3.6 are valid for the high-dimensional boxes as well (with the k-frame partition defined just as in Definition 3.4). All we need to use now is Lemma 3.6(2), according to which multiple eigenvalues[4] cannot be Courant-sharp. Alternatively, we may use Lemma B.1 which is a generalization of Lemma 3.6(2).

We are left to check the Courant-sharpness of simple eigenvalues. Since the nodal set of the basis eigenfunctions, $\varphi_{\vec{m}}$, is determined by

$$\prod_{j=1}^{n} \cos(\gamma_n^{j-1} m_j x_j) = 0; \;\; 0 \le x_j \le \pi/\gamma_n^{j-1},$$

it is straightforward to deduce that

$$\nu(\varphi_{\vec{m}}) = \prod_{j=1}^{n} (m_j + 1).$$

This is compared with the spectral position in the next proposition which rules out the Courant-sharpness of all eigenvalues not appearing in Theorem 1.2.

Proposition 4.8

(1) *For $n \ge 3$ and $N \ge 2$, λ_N is not a Courant-sharp eigenvalue of $\mathcal{B}^{(n)}$.*
(2) *For $n = 2$ and $N \notin \{1, 2, 4, 6\}$, λ_N is not a Courant-sharp eigenvalue for $\mathcal{B}^{(2)}$.*

[4] See "Appendix A", where we discuss the possible eigenvalue multiplicities of the problem.

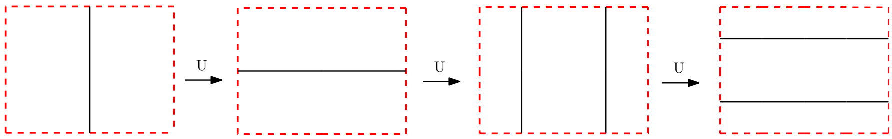


Fig. 9 The first four k-frames of $\mathcal{B}^{(2)}$

Proof By the analogue of Lemma 3.6(2) (see discussion before this proposition), we only need to rule out the Courant-sharpness of simple eigenvalues. Let $\lambda_{\vec{m}} \in \sigma\left(\mathcal{B}^{(n)}\right)$ be a simple eigenvalue.

Let $\mathcal{B}_{\mathcal{Q}}(\lambda) = \mathcal{Q} \cap \left\{\left((\tilde{m}_1, \ldots, \tilde{m}_n) \middle| \tilde{m}_j \leq m_j\ ,\ \forall j\right)\right\}$. Note that $\mathcal{B}_{\mathcal{Q}}(\lambda)$ contains all $\mathcal{Q}$-points contained in an n-dimensional box and $\mathcal{Q}(\lambda)$ forms all the $\mathcal{Q}$ points contained within an n-dimensional ellipsoid (see Fig. 10 for the $n = 2$ case).
In the sequel, we show $\mathcal{B}_{\mathcal{Q}}(\lambda) \subsetneq \mathcal{Q}(\lambda) \cup (m_1, \ldots, m_n)$ which rules out Courant-sharpness, since it gives

$$N(\lambda) = |\mathcal{Q}(\lambda) \cup (m_1, \ldots, m_n)| > \left|\mathcal{B}_{\mathcal{Q}}(\lambda_{\vec{m}})\right| = \nu(\varphi_{\vec{m}}).$$

It is easily seen that

$$\mathcal{B}_{\mathcal{Q}}(\lambda) \subseteq \mathcal{Q}(\lambda) \cup (m_1, \ldots, m_n),$$

and to show that

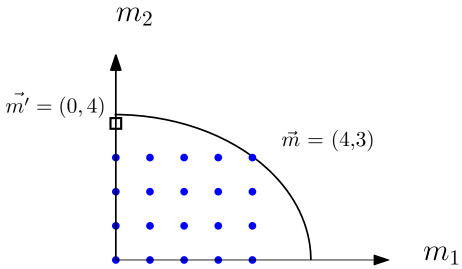
$$\mathcal{B}_{\mathcal{Q}}(\lambda) \subsetneq \mathcal{Q}(\lambda) \cup (m_1, \ldots, m_n),$$

we point out $\vec{m}' \in \mathcal{Q}$, such that $\vec{m}' \in \mathcal{Q}(\lambda) \backslash \mathcal{B}_{\mathcal{Q}}(\lambda)$. Note that this proof technique resembles the one which is used in the proof of Proposition 3.1(3) and the set $\mathcal{B}_{\mathcal{Q}}(\lambda)$ plays the same role as the set $\mathcal{T}_{\mathcal{Q}}(\lambda)$ there.

Start by assuming that $\vec{m}$ is such that there exists k for which $m_k < m_{k+1}$. Choosing

$$\vec{m}' = (0, \ldots, 0, \underbrace{m_k + 1}_{k-th\ position} 0, \ldots, 0), \tag{4.17}$$

Fig. 10 Illustration of $\mathcal{B}_{\mathcal{Q}}(\lambda_{4,3})$



satisfies

$$\lambda_{\vec{m}'} < \lambda_{(0,\ldots,0,\underbrace{m_{k+1}}_{k+1-th\ position}0,\ldots,0)} \leq \lambda_{\vec{m}}.$$

We may, therefore, proceed by assuming that the entries of $\vec{m}$ form a non-increasing ordered set.

We distinguish the non-increasing sequences by setting

$$I := \min\left\{ j \,\middle|\, m_j = \min_{1\leq q\leq n} m_q \right\},$$

so that I is the first index starting from which all entries are equal.

(1) $I = 1$. In this case, $\vec{m}$ is a constant sequence, that is

$$\vec{m} = (m_1, \ldots, m_1)\,.$$

Choose

$$\vec{m}' = (m_1 + 1, 0, \ldots, 0)\,.$$

We have

$$\lambda_{\vec{m}} = m_1^2 \sum_{j=1}^{n} \gamma_n^{2(j-1)} = m_1^2 \frac{3}{\gamma_n^2 - 1} \quad \text{and} \quad \lambda_{\vec{m}'} = (m_1 + 1)^2\,.$$

An easy calculation shows that $\lambda_{\vec{m}'} < \lambda_{\vec{m}}$ holds for all values of n and m_1 with the only exceptions being $\vec{m} = (1, 1)$ and $\vec{m} = \vec{0}$. Indeed, we see later (Lemma 4.9) that those are Courant-sharp.

(2) $I \geq 3$. With

$$\vec{m} = (m_1, \ldots, \underbrace{m_I}_{I-th\ position} \ldots, m_I),$$

choose

$$\vec{m}' = (0, \ldots, 0, \underbrace{m_I + 1}_{I-th\ position} 0, \ldots, 0)$$

and consider an auxiliary point

$$\vec{m}'' := (m_I + 1, \ldots, m_I + 1, \underbrace{0}_{I} \ldots, 0).$$

Clearly, we have

$$\lambda_{\vec{m}''} \leq \lambda_{\vec{m}},$$

and it is left to show

$$\lambda_{\vec{m}'} < \lambda_{\vec{m}''}. \tag{4.18}$$

To get (4.18), simply note that

$$\gamma_n^{2(k-1)}\,(m_I+1)^2 < \frac{1-\gamma_n^{2(k-1)}}{1-\gamma_n^2}\,(m_I+1)^2 = \sum_{j=1}^{k-1}\gamma_n^{2(j-1)}\,(m_I+1)^2\,,$$

for all $n \geq k \geq 3$.

(3) $I = 2$. With

$$\vec{m} = (m_1, m_2, \ldots, m_2),$$

choose

$$\vec{m}' = (0, m_2+1, 0, \ldots, 0)$$

and consider an auxiliary point

$$\vec{m}'' := (m_2+1, m_2, \ldots, m_2).$$

As

$$\lambda_{\vec{m}''} \leq \lambda_{\vec{m}},$$

it is left to show

$$\lambda_{\vec{m}'} < \lambda_{\vec{m}''}. \tag{4.19}$$

We get that (4.19) is equivalent to

$$m_2 > \sqrt{\gamma_n^2 - 1}\left(\sqrt{\sum_{j=2}^{n}\gamma_n^{2(j-1)}} - \sqrt{\gamma_n^2-1}\right)^{-1}. \tag{4.20}$$

This inequality holds if either $n \geq 3$ and $m_2 \geq 1$ or $n = 2$ and $m_2 \geq 3$ (the case $n = 2,\ m_2 = 3$ is demonstrated in Fig. 10).
The remaining subcases are as follows:
(a) For $n \geq 3,\ m_2 = 0$, we have that $\vec{m} = (m_1, 0, \ldots, 0)$ and

$$\lambda_{\vec{m}'} = \lambda_{(0,1,0,\ldots,0)} < \lambda_{(m_1,0,\ldots,0)} = \lambda_{\vec{m}},$$

for all $m_1 \geq 2$.

Note that $n \geq 3,\ m_2 = 0,\ m_1 \in \{0, 1\}$ correspond to Courant-sharp eigenvalues (Lemma 4.9).

(b) For $n = 2,\ m_2 \in \{0, 1, 2\}$, we have the following subcases:

(i) If $m_2 = 2$ and $m_1 > 3$, then $\lambda_{\vec{m}'} = \lambda_{(0,3)} = 18 < m_1^2 + 8 = \lambda_{(m_1,2)}$.

(ii) If $m_2 = 2$ and $m_1 = 3$, then $\lambda_{\vec{m}'} = \lambda_{(4,0)} = 16 < 17 = \lambda_{(3,2)}$.

(iii) If $m_2 = 1$ and $m_1 \geq 3$, then $\lambda_{\vec{m}'} = \lambda_{(0,2)} = 8 < m_1^2 + 2 = \lambda_{(m_1,1)}$.

(iv) If $m_2 = 0$ and $m_1 \geq 2$, then $\lambda_{\vec{m}'} = \lambda_{(0,1)} = 2 < m_1^2 = \lambda_{(m_1,0)}$.

Note that $\lambda_{(1,0)}$ and $\lambda_{(2,1)}$ correspond to Courant-sharp eigenvalues (Lemma 4.9). □

Finally, Theorem 1.2 is proven by validating the Courant-sharpness of the remaining eigenvalues.

Lemma 4.9

(1) *Let* $n \geq 3$. λ_1 *and* λ_2 *are Courant-sharp eigenvalues of* $\mathcal{B}^{(n)}$.

(2) *For n=2 (the rectangle case),* $\lambda_1, \lambda_2, \lambda_4,$ *and* λ_6 *are Courant-sharp eigenvalues of* $\mathcal{B}^{(2)}$.

Proof By Courant's bound and orthogonality of eigenfunctions, λ_1 and λ_2 are always Courant-sharp. For the rectangle, $\mathcal{B}^{(2)}$, one counts that the eigenfunction $\varphi_{(1,1)}$, which corresponds to λ_4 has four nodal domains and the eigenfunction $\varphi_{(2,1)}$ which corresponds to λ_6 has six nodal domains. □

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Appendix A: On multiplicity of eigenvalues of high-dimensional boxes

We start by relating the multiplicity function to the following classical problem. Denote the sum of squares function by

$$r_2(z) = \left|\{(m, n) \in \mathbb{Z}^2, \text{ such that } m^2 + n^2 = z\}\right|.$$

The equality $\mathrm{d}(\lambda) = \mathrm{d}\left(\lambda^{(0)}\right)$ in Corollary 2.6 implies that $r_2(z) = r_2(2^k z)$, for all $k \in \mathbb{N}$, (see also [20], Chapter 2, Section 4). This fact nicely generalizes in Corollary 4.7 by the same equality. Indeed, defining the following quadratic form:

$$q(x_1, \ldots x_n) = \sum_{j=1}^{n} \left(\gamma_n^{j-1} x_j\right)^2,$$

and denoting

$$r_n^q(z) = \left|\{(m_1, \ldots, m_n) \in \mathbb{Z}^n, \text{ such that } q(m_1, \ldots, m_n) = z\}\right|,$$

we get $r_n^q(z) = r_n^q(\gamma_n^{2k} z)$ for all $k \in \mathbb{N}$. In particular, this relation seems more interesting for even values of n, as can be interpreted from the following.

Proposition A.1

(1) *If n is odd, then all eigenvalues $\lambda \in \sigma\left(\mathcal{B}^{(n)}\right)$ are simple.*
(2) *Let n be even, and let $\lambda \in \sigma\left(\mathcal{B}^{(n)}\right)$. Then*
   (a) *λ is uniquely written as $\lambda = \sum_{j=0}^{\frac{n}{2}-1} \lambda^{(j)} \gamma_n^{2j}$.*
   (b) *Each $\lambda^{(j)}$ is some eigenvalue of the rectangle problem ($\lambda^{(j)} \in \sigma\left(\mathcal{B}^{(2)}\right)$).*
   (c) *The multiplicity of λ equals to the product over multiplicities of all $\lambda^{(j)}$'s as eigenvalues of the rectangle problem.*

Proof

(1) Assume n is odd. Assume that there exist $\vec{m}^{(1)}, \vec{m}^{(2)} \in \mathcal{Q}$, such that $\lambda_{\vec{m}^{(1)}} = \lambda_{\vec{m}^{(2)}}$. We get

$$
\begin{aligned}
\sum_{j=1}^{n}\left(\gamma_n^{j-1} m_j^{(1)}\right)^2 = \sum_{j=1}^{n}\left(\gamma_n^{j-1} m_j^{(2)}\right)^2 \quad &\Leftrightarrow \\
0 = \sum_{j=1}^{n} \gamma_n^{2(j-1)} &\left(\left(m_j^{(1)}\right)^2 - \left(m_j^{(2)}\right)^2\right) \\
= \sum_{j=1}^{\frac{n+1}{2}} \gamma_n^{2(j-1)} &\left(\left(m_j^{(1)}\right)^2 - \left(m_j^{(2)}\right)^2\right) \\
+ \sum_{j=1}^{\frac{n-1}{2}} 2\gamma_n^{2j-1} &\left(\left(m_{\frac{n+1}{2}+j}^{(1)}\right)^2 - \left(m_{\frac{n+1}{2}+j}^{(2)}\right)^2\right),
\end{aligned}
$$

using $\gamma_n^n = 2$, in the reordering of terms in the last line. The right-hand side of the above is a linear combination of the basis $\left\{\gamma_n^j\right\}_{j=0}^{n-1}$, so that we conclude for all j, $m_j^{(1)} = m_j^{(2)}$, as required.

(2) Assume n is even. Let $\vec{m} \in \mathcal{Q}$, such that $\lambda_{\vec{m}} = \lambda$. We have

$$
\lambda = \sum_{j=1}^{n}\left(\gamma_n^{j-1} m_j\right)^2 = \sum_{j=1}^{\frac{n}{2}} \gamma_n^{2(j-1)}\left(m_j^2 + 2m_{j+\frac{n}{2}}^2\right), \tag{A.1}
$$

and conclude

$$
\forall 1 \le j \le \frac{n}{2} \quad \lambda^{(j-1)} = m_j^2 + 2m_{j+\frac{n}{2}}^2. \tag{A.2}
$$

Hence, we have shown (a), where the uniqueness comes from $\left\{\gamma_n^{2j}\right\}_{j=0}^{\frac{n}{2}-1}$ being a basis. From (A.2), it is easily verified that $\lambda^{(j-1)} \in \sigma\left(\mathcal{B}^{(2)}\right)$, and combining with

(A.1), we deduce that the multiplicity of $\lambda \in \sigma\left(\mathcal{B}^{(n)}\right)$ is obtained as a product over all multiplicities of $\left\{\lambda^{(j-1)}\right\}_{j=1}^{\frac{n}{2}}$.

□

Remark The second part of the proposition above may be explained as the following. One may express $\mathcal{B}^{(2k)}$ as a direct product of k scaled copies of $\mathcal{B}^{(2)}$. Denoting the edge lengths of $\mathcal{B}^{(2k)}$ by $l_1, \ldots l_{2k}$, the jth copy of $\mathcal{B}^{(2k)}$ has edge lengths l_j, l_{k+j}. Each eigenfunction on $\mathcal{B}^{(2k)}$ can be expressed as a product of eigenfunctions on all different k scaled copies of $\mathcal{B}^{(2)}$. Hence, each eigenvalue of $\mathcal{B}^{(2k)}$ is a sum over eigenvalues of all the $\mathcal{B}^{(2)}$ copies.

Appendix B: Nodal deficiency

The importance of nodal deficiency of eigenfunctions has been recognized in recent studies [5,10–13]. It is the nodal deficiency that has been exactly expressed by variations over partitions and eigenvalues. These recent works concern manifolds, as well as quantum and discrete graphs. We bring here some interesting bounds on the nodal deficiencies of the spectral problems studied in this paper.

We define the nodal deficiency of an eigenfunction φ of an eigenvalue λ by

$$\delta(\varphi) = N(\lambda) - \nu(\varphi),$$

and the nodal deficiency of an eigenvalue by

$$\delta(\lambda) := \min_{\varphi \in E(\lambda)} \delta(\varphi). \tag{B.1}$$

Here $E(\lambda)$ is the eigenspace associated with λ. In the following, we use the analysis of Sect. 3 to derive lower bounds of the nodal deficiency of eigenvalues. The following lemmata holds for all the domains treated in the paper. We use the notation Ω to indicate both $\mathcal{D}$ and $\mathcal{B}^{(n)}$ and denote

$$\gamma(\Omega) := \begin{cases} \sqrt{2} & \Omega = \mathcal{D} \\ \gamma_n & \Omega = \mathcal{B}^{(n)} \end{cases}.$$

The next lemma provides a lower bound on the nodal deficiency of multiple eigenvalues.

Lemma B.1 *Let $\lambda^{(0)} \in \sigma_{odd}(\Omega)$ and $k \in \mathbb{N}_0$. The nodal deficiency of the eigenvalue $\gamma(\Omega)^{2k} \cdot \lambda^{(0)}$ obeys*

$$\delta\left(\gamma(\Omega)^{2k} \cdot \lambda^{(0)}\right) \geq \left(\mathrm{d}\left(\lambda^{(0)}\right) - 1\right) \cdot (M(k, \Omega) - 1),$$

where $M(k, \Omega)$ is the number of the subdomains of the k-frame partition of Ω and $d\left(\lambda^{(0)}\right)$ is the multiplicity of $\lambda^{(0)}$.

Proof For the sake of convenience, we abbreviate notations by writing γ instead of $\gamma(\Omega)$ and $M(k)$ instead of $M(k,\Omega)$. Note that the following arguments below are similar to those we have used in the proof of Lemma 3.6. We have

$$\begin{aligned}
\nu(\varphi) &= \sum_{i=1}^{M(k)} \nu\left(\varphi|_{\Omega_i^{(k)}}\right) \underbrace{\leq}_{\text{Courant's nodal theorem}} \sum_{i=1}^{M(k)} N_i^{(k)}\left(\gamma^{2k}\cdot\lambda^{(0)}\right)\\
&= \sum_{i=1}^{M(k)} \overline{N}_i^{(k)}\left(\gamma^{2k}\cdot\lambda^{(0)}\right) + \sum_{i=1}^{M(k)}\left(N_i^{(k)}\left(\gamma^{2k}\cdot\lambda^{(0)}\right) - \overline{N}_i^{(k)}\left(\gamma^{2k}\cdot\lambda^{(0)}\right)\right)\\
&\underbrace{\leq}_{\text{variational principle (3.20)}} \overline{N}\left(\gamma^{2k}\cdot\lambda^{(0)}\right) + \sum_{i=1}^{M(k)}\left(N_i^{(k)}\left(\gamma^{2k}\cdot\lambda^{(0)}\right) - \overline{N}_i^{(k)}\left(\gamma^{2k}\cdot\lambda^{(0)}\right)\right)\\
&\underbrace{\leq}_{\text{Lemma 3.5}} \overline{N}\left(\gamma^{2k}\cdot\lambda^{(0)}\right) + \sum_{i=1}^{M(k)}\left(N\left(\gamma^{2k}\cdot\lambda^{(0)}\right) - \overline{N}\left(\gamma^{2k}\cdot\lambda^{(0)}\right)\right)\\
&= N\left(\gamma^{2k}\cdot\lambda^{(0)}\right) + (M(k)-1)\cdot\left(N\left(\gamma^{2k}\cdot\lambda^{(0)}\right) - \overline{N}\left(\gamma^{2k}\cdot\lambda^{(0)}\right)\right).
\end{aligned}$$

Thus

$$N\left(\gamma^{2k}\cdot\lambda^{(0)}\right) - \nu(\varphi) \geq (M(k)-1)\cdot\left(\overline{N}\left(\gamma^{2k}\cdot\lambda^{(0)}\right) - N\left(\gamma^{2k}\cdot\lambda^{(0)}\right)\right),$$

and

$$\delta(\varphi) \geq \left(\mathrm{d}\left(\gamma^{2k}\cdot\lambda^{(0)}\right) - 1\right)\cdot(M(k)-1).$$

Since the right-hand side does not depend on φ, we get

$$\delta\left(\gamma^{2k}\cdot\lambda^{(0)}\right) \geq \left(\mathrm{d}\left(\gamma^{2k}\cdot\lambda^{(0)}\right) - 1\right)\cdot(M(k)-1).$$

Finally, use

$$\mathrm{d}\left(\lambda^{(0)}\right) = \mathrm{d}\left(\gamma^{2k}\cdot\lambda^{(0)}\right),$$

(see Corollary 2.6 for the triangle or Corollary 4.7 for the boxes) to finish the proof. □

We may obtain even more explicit bounds on the nodal deficiency by computing $M(k,\Omega)$, as explained in the following. In the case of $\mathcal{B}^{(n)}$, we can get an explicit expression for $M\left(k,\mathcal{B}^{(n)}\right)$, noticing the relations

$$\begin{aligned}
\forall 0\leq k\leq n-1;\quad & M\left(k,\mathcal{B}^{(n)}\right) = 2\\
\forall k\geq 0;\quad & M\left(k+n,\mathcal{B}^{(n)}\right) = 2M\left(k,\mathcal{B}^{(n)}\right) - 1.
\end{aligned}$$

Those relations may be obtained by noticing that all k frames are formed by hyperplanes, all parallel to each other. The number of those hyperplanes determines $M\left(k, \mathcal{B}^{(n)}\right)$, and this number may be deduced by working out the definition of k-frames (Definition 3.4 with (4.4), (4.5), and (4.11), and see, as an example, Fig. 7). From those relations, we obtain

$$M\left(k, \mathcal{B}^{(n)}\right) = 2^{\left\lfloor \frac{k}{n} \right\rfloor} + 1, \quad \forall k \geq 0.$$

In the case of the triangle, we may also obtain the explicit expression for $M(k, \mathcal{D})$. Yet, as the calculation is somewhat cumbersome, we chose to provide the following estimate. A square subdomain appears on the 4-frame partition (see Fig. 2). Getting to the next k-frames, each unfolding at least doubles the number of this particular subdomain (up to scaling), and hence, $M(k, \mathcal{D}) > c2^k$, for some constant c. In effect, the exact calculation gives the same order of magnitude, i.e., $M(k, \mathcal{D}) = \Theta(2^k)$. Applying Lemma B.1 for odd eigenvalues, where $k = 0$ and $M(0, \Omega) = 2$, we get $\delta\left(\lambda^{(0)}\right) \geq \mathrm{d}\left(\lambda^{(0)}\right) - 1$. We may actually improve this bound by relating the nodal deficiency with the count of boundary lattice points, as follows.

Lemma B.2 *Let* $\lambda^{(0)} \in \sigma_{odd}(\Omega)$*, then*

$$\delta\left(\lambda^{(0)}\right) \geq \left| \underset{\rightarrow}{\partial} \mathcal{Q}\left(\lambda^{(0)}\right) \cap \mathcal{E} \right| - 1.$$

Proof Let φ be an eigenfunction that corresponds to $\lambda^{(0)} \in \sigma_{\text{odd}}(\Omega)$. For the triangle, we have by Eq. (3.13) that

$$\delta(\varphi) \geq \left| \underset{\rightarrow}{\partial} \mathcal{Q}\left(\lambda^{(0)}\right) \cap \mathcal{E} \right| - 1,$$

and the same bound for the boxes, defining $\underset{\rightarrow}{\partial} \mathcal{Q}\left(\lambda^{(0)}\right)$ by generalizing (3.3). The lemma now follows, since the right-hand side does not depend on φ. □

Note that in the course of the proof of Proposition 3.1(1) [see (3.14)], it is shown that $\left| \underset{\rightarrow}{\partial} \mathcal{Q}\left(\lambda^{(0)}\right) \cap \mathcal{E} \right| > 1$, so that the bound of the lemma above is not trivial. In fact, by a lattice analysis, one may further get that the size of this set is of order $\sqrt{\lambda^{(0)}}$.

Appendix C: The Dirichlet problem

We shortly discuss below how the methods of this work may be applied to examine the Dirichlet eigenvalue problem on the domains treated herein. The Courant-sharp eigenvalues of the Dirichlet right-angled isosceles triangle are already determined in [8] using the analysis of the corresponding Dirichlet eigenvalue problem [3], done by two of the authors of the current paper together with Aronovitch and Gnutzmann.

Let us lay the framework for examining the Dirichlet 2-rep-tiles. The quantum number set of the Dirichlet triangle is

$$\mathcal{Q} := \{(m,n) \in \mathbb{N}\times\mathbb{N} |\, m > n\}, \tag{C.1}$$

and the corresponding eigenvalues are

$$\lambda_{m,n} = \|(m,n)\|^2;\ \ (m,n) \in \mathcal{Q}.$$

For the boxes, the quantum number set is

$$\mathcal{Q} := \left\{\vec{m} \in \mathbb{N}^n\right\}, \tag{C.2}$$

and the corresponding eigenvalues are

$$\lambda_{\vec{m}} = \sum_{j=1}^{n} \left(\gamma_n^{j-1} m_j\right)^2;\ \ \vec{m} \in \mathcal{Q}. \tag{C.3}$$

Note that these sets of quantum numbers are included in those defined for the Neumann problems. We exploit this to define $\mathcal{O}$, $\mathcal{E}$, and, in turn, $\sigma_{\text{odd}}(\Omega)$ and $\sigma_{\text{even}}(\Omega)$ for Ω being either $\mathcal{D}$ or $\mathcal{B}^{(n)}$ exactly in the same manner as we did for the Neumann problem (see Definitions 2.2 and 4.3). We obtain for the Dirichlet problem the following lemma, which may be proved similarly to its Neumann analogues, Lemmata 2.1 and 4.2.

Lemma C.1 *Let* $\lambda \in \sigma_{odd}(\Omega)$ $(\lambda \in \sigma_{even}(\Omega)$ *), then its corresponding eigenfunctions are even* (*odd*) *w.r.t.* L *if and only if* λ *is odd* (*even*).

One should pay careful attention to the difference in phrasing of this lemma comparing to its Neumann analogues. Here, an eigenvalue belongs to the *even* spectrum if and only if its eigenfunctions are *odd*. In turn, the folding transformation may be applied only on even eigenvalues (alternatively, on odd eigenfunctions).

We use Lemma C.1 to express the nodal deficiency in terms of boundary lattice points, adopting the notation $\gamma(\Omega)$ of the previous appendix.

Lemma C.2 *Let* $\lambda \in \sigma_{even}(\Omega)$, *then we have*

$$\delta(\lambda) = 2\cdot\delta\left(\gamma(\Omega)^{-2}\cdot\lambda\right) + \left|\underset{\rightarrow}{\partial}\mathcal{Q}(\lambda)\cap\mathcal{O}\right| - 1.$$

Proof Start by noting that $U_{\mathcal{Q}}\big|_{\mathcal{Q}(\gamma(\Omega)^{-2}\cdot\lambda)}$ is an injection and its image is $\mathcal{E}(\lambda)$, thus

$$|\mathcal{E}(\lambda)| = \underline{N}\left(\gamma(\Omega)^{-2}\cdot\lambda\right). \tag{C.4}$$

In addition, note that

$$B:\ \mathcal{E}(\lambda) \to \mathcal{O}(\lambda)\backslash\left(\underset{\rightarrow}{\partial}\mathcal{Q}(\lambda)\cap\mathcal{O}\right)$$
$$B(m_1,\ldots,m_n) = (m_1-1,\ldots,m_n),$$

with $n = 2$ in the triangle case, is a bijection [cf. (3.9)], which gives

$$|\mathcal{O}(\lambda)| = |\mathcal{E}(\lambda)| + \left|\underset{\rightarrow}{\partial}\mathcal{Q}(\lambda)\cap\mathcal{O}\right|. \tag{C.5}$$

We get

$$N(\lambda) = |\mathcal{E}(\lambda)| + |\mathcal{O}(\lambda)| + 1 \underbrace{=}_{(C.4),\,(C.5)} 2\cdot N\left(\gamma(\Omega)^{-2}\cdot\lambda\right) + \left|\underset{\rightarrow}{\partial}\mathcal{Q}(\lambda)\cap\mathcal{O}\right| - 1.$$

Choosing some eigenfunction φ of λ, we have

$$N(\lambda) = \nu(\varphi) + \delta(\varphi) \underbrace{=}_{\varphi\text{ is odd}} 2\cdot\nu\left(\varphi|_{\frac{1}{2}\Omega}\right) + \delta(\varphi). \tag{C.6}$$

Hence

$$2\cdot N\left(\gamma(\Omega)^{-2}\cdot\lambda\right) + \left|\underset{\rightarrow}{\partial}\mathcal{Q}(\lambda)\cap\mathcal{O}\right| - 1 = 2\nu\left(\varphi|_{\frac{1}{2}\Omega}\right) + \delta(\varphi),$$

which, in turn, leads to

$$\delta(\varphi) = 2\cdot\delta\left(\varphi|_{\frac{1}{2}\Omega}\right) + \left|\underset{\rightarrow}{\partial}\mathcal{Q}(\lambda)\cap\mathcal{O}\right| - 1. \tag{C.7}$$

Since the nodal deficiency of an eigenvalue is the minimal deficiency over all corresponding eigenfunctions, we get

$$\delta(\varphi) \geq 2\cdot\delta\left(\gamma(\Omega)^{-2}\cdot\lambda\right) + \left|\underset{\rightarrow}{\partial}\mathcal{Q}(\lambda)\cap\mathcal{O}\right| - 1.$$

The right-hand side is independent of φ, and therefore

$$\delta(\lambda) \geq 2\cdot\delta\left(\gamma(\Omega)^{-2}\cdot\lambda\right) + \left|\underset{\rightarrow}{\partial}\mathcal{Q}(\lambda)\cap\mathcal{O}\right| - 1.$$

We note that the opposite inequality follows by the same method, which finishes the proof. $\square$

We end by noting that as $\left|\underset{\rightarrow}{\partial}\mathcal{Q}\left(\lambda^{(0)}\right)\cap\mathcal{E}\right| > 1$ (see (3.14) and discussion at the end of the previous appendix), the result of the lemma both supplies a non-trivial bound on the deficiency and also rules out all even eigenvalues from being Courant-sharp (the argument is actually the same as the one used in the proof of Proposition 3.1(1) to rule out the Courant-sharpness of odd eigenvalues in the Neumann case).

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