IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **43** (2010) 415201 (17pp) doi:10.1088/1751-8113/43/41/415201

Scattering from isospectral quantum graphs

R Band[1], **A Sawicki**[1,2] **and U Smilansky**[1,3]

[1] Department of Physics of Complex Systems, The Weizmann Institute of Science, Rehovot 76100, Israel
[2] Center for Theoretical Physics, Polish Academy of Sciences Al. Lotników 32/46, 02-668 Warszawa, Poland
[3] Cardiff School of Mathematics and WIMCS, Cardiff University, Senghennydd Road, Cardiff CF24 4AG, UK

E-mail: rami.band@weizmann.ac.il, assawi@cft.edu.pl and uzy.smilansky@weizmann.ac.il

Received 9 July 2010, in final form 24 August 2010
Published 27 September 2010
Online at stacks.iop.org/JPhysA/43/415201

Abstract
Quantum graphs can be extended to scattering systems when they are connected by leads to infinity. It is shown that for certain extensions, the scattering matrices of isospectral graphs are conjugate to each other and their poles distributions are therefore identical. The scattering matrices are studied using a recently developed isospectral theory (Band *et al* 2009 *J. Phys. A: Math. Theor.* **42** 175202 and Parzanchevski and Band 2010 *J. Geom. Anal.* **20** 439–71). At the same time, the scattering approach offers a new insight on the mentioned isospectral construction.

PACS numbers: 02.20.−a, 02.30.Jr, 03.65.Nk

1. Introduction

The investigation of spectral inverse problems goes back to the famous question of Marc Kac, ‘Can one hear the shape of a drum?’ [3]. This question can be rephrased as: ‘does the Laplacian on every planar domain with Dirichlet boundary conditions have a unique spectrum?’. Ever since the time when Kac posed this fascinating question, physicists and mathematicians alike have attacked the problem from various angles. Attempts were made both to reconstruct the shape of an object from its spectrum and to find different objects that are isospectral, i.e. have the same spectrum. In 1985, Sunada presented a method for constructing isospectral Riemannian manifolds [4]. Over the years, several pairs of isospectral objects were found, but these were not planar domains, and therefore did not serve as an exact answer to Kac’s question. In 1992, by applying an extension of Sunada’s theorem, Gordon, Webb and Wolpert were able to finally answer Kac’s question as it related to drums, presenting the first pair of isospectral two-dimensional planar domains [5, 6]. At the same time, the research of inverse spectral problems extended to include the examination of scattering data. Examples of objects that share the same scattering information were found both for finite area [7, 8] and infinite area [9, 10] Riemann surfaces. The search for isospectral and isoscattering examples went further than expected by the original question of Kac—it now includes various approaches and a wealth of objects to consider, ranging from Riemannian manifolds to discrete graphs. The interested reader can find more about the field in the reviews [11–13] and the references therein.

The main result in the field of isospectrality and isoscattering of quantum graphs is that of Gutkin and Smilansky [14], where they use the trace formula to solve the inverse spectral problem of a general quantum graph. They show that under certain conditions a quantum graph can be recovered either from the spectrum of its Laplacian or from the overall phase of its scattering matrix. The necessary conditions include the graph being simple and its edges having rationally independent lengths. When these conditions are not satisfied, isospectral quantum graphs indeed arise. Examples of isospectral and isoscattering quantum graphs appear in [14–24]. A recent work [1, 2] presents a new method for constructing isospectral objects. This method is a generalization of Sunada's theorem of isospectrality [4], as it relaxes its hypothesis.

The present paper makes use of the above-mentioned isospectral theory in order to investigate isoscattering problems and relate them to the isospectral research. We were motivated by [25], in which scattering from the exterior of isospectral domains in $\mathbb{R}^2$ is discussed. Okada *et al* conjecture in this paper that one may distinguish isospectral drums by measuring sound scattered by the drums. In other words, in spite of the fact that the two domains are isospectral, the authors suggest that when looked from the exterior, the corresponding scattering matrices have different poles distributions, i.e. the domains are not isoscattering. The results that we obtain in this manuscript for quantum graphs shed a new light on the above conjecture.

The paper is arranged as follows. In this section we shortly review quantum graphs in the context of isospectrality and scattering matrices. We then bring a basic example which we use throughout the paper to demonstrate the obtained results. Section 3 develops a theory which connects scattering matrices to isospectrality and the following section applies it to a few examples. We end by indicating the link to [25], summarizing and suggesting future research directions.

1.1. Quantum graphs

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a finite graph which consists of $|\mathcal{V}|$ vertices connected by $|\mathcal{E}|$ edges. Each edge $e \in \mathcal{E}$ is a one-dimensional segment of finite length L_e with a coordinate $x_e \in [0, L_e]$ and this makes Γ metric graph. The metric graph becomes quantum, when we supply it with a differential operator. In this paper we choose our operator to be the free Schrödinger operator and denote it by Δ. This is merely the one-dimensional Laplacian which equals $-\frac{\mathrm{d}^2}{\mathrm{d}x_e^2}$ on each of the edges $e \in \mathcal{E}$, and its domain is $\bigoplus_{e\in\mathcal{E}} H^2(e)$. The coupling between the edges is introduced by supplementing vertex conditions at the vertices. There are many choices of vertex conditions which render the resulting operator self-adjoint, and the most common ones are the Neumann vertex conditions, described below.

Let $v \in \mathcal{V}$, and $\mathcal{E}_v$ be the set of edges incident to v. A function f on Γ obeys the Neumann vertex conditions at v if and only if

(i) f is continuous at v, i.e.

$$\forall e_1, e_2 \in \mathcal{E}_v, \qquad f_{e_1}(v) = f_{e_2}(v);$$

(ii) the sum of outgoing derivatives of f at the vertex v equals zero, i.e.

$$\sum_{e\in\mathcal{E}_v} \frac{\mathrm{d}f}{\mathrm{d}x_e}(v) = 0.$$

The spectrum of a graph all of whose vertices obey the Neumann conditions is discrete, non-negative and unbounded.

Other typical conditions, which are called Dirichlet, set the values of the function to zero at the vertex,

$$\forall e \in \mathcal{E}_v, \qquad f_e(v) = 0,$$

and do not put any requirement on the derivatives.

In general, the vertex conditions at a certain vertex $v \in \mathcal{V}$ can be described in one of two ways as follows.

(i) Stating linear equations for the values and the derivatives of the function at the vertex. Specifically, we use $|\mathcal{E}_v|$ equations for the $2|\mathcal{E}_v|$ variables: $\{f_e(v)\}_{e\in\mathcal{E}_v}$ and $\left\{\frac{\mathrm{d}f}{\mathrm{d}x_e}(v)\right\}_{e\in\mathcal{E}_v}$.

(ii) Representing the function on each edge $e \in \mathcal{E}_v$ as a linear combination of two exponents,

$$f_e(x_e) = a_e^{\mathrm{in}} \exp(-\mathrm{i}kx_e) + a_e^{\mathrm{out}} \exp(\mathrm{i}kx_e),$$

and dictating linear relations between the vectors of amplitudes $\vec{a}^{\mathrm{in}} \in \mathbb{C}^{|\mathcal{E}_v|}$ and $\vec{a}^{\mathrm{out}} \in \mathbb{C}^{|\mathcal{E}_v|}$. The vertex conditions are then expressed using an *a priori* given unitary matrix σ_v as

$$\vec{a}^{\mathrm{out}} = \sigma_v \vec{a}^{\mathrm{in}}.$$

A more detailed description on these characterizations of vertex conditions and the relations between them appears in [26, 27].

1.2. Isospectral graphs and their transplantation

The recent papers [1, 2] present a new isospectral construction method which is based on basic elements of representation theory and can be applied for any geometric object. We bring here the relevant aspects of the theory as it applies to quantum graphs[4].

Let Γ be a graph which obeys a certain symmetry group G. This means that each element of G is a graph automorphism which preserves both the lengths of the edges and the vertex conditions. Denote by $\Phi_\Gamma(k)$ the eigenspace of the Laplacian on Γ with eigenvalue k^2. The action of G on Γ induces an action of G on $\Phi_\Gamma(k)$ by

$$\forall x \in \Gamma, \qquad \forall g \in G, \qquad \forall f \in \Phi_\Gamma(k); \qquad (g\cdot f)(x) = f(g^{-1}x).$$

Since $\Phi_\Gamma(k)$ is closed under the action of G, we have that $\Phi_\Gamma(k)$ is a carrier space of a certain representation of G. Let R be another representation of G with some abstract carrier space V^R. We consider all linear transformations $\varphi : V^R \to \Phi_\Gamma(k)$ that respect the action of the group G, i.e.

$$\forall g \in G, \qquad \forall v \in V^R; \qquad \varphi(g\cdot v) = g\cdot\varphi(v).$$

These linear transformations are called intertwiners and they form a vector space which is denoted by $\mathrm{Hom}_{\mathbb{C}G}(V^R, \Phi_\Gamma(k))$. When R is an irreducible representation, each such intertwiner φ is an embedding of V^R in the eigenspace $\Phi_\Gamma(k)$. The dimension of $\mathrm{Hom}_{\mathbb{C}G}(V^R, \Phi_\Gamma(k))$ in this case equals the number of copies of V^R that are contained within $\Phi_\Gamma(k)$.

[4] The interested reader is referred to appendix A in [1] for a short review of the algebra used in this section.

It was shown in [1, 2] that there exists a graph, denoted by $\frac{\Gamma}{R}$, which obeys

$$\forall k \quad \Phi(k) \cong \mathrm{Hom}_{\mathbb{C}G}(V^R, \Phi_\Gamma(k)). \tag{1}$$

These papers supply a theorem which gives a sufficient condition for two quotient graphs $\frac{\Gamma}{R_1}$ and $\frac{\Gamma}{R_2}$ to be isospectral, where R_1 and R_2 are representations of two symmetry groups $H_1, H_2 \leqslant G$ of Γ. The condition is $\mathrm{Ind}_{H_1}^G R_1 \cong \mathrm{Ind}_{H_2}^G R_2$.

An important element of the isospectral research is the concept of *transplantation*. This is a transformation between the eigenspace of one object to the eigenspace of its isospectral partner, with the same eigenvalue. Specifically, if Γ_1 and Γ_2 are isospectral graphs, then a transplantation is an isomorphism $T : \Phi_{\Gamma_1}(k) \xrightarrow{\cong} \Phi_{\Gamma_2}(k)$. A transplantation is therefore quite a useful tool to prove the isospectrality of objects. The isospectral construction method described in [1, 2] guarantees a corresponding transplantation. In general, isospectral objects consist of some elementary building blocks that are attached to each other in a prescribed way to form each of the objects. The transplantation can be described in terms of these building blocks. It expresses the restriction of an eigenfunction to a building block of the first object as a linear combination of the restrictions of an eigenfunction to building blocks of the second one. The transplantation can be therefore described by a matrix whose dimension equals the number of the building blocks. Furthermore, this matrix is independent of the spectral parameter k.

It is important to note that isospectrality does not necessarily imply transplantability. Namely, there are examples of isospectral objects for which there is no known transplantation. Such an example appears in section 4.2.

1.3. The scattering matrix of a quantum graph

To convert a compact graph Γ into a scattering system, one can connect its vertices (all or a subset) by leads which extend to infinity. We will denote by $\tilde{\Gamma}$ the extended quantum graph which consists of the original graph Γ and the external leads. Given Γ, the additional information that one needs in order to describe its extension $\tilde{\Gamma}$ consists of the set of vertices to which the leads are connected and also the modified vertex conditions at these vertices. The vertices which are not connected to leads in $\tilde{\Gamma}$ have the same vertex conditions as they had in Γ. It is therefore clear that there is more than one possible way to turn a compact graph Γ into a scattering system.

The graph $\tilde{\Gamma}$ has a continuous spectrum and we denote by $\Phi_{\tilde{\Gamma}}(k)$ the space of all generalized eigenfunctions of $\tilde{\Gamma}$ with the eigenvalue k^2: they are called generalized since they do not necessarily have a bounded L^2-norm. Let $f \in \Phi_{\tilde{\Gamma}}(k)$ and $\mathcal{L}$ be the set of leads connected to Γ. Then the restriction of f to the lead $l \in \mathcal{L}$ can be written in the form

$$f_l(x_l) = a_l^{\mathrm{in}} \exp(-\mathrm{i}kx_l) + a_l^{\mathrm{out}} \exp(\mathrm{i}kx_l). \tag{2}$$

Collecting all the variables $\{a_l^{\mathrm{in}}\}_{l\in\mathcal{L}}$ and $\{a_l^{\mathrm{out}}\}_{l\in\mathcal{L}}$ into the vectors $\vec{a}^{\mathrm{in}}$ and $\vec{a}^{\mathrm{out}}$, we introduce the shorthand notation

$$f|_{\mathcal{L}} = \vec{a}^{\mathrm{in}} \exp(-\mathrm{i}kx) + \vec{a}^{\mathrm{out}} \exp(\mathrm{i}kx).$$

Using the vertex conditions on all the vertices of the graph, we get equations whose variables contain $\{a_l^{\mathrm{in}}\}_{l\in\mathcal{L}}$ and $\{a_l^{\mathrm{out}}\}_{l\in\mathcal{L}}$. Simple algebraic manipulations allow us to find a relation of the following type:

$$\vec{a}^{\mathrm{out}} = S_{\tilde{\Gamma}}(k)\vec{a}^{\mathrm{in}}, \tag{3}$$

where $S_{\tilde{\Gamma}}(k)$ is a square matrix of dimension $|\mathcal{L}|$ and it is unitary for every $k \in \mathbb{R}$. This is the scattering matrix of the graph $\tilde{\Gamma}$. The existence and uniqueness of $S_{\tilde{\Gamma}}(k)$ for every value of k and its unitarity on the real axis are proved in [28].

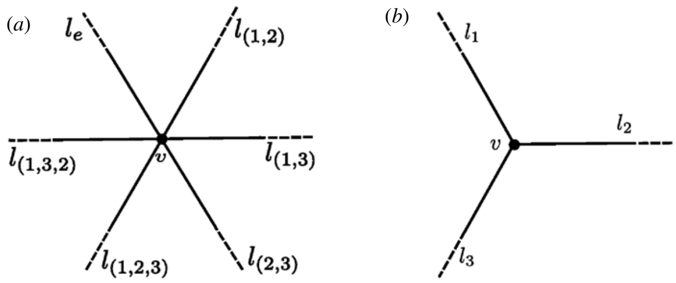

Figure 1. (*a*) The graph $\tilde{\Gamma}$ with leads labelled by the elements of $G = S_3$, (*b*) the graph $\frac{\tilde{\Gamma}}{\mathbf{1}_H}$.

Once $S_{\tilde{\Gamma}}(k)$ is given, we may consider a graph $\tilde{\Gamma}_0$ which consists of a single vertex, connected to the same number of leads as $\tilde{\Gamma}$ and with vertex conditions given by $S_{\tilde{\Gamma}}(k)$. The graphs $\tilde{\Gamma}_0$ and $\tilde{\Gamma}$ share the same spectral properties and their eigenfunctions coincide on the leads. Some of the information about the graph's complex structure is contained within the function $S_{\tilde{\Gamma}}(k)$. In particular, $S_{\tilde{\Gamma}}(k)$ contains the spectral information of Γ as is presented by the 'exterior–interior' duality for graphs [29]. This means that the eigenvalues of Γ can be identified as the solutions of the equation

$$\det(I - S_{\tilde{\Gamma}}(k)) = 0, \tag{4}$$

where $\tilde{\Gamma}$ is any extension of Γ, with the Neumann conditions at the vertices attached to the leads.

All of the above makes one wonder about the data that are contained in the scattering matrix and their comparison to the spectral information. This motivates the following definitions.[5]

Definition 1. *Let $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ be two quantum graphs. They are called isopolar if their scattering matrices $S_{\tilde{\Gamma}_1}(k)$ and $S_{\tilde{\Gamma}_2}(k)$ share the same poles on the complex plane.*

Definition 2. *Let $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ be two quantum graphs. They are called isophasal if they satisfy the following condition:*

$$\forall k \in \mathbb{R} \quad \frac{1}{\mathrm{i}} \log(\det S_{\tilde{\Gamma}_1}(k)) = \frac{1}{\mathrm{i}} \log(\det S_{\tilde{\Gamma}_2}(k)). \tag{5}$$

We will use the term *isoscattering* to refer to either *isopolar* or *isophasal* quantum graphs. This paper treats the construction of isoscattering graphs and the relation of isospectrality to isoscattering.

2. A basic example

In this section we explain using a simple example the idea of the quotient graph construction. We then use this example throughout the paper to illustrate its main results. Following the discussion after equation (3) we can restrict our attention to graphs with a single vertex attached to leads. Let us consider the quantum graph $\tilde{\Gamma}$ that consists of six semi-infinite leads which are connected to a single vertex v. The vertex conditions at v are the Neumann conditions (see figure 1(*a*)).

[5] Since there are various definitions in the literature, we specify what isopolar, isophasal and isoscattering mean in our setting.

We label Γ's leads by the elements of the group $G = S_3$ and parameterize each lead l_g by a coordinate $x_g \in [0, \infty[$ such that $x_g(v) = 0$. The permutation group G acts on the graph in the following way:

$$\forall g, h \in G, \qquad \forall x_h \in l_h : \qquad g x_h = x_{gh} \in l_{gh}. \tag{6}$$

The action of $g \in G$ on $f \in \Phi_{\tilde{\Gamma}}(k)$ gives the function $gf \in \Phi_{\tilde{\Gamma}}(k)$ defined by

$$(gf)(x) = f(g^{-1}x). \tag{7}$$

Let us consider the subgroup $H = \{e, (1, 2)\}$ of G. The action of H on the leads of $\tilde{\Gamma}$ gives the following orbits:

$$\mathcal{O}_e = \{l_e, l_{(1,2)}\}, \qquad \mathcal{O}_{(1,3)} = \{l_{(1,3)}, l_{(1,3,2)}\}, \qquad \mathcal{O}_{(1,2,3)} = \{l_{(1,2,3)}, l_{(2,3)}\}.$$

Let us restrict our attention to those functions $f \in \Phi_{\tilde{\Gamma}}(k)$ which transform under the action of H according to the trivial representation $\mathbf{1}_H$ of H. This means that

$$f|_{l_e} = f|_{l_{(1,2)}}, \qquad f|_{l_{(1,3)}} = f|_{l_{(1,3,2)}}, \qquad f|_{l_{(1,2,3)}} = f|_{l_{(2,3)}}.$$

Therefore, it is enough to know the values of the function f on the three leads $l_e, l_{(1,3)}$ and $l_{(1,2,3)}$, being representatives of each orbit, to deduce its values on the whole of $\tilde{\Gamma}$. We wish to encode the information about such a function and may do so by constructing the so-called quotient graph. The quotient, which we denote by $\frac{\tilde{\Gamma}}{\mathbf{1}_H}$ has three leads, each is a representative of an orbit of the H-action (figure 1(*b*)). The encoding is described by the map ϕ that acts on functions which transforms under the action of H according to the representation $\mathbf{1}_H$. It takes the values of such $f \in \Phi_{\tilde{\Gamma}}(k)$ restricted to the leads l_e, $l_{(1,2)}$ and $l_{(1,2,3)}$ and assigns them to the function ϕf on the leads l_1, l_2, l_3 of $\frac{\tilde{\Gamma}}{\mathbf{1}_H}$. By the construction we see that ϕf satisfies the following vertex conditions:

$$(\phi f)|_{l_1}(v) = (\phi f)|_{l_2}(v) = (\phi f)|_{l_3}(v)$$

$$\sum_{i=1}^{3} (\phi f)|'_{l_i}(v) = f|'_{l_e}(v) + f|'_{l_{(1,3)}}(v) + f|'_{l_{(1,2,3)}}(v) = \frac{1}{2}\sum_{g\in G} f|'_{l_g}(v) = 0.$$

We therefore equip $\frac{\tilde{\Gamma}}{\mathbf{1}_H}$ with the Neumann condition at its vertex. In general, the vertex conditions at each vertex of a quotient graph $\frac{\tilde{\Gamma}}{R}$ are determined by the vertex conditions of the corresponding vertices in $\tilde{\Gamma}$ combined with the group action and the representation R. The exact formula for the general case is given in [2].

We now construct another quotient graph which will be shown to be isoscattering to $\frac{\tilde{\Gamma}}{\mathbf{1}_H}$. The quotient will be calculated with respect to the representation $R = \mathbf{1}_G \oplus R_{2d}$, where $\mathbf{1}_G$ is the trivial representation of G and R_{2d} is the two-dimensional irreducible representation of G. It was shown in [1] that $\frac{\tilde{\Gamma}}{R} = \frac{\tilde{\Gamma}}{\mathbf{1}_G} \cup \frac{\tilde{\Gamma}}{R_{2d}}$ and therefore we may construct each of the quotients on the right-hand side separately. The quotient $\frac{\tilde{\Gamma}}{\mathbf{1}_G}$ can be inferred using similar reasoning as in the case of $\frac{\tilde{\Gamma}}{\mathbf{1}_H}$, and it turns out to be a single lead with the Neumann vertex condition (figure 2(*a*)). In order to construct $\frac{\tilde{\Gamma}}{R_{2d}}$, we choose a basis in $V^{R_{2d}} \cong \mathbb{C}^2$ and write the corresponding matrix representation $\rho^{R_{2d}} : G \to Gl(2, \mathbb{C})$ of R_{2d}. Such a choice gives, for example,

$$\rho^{R_{2d}}(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho^{R_{2d}}((1,2)) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho^{R_{2d}}((1,3)) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$
$$\rho^{R_{2d}}((2,3)) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \quad \rho^{R_{2d}}((1,2,3)) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad \rho^{R_{2d}}((1,3,2)) = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}. \tag{8}$$

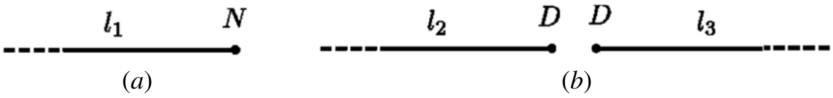

Figure 2. (*a*) The graph $\frac{\tilde{\Gamma}}{\mathbf{1}_G}$, (*b*) the graph $\frac{\tilde{\Gamma}}{R_{2d}}$. In both cases N stands for the Neumann vertex conditions and D for the Dirichlet vertex conditions.

The representation R_{2d} is two dimensional, which means we need to consider two linearly independent functions $f_1, f_2 \in \Phi_{\tilde{\Gamma}}(k)$ that transform under the action of G according to the representation R_{2d}, and in particular

$$\forall g \in G \quad [f_1|_{l_g}, f_2|_{l_g}]^T = (\rho^{R_{2d}}(g))[f_1|_{l_e}, f_2|_{l_e}]^T. \tag{9}$$

The restriction of f_1 and f_2 to l_e together with property (9) can be used to determine the values of $f_1$andf_2 on the whole of $\tilde{\Gamma}$. The encoding process therefore requires two leads, which is the number of leads in the quotient graph $\frac{\tilde{\Gamma}}{R_{2d}}$. It is only left to conclude the vertex conditions of this graph. Plugging $g = (1, 3)$ and $g = (2, 3)$ in (9) we obtain

$$[f_1|_{l_{(1,3)}}, f_2|_{l_{(1,3)}}]^T = [-f_2|_{l_e}, -f_1|_{l_e}]^T \tag{10}$$

$$[f_1|_{l_{(2,3)}}, f_2|_{l_{(2,3)}}]^T = [f_1|_{l_e}, f_1|_{l_e} - f_2|_{l_e}]^T. \tag{11}$$

Restricting (10) and (11) to the vertex v which obeys the Neumann conditions we see that their left-hand sides are equal to each other. Carrying this to the right-hand sides we get

$$-f_2|_{l_e}(v) = f_1|_{l_e}(v) \tag{12}$$

$$-f_1|_{l_e}(v) = f_1|_{l_e} - f_2|_{l_e}(v). \tag{13}$$

These two relations give $f_1(v) = 0$ and $f_2(v) = 0$. One can check that using the other group elements we do not add any more linearly independent conditions on the values or on the derivatives of f_1 and f_2. We therefore conclude that $\frac{\tilde{\Gamma}}{R_{2d}}$ is the union of two leads with the Dirichlet vertex conditions as is shown in figure 2(*b*).

3. Isospectrality and scattering matrices

3.1. *The scattering matrix of the quotient graph*

In the following section we apply the isospectral construction presented in [1, 2] to quantum graphs with leads. This allows us to investigate the relation between scattering matrices of isospectral graphs. Let Γ be a quantum graph and $\tilde{\Gamma}$ some extension of Γ to a scattering system. Let G be a symmetry group of $\tilde{\Gamma}$ and let R be a matrix representation of G with some carrier space V^R. We may then apply the isospectral theory to have the analogue of (1) for graphs with leads

$$\Psi : \Phi_{\tilde{\Gamma}/R}(k) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{C}G}(V^R, \Phi_{\tilde{\Gamma}}(k)). \tag{14}$$

Choosing some $v \in V^R$, we define

$$\Psi^{(v)} : \Phi_{\tilde{\Gamma}/R}(k) \to \Phi_{\tilde{\Gamma}}(k)$$

$$\Psi^{(v)}(f) := \Psi(f)(v),$$

and denote the image of $\Psi^{(v)}$ by

$$\Phi^{(v)}_{\tilde{\Gamma}}(k) := \left\{\Psi(f)(v) | f \in \Phi_{\tilde{\Gamma}/R}(k)\right\}.$$

It is easy to see that $\Phi^{(v)}_{\tilde{\Gamma}}(k)$ is a vector space. One can also show that if $\mathrm{Hom}_{\mathbb{C}G}(V^R, \Phi_{\tilde{\Gamma}}(k))$ is non-trivial and R is an irreducible representation, then the subset $\mathcal{F} \subset \Phi_{\tilde{\Gamma}/R}(k)$ is linearly independent if and only if $\{\Psi^{(v)}(f)\}_{f\in\mathcal{F}}$ is linearly independent. We therefore restrict ourselves for the moment to the case of irreducible representations, for which we get

$$\Psi^{(v)} : \Phi_{\tilde{\Gamma}/R}(k) \xrightarrow{\cong} \Phi^{(v)}_{\tilde{\Gamma}}(k). \tag{15}$$

Having in mind that this isospectral theory is motivated by an encoding scheme, we give such an interpretation to (15). Each eigenfunction of the quotient $\frac{\tilde{\Gamma}}{R}$ encodes information on some eigenfunction of $\tilde{\Gamma}$ which transforms just like v transforms, under the group action, as is described by the matrix representations R.

In order to relate this construction to the corresponding scattering matrices, we need to rephrase the equations above in terms of the spaces of ingoing and outgoing amplitudes on the leads rather then the eigenspaces. In order to do so, we introduce the notation $L_{\tilde{\Gamma}}$ for the vector space in which $\vec{a}^{\mathrm{in}}$ lies. It is easy to see that given $f \in \Phi_{\tilde{\Gamma}}(k)$, the corresponding vector $\vec{a}^{\mathrm{in}} \in L_{\tilde{\Gamma}}$ can be uniquely determined by restricting f to the leads and reading the coefficients of the exponents $\exp(-\mathrm{i}kx_l)$. In addition, it was shown in [28] that the opposite also holds, i.e. $\vec{a}^{\mathrm{in}} \in L_{\tilde{\Gamma}}$ uniquely determines the corresponding $f \in \Phi_{\tilde{\Gamma}}(k)$.[6] We therefore have a natural (k-dependent) isomorphism between those spaces

$$L_{\tilde{\Gamma}} \cong \Phi_{\tilde{\Gamma}}(k). \tag{16}$$

One should note that there is a well-defined action of G on $L_{\tilde{\Gamma}}$, as it induces the action of G on the leads of $\tilde{\Gamma}$. The isomorphism in (16) is actually an intertwiner, and as such, we can rewrite all the equalities in this section in terms of $L_{\tilde{\Gamma}}$ and $L_{\tilde{\Gamma}/R}$. In particular, we get the following analogue of (15):

$$\Upsilon^{(v)} : L_{\tilde{\Gamma}/R} \xrightarrow{\cong} L^{(v)}_{\tilde{\Gamma}}, \tag{17}$$

where $\Upsilon^{(v)}$ is the restriction of $\Psi^{(v)}$ on the spaces $L_{\tilde{\Gamma}/R}$ and $L_{\tilde{\Gamma}}$, and $L^{(v)}_{\tilde{\Gamma}}$ is the pre-image of $\Psi^{(v)}$ under the isomorphism in (16).

An important property of the encoding (15) (also (14)) is that it does not mix between the ingoing amplitudes $\vec{a}^{\mathrm{in}}$ and the outgoing ones, $\vec{a}^{\mathrm{out}}$. This results from the nature of the encoding and means that the ingoing and the outgoing amplitudes are encoded in exactly the same manner. Namely, for some $f \in \Phi_{\tilde{\Gamma}/R}(k)$ whose values on the leads are

$$f|_{\mathcal{L}} = \vec{a}^{\mathrm{in}} \exp(-\mathrm{i}kx) + \vec{a}^{\mathrm{out}} \exp(\mathrm{i}kx),$$

we have

$$(\Psi^{(v)} f)|_{\mathcal{L}} = \Upsilon^{(v)}(\vec{a}^{\mathrm{in}}) \exp(-\mathrm{i}kx) + \Upsilon^{(v)}(\vec{a}^{\mathrm{out}}) \exp(\mathrm{i}kx). \tag{18}$$

Note that in (18) we consider the spaces $L_{\tilde{\Gamma}}$ and $L_{\tilde{\Gamma}/R}$ as spaces which contain not only the ingoing amplitudes vectors $\vec{a}^{\mathrm{in}}$, but also the outgoing ones, $\vec{a}^{\mathrm{out}}$. We now have all that is required to express $S_{\tilde{\Gamma}/R}$ in terms of $S_{\tilde{\Gamma}}$. Let $\vec{a}^{\mathrm{in}}_{\tilde{\Gamma}/R} \in L_{\tilde{\Gamma}/R}$. There exists a corresponding function $f \in \Phi_{\tilde{\Gamma}/R}(k)$, whose restriction to the leads is given by

$$f|_{\mathcal{L}} = \vec{a}^{\mathrm{in}}_{\tilde{\Gamma}/R} \exp(-\mathrm{i}kx) + S_{\tilde{\Gamma}/R}\vec{a}^{\mathrm{in}}_{\tilde{\Gamma}/R} \exp(\mathrm{i}kx).$$

Applying $\Psi^{(v)}$ (with an arbitrary $v \in V^R$) to f and restricting again to the leads (this time the leads of $\tilde{\Gamma}$), we get

$$(\Psi^{(v)} f)|_{\mathcal{L}} = \vec{a}^{\mathrm{in}}_{\tilde{\Gamma}} \exp(-\mathrm{i}kx) + S_{\tilde{\Gamma}}\vec{a}^{\mathrm{in}}_{\tilde{\Gamma}} \exp(\mathrm{i}kx), \tag{19}$$

[6] This is true under some generality conditions which are stated in [28].

where $\vec{a}^{\text{in}}_{\tilde{\Gamma}}$ is an appropriately chosen vector in $L_{\tilde{\Gamma}}$. Comparing (19) to (18) gives

$$\vec{a}^{\text{in}}_{\tilde{\Gamma}} = \Upsilon^{(v)}\left(\vec{a}^{\text{in}}_{\tilde{\Gamma}/R}\right)$$
$$S_{\tilde{\Gamma}}\vec{a}^{\text{in}}_{\tilde{\Gamma}} = \Upsilon^{(v)}\left(S_{\tilde{\Gamma}/R}\vec{a}^{\text{in}}_{\tilde{\Gamma}/R}\right).$$

From the above we arrive at

$$S_{\tilde{\Gamma}/R}\vec{a}^{\text{in}}_{\tilde{\Gamma}/R} = (\Upsilon^{(v)})^{-1}S_{\tilde{\Gamma}}\Upsilon^{(v)}\left(\vec{a}^{\text{in}}_{\tilde{\Gamma}/R}\right),$$

which holds for every $\vec{a}^{\text{in}}_{\tilde{\Gamma}/R} \in L_{\tilde{\Gamma}/R}$ and therefore

$$S_{\tilde{\Gamma}/R} = (\Upsilon^{(v)})^{-1}S_{\tilde{\Gamma}}\Upsilon^{(v)}. \tag{20}$$

We can generalize (20) to any representation $R = \oplus_i n_i R_i$, where $\{R_i\}$ are the irreducible representations of G. We choose a vector v_i from each of the abstract carrier spaces $\{V^{R_i}\}$ of the representations $\{R_i\}$. The scattering matrix $S_{\tilde{\Gamma}/R}$ is then composed of the sub-matrices $(\Upsilon^{(v_i)})^{-1}S_{\tilde{\Gamma}}\Upsilon^{(v_i)}$, each appearing n_i times.

3.2. Back to the example

We now implement (20) in order to calculate the scattering matrices of the quotient graphs $\frac{\tilde{\Gamma}}{\mathbf{1}_H}$, $\frac{\tilde{\Gamma}}{R}$ presented in section 2. The scattering matrix of a star graph with the Neumann vertex conditions is given by

$$S_{ij} = \frac{2}{d} - \delta_{ij}, \tag{21}$$

where d is the valency of the graph's vertex. Applying this formula to the graph $\tilde{\Gamma}$ (where $d = 6$) we get the scattering matrix $(S_{\tilde{\Gamma}})_{ij} = \frac{1}{3} - \delta_{ij}$. The space of ingoing\outgoing amplitudes in this case is $L_{\tilde{\Gamma}} = \mathbb{C}^6$. We denote the standard basis elements of $\mathbb{C}^6$ by the elements of $G = S_3$ and infer using equation (6) and figure 1(*a*) that the subgroup $H = \{e, (1, 2)\}$ is acting on $L_{\tilde{\Gamma}}$ by

$$e \to \mathbf{1}_{L_{\tilde{\Gamma}}}, \qquad (1, 2) \to \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \tag{22}$$

In order to compute $L_{\tilde{\Gamma}/\mathbf{1}_H}$ we need to find a subspace of linearly independent vectors in $L_{\tilde{\Gamma}}$ that transform according to the trivial representation under the action of H. In other words, we are looking for eigenvectors of the matrix assigned to (1) and (2) by (22) with the eigenvalue 1. This subspace is

$$L^{(v)}_{\tilde{\Gamma}} = \text{Span}([1, 1, 0, 0, 0, 0]^T, [0, 0, 1, 0, 0, 1]^T, [0, 0, 0, 1, 1, 0]^T).$$

This is actually the image of $\Upsilon^{(v)}$ acting on $L_{\tilde{\Gamma}/\mathbf{1}_H}$ and therefore a possible basis for $L_{\tilde{\Gamma}/\mathbf{1}_H}$ is given by the action of encoding map $(\Upsilon^{(v)})^{-1}$ on the vectors above:

$$(\Upsilon^{(v)})^{-1}[1, 1, 0, 0, 0, 0]^T = [1, 0, 0]^T,$$
$$(\Upsilon^{(v)})^{-1}[0, 0, 1, 0, 0, 1]^T = [0, 1, 0]^T,$$
$$(\Upsilon^{(v)})^{-1}[0, 0, 0, 1, 1, 0]^T = [0, 0, 1]^T,$$

where the encoding is just a projection on the first, third and fifth coordinates of $L_{\tilde{\Gamma}}$ as was explained in section 2. Making use of (20) we get

$$S_{\tilde{\Gamma}/\mathbf{1}_H}(k) = (\Upsilon^{(v)})^{-1} S_{\tilde{\Gamma}}(k) \Upsilon^{(v)} = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}. \tag{23}$$

As expected, the above-mentioned equation corresponds the general formula for the scattering matrix of a star graph with the Neumann conditions (21).

We know that for $\frac{\tilde{\Gamma}}{R}$ the scattering matrix is the direct sum $S_{\tilde{\Gamma}/\mathbf{1}_G} \oplus S_{\tilde{\Gamma}/R_{2d}}$. The first element in the sum is $S_{\tilde{\Gamma}/\mathbf{1}_G} = 1$ (by similar reasoning as for $\frac{\tilde{\Gamma}}{\mathbf{1}_H}$). In order to compute $L_{\tilde{\Gamma}/R_{2d}}$ we need to choose any vector v from the abstract carrier space $V^{R_{2d}}$ and to find a subspace of the linearly independent vectors in $L_{\tilde{\Gamma}}$ that transform under the action of G as v under the action of R_{2d} (given by the matrices (8)). The vector that we choose is the first basis vector for which R_{2d} is represented by the matrices (8). It can be easily checked that there are exactly two linearly independent vectors in $L_{\tilde{\Gamma}}$ that fulfil these conditions:

$$L_{\tilde{\Gamma}}^{(v)} = \mathrm{Span}([1, -1, 0, 1, 1, -1]^T, [0, 1, -1, 0, -1, 1]^T).$$

The action of the encoding map $(\Upsilon^{(v)})^{-1}$ on these vectors gives

$$(\Upsilon^{(v)})^{-1}[1, -1, 0, 1, 1, -1]^T = [1, 0]^T,$$
$$(\Upsilon^{(v)})^{-1}[0, 1, -1, 0, -1, 1]^T = [0, 1]^T.$$

Finally,

$$S_{\tilde{\Gamma}/R_{2d}}(k) = (\Upsilon^{(v)})^{-1} S_{\tilde{\Gamma}}(k) \Upsilon^{(v)} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix};$$

hence,

$$S_{\tilde{\Gamma}/R}(k) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{24}$$

In the example above we have calculated the scattering matrices of the quotient graphs $\frac{\tilde{\Gamma}}{\mathbf{1}_H}$, $\frac{\tilde{\Gamma}}{\mathbf{1}_G}$ and $\frac{\tilde{\Gamma}}{R_{2d}}$ using formula (20). Furthermore, this is actually a reconstruction of the graphs themselves, as we now explain. Each of the graphs $\frac{\tilde{\Gamma}}{\mathbf{1}_H}$, $\frac{\tilde{\Gamma}}{\mathbf{1}_G}$ and $\frac{\tilde{\Gamma}}{R_{2d}}$ consists of a single vertex attached to leads and therefore its scattering matrix dictates the vertex conditions and gives a complete description of the graph. This can be generalized to form an alternative description for the quotient graph construction in [1, 2]. A fundamental element of the quotient construction is the ability to express the vertex conditions of the quotient in terms of the vertex conditions of the original graph and the group action. The vertex conditions in [1, 2] are described by stating linear sets of equations on the values and the derivatives of the function at each vertex. The quotient construction relates the quotient equation sets to those of the original graphs. However, as was explained in section 1.1, vertex conditions can also be represented by unitary scattering matrices at the vertices. Using this approach one can construct a quotient graph by relating the scattering matrices at the vertices of the quotient to the ones at the vertices of the original graph, using formula (20) and its generalization. The method of determining the exact structure of the quotient graph, i.e. its edges, vertices and their connectivity remains the same as in [1, 2]. The difference in the current approach is the ability to use the scattering description of the vertex conditions in the construction process. This supplies us with more graphs to which the construction can be applied, as there are vertex conditions that can be described only using the scattering approach. An example for such vertex conditions is given by the discrete Fourier transform matrices

$$\sigma_{p,q} = \frac{1}{\sqrt{d}} \exp\left(2\pi \mathrm{i}\frac{pq}{d}\right),$$

where d is the degree of the vertex (more details can be found in [26]).

3.3. Isoscattering graphs

Let Γ_1 and Γ_2 be isospectral graphs constructed by the method given in [1, 2]. Let Γ be the graph from which Γ_1 and Γ_2 are obtained as quotients. Namely, there is a group G and two subgroups $H_1, H_2 < G$ with the corresponding representations R_1 and R_2, such that $\Gamma_1 = \frac{\Gamma}{R_1}, \Gamma_2 = \frac{\Gamma}{R_2}$ and $\mathrm{Ind}_{H_1}^G R_1 \cong \mathrm{Ind}_{H_2}^G R_2$. Let $\tilde{\Gamma}$ be a quantum graph which is obtained by attaching leads to Γ in a way that conserves both symmetries H_1 and H_2. We may therefore construct the quotients $\tilde{\Gamma}_1 = \frac{\tilde{\Gamma}}{R_1}$ and $\tilde{\Gamma}_2 = \frac{\tilde{\Gamma}}{R_2}$ according to the method presented in [1, 2] and obtain the existence of the following transplantation:

$$T : \Phi_{\tilde{\Gamma}_1}(k) \xrightarrow{\cong} \Phi_{\tilde{\Gamma}_2}(k).$$

It was already mentioned in section 3.1 that the maps we consider can be restricted from the eigenspaces $\Phi_{\tilde{\Gamma}_1}$ and $\Phi_{\tilde{\Gamma}_2}$ to the leads' amplitudes spaces $L_{\tilde{\Gamma}_1}$ and $L_{\tilde{\Gamma}_2}$. Furthermore, they act the same on the ingoing and outgoing amplitudes (recall (18) and the explanation that precedes it). Exploiting this, we may apply the transplantation on

$$f|_{\mathcal{L}} = \vec{a}^{\,\mathrm{in}}_{\tilde{\Gamma}_1} \exp(-\mathrm{i}kx) + S_{\tilde{\Gamma}_1}\vec{a}^{\,\mathrm{in}}_{\tilde{\Gamma}_1} \exp(\mathrm{i}kx),$$

and get

$$(Tf)|_{\mathcal{L}} = \Pi\vec{a}^{\,\mathrm{in}}_{\tilde{\Gamma}_1} \exp(-\mathrm{i}kx) + \Pi S_{\tilde{\Gamma}_1}\vec{a}^{\,\mathrm{in}}_{\tilde{\Gamma}_1} \exp(\mathrm{i}kx), \tag{25}$$

where

$$\Pi : L_{\tilde{\Gamma}_1} \xrightarrow{\cong} L_{\tilde{\Gamma}_2}(k),$$

is the restriction of the transplantation to the spaces $L_{\tilde{\Gamma}_1}$ and $L_{\tilde{\Gamma}_2}$.

We can write (25) in the form

$$(Tf)|_{\mathcal{L}} = \vec{a}^{\,\mathrm{in}}_{\tilde{\Gamma}_2} \exp(-\mathrm{i}kx) + S_{\tilde{\Gamma}_2}\vec{a}^{\,\mathrm{in}}_{\tilde{\Gamma}_2} \exp(\mathrm{i}kx),$$

where $\vec{a}^{\,\mathrm{in}}_{\tilde{\Gamma}_2}$ is an appropriately chosen vector in $L_{\tilde{\Gamma}_2}$. Comparing the two expressions for $(Tf)|_{\mathcal{L}}$ we obtain

$$\vec{a}^{\,\mathrm{in}}_{\tilde{\Gamma}_2} = \Pi\vec{a}^{\,\mathrm{in}}_{\tilde{\Gamma}_1}$$
$$S_{\tilde{\Gamma}_2}\vec{a}^{\,\mathrm{in}}_{\tilde{\Gamma}_2} = \Pi S_{\tilde{\Gamma}_1}\vec{a}^{\,\mathrm{in}}_{\tilde{\Gamma}_1},$$

which holds for every $\vec{a}^{\,\mathrm{in}}_{\tilde{\Gamma}_1} \in L_{\tilde{\Gamma}}$ and therefore

$$S_{\tilde{\Gamma}_1} = \Pi^{-1} S_{\tilde{\Gamma}_2} \Pi. \tag{26}$$

We conclude that the scattering matrices $S_{\tilde{\Gamma}_1}(k)$ and $S_{\tilde{\Gamma}_2}(k)$ of the quotients $\tilde{\Gamma}_1 = \frac{\tilde{\Gamma}}{R_1}, \tilde{\Gamma}_2 = \frac{\tilde{\Gamma}}{R_2}$ are conjugate to each other. Note that the conjugating matrix Π does not depend on k, as it inherits this property from the transplantation. In particular, we may conclude that $S_{\tilde{\Gamma}_1}(k)$ and $S_{\tilde{\Gamma}_2}(k)$ have the same phases and the same pole structure, i.e. the graphs $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are both isophasal and isopolar. The graphs Γ_1 and Γ_2 are therefore not only isospectral, but also isoscattering (with respect to attaching leads as in $\tilde{\Gamma}_1 = \frac{\tilde{\Gamma}}{R_1}$ and $\tilde{\Gamma}_2 = \frac{\tilde{\Gamma}}{R_2}$).

We return to the example from section 2. The quotients there are constructed according to the isospectral method in [1] which guarantees the existence of a transplantation T between

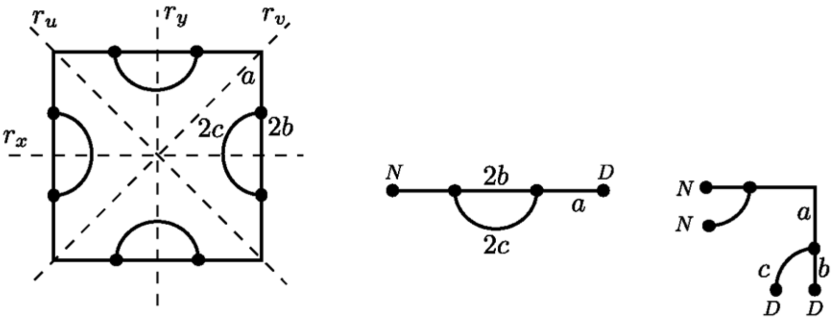

Figure 3. (*a*) The graph Γ that obeys the dihedral symmetry of the square D_4. The lengths of some edges and the axes of the reflection elements in D_4 are marked. (*b*) The graph $\frac{\Gamma}{R_1}$. (*c*) The graph $\frac{\Gamma}{R_2}$. Neumann conditions are default conditions in every unmarked vertex.

the graphs $\frac{\tilde{\Gamma}}{\mathbf{1}_H}$ and $\frac{\tilde{\Gamma}}{R}$. We note that both graphs are composed of three leads, which form the so-called 'building blocks' on which the transplantation acts:

$$\begin{pmatrix} (Tf)|_{l_1} \\ (Tf)|_{l_2} \\ (Tf)|_{l_3} \end{pmatrix} = \Pi \begin{pmatrix} f|_{l_1} \\ f|_{l_2} \\ f|_{l_3} \end{pmatrix}, \qquad \Pi = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}. \tag{27}$$

It can be easily checked that starting from any $f \in \Phi_{\tilde{\Gamma}/\mathbf{1}_H}(k)$ and applying T the function Tf obeys the vertex condition of $\frac{\tilde{\Gamma}}{R}$ and therefore $Tf \in \Phi_{\tilde{\Gamma}/R}(k)$. In addition, T is invertible as can be seen from the matrix above.

We now treat the scattering properties of $\frac{\tilde{\Gamma}}{\mathbf{1}_H}$ and $\frac{\tilde{\Gamma}}{R}$. Using (23), (24) and (27) we confirm the validity of (26) for our example:

$$\Pi^{-1} S_{\tilde{\Gamma}/R}(k) \Pi = S_{\tilde{\Gamma}/\mathbf{1}_H}(k). \tag{28}$$

4. More examples

4.1. Various extensions to scattering systems

In this section we give further examples for the different ways to extend isospectral graphs to isoscattering systems. Let Γ be the quantum graph shown in figure 3(*a*). The dihedral group $G = D_4$ is the symmetry group of this graph.

Examine the following subgroups of G:

$$H_1 = \{e, r_x, r_y, \sigma^2\}, \qquad H_2 = \{e, r_u, r_v, \sigma^2\},$$

where r_x, r_y, r_u, r_v denote reflections by the axes x, y, u, v and σ is the counterclockwise rotation by $\frac{\pi}{2}$. Consider the following one-dimensional representations R_1 and R_2 of H_1 and H_2 respectively:

$$R_1 = \{e \to (1), \sigma \to (-1), r_v \to (-1), r_u \to (1)\},$$
$$R_2 = \{e \to (1), \sigma \to (-1), r_y \to (1), r_x \to (-1)\}.$$

We have $\mathrm{Ind}_{H_1}^G R_1 = \mathrm{Ind}_{H_2}^G R_2$ and therefore may use isospectral construction to obtain the two isospectral quotient graphs $\frac{\tilde{\Gamma}}{R_1}$ and $\frac{\tilde{\Gamma}}{R_2}$ shown in figures 3(*b*) and (*c*). We now consider two

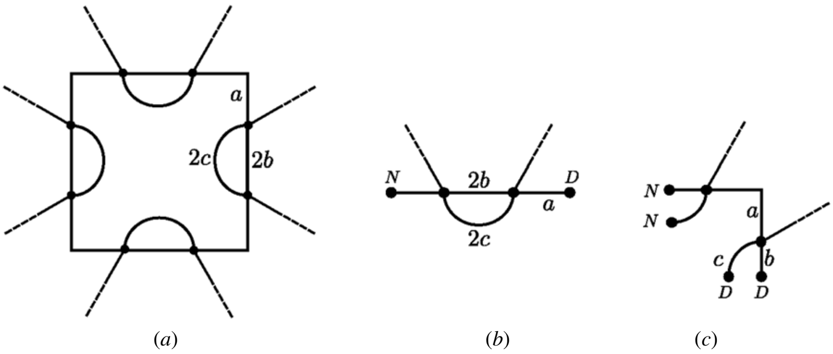

Figure 4. (a) The graph $\tilde{\Gamma}$ with leads attached, (b) the graph $\frac{\tilde{\Gamma}}{R_1}$ and (c) the graph $\frac{\tilde{\Gamma}}{R_2}$.

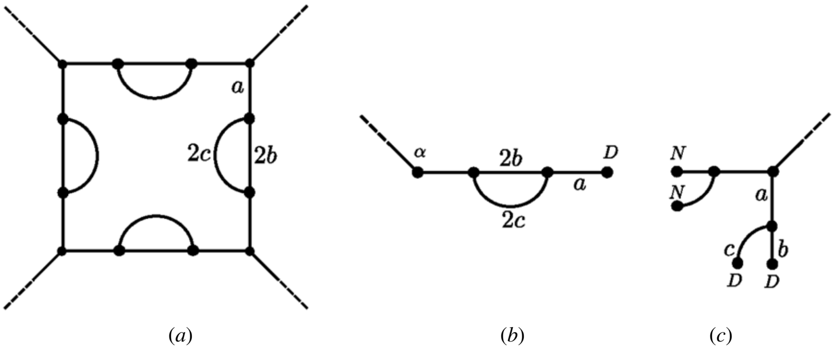

Figure 5. (a) The graph $\tilde{\Gamma}$ with leads attached, (b) the graph $\frac{\tilde{\Gamma}}{R_1}$ and (c) the graph $\frac{\tilde{\Gamma}}{R_2}$.

possible extensions of Γ to $\tilde{\Gamma}$ by adding infinite leads (figures 4(a) and 5(a)). The action of G on the graph in figure 4(a) is a free action. This causes the vertices to which the leads are attached keep their original (Neumann) vertex conditions from Γ. The isoscattering quotients in this case appear in figures 4(b) and (c). The case is different for the graph shown in figure 5(a) in which the action of r_u, r_v on the leads is not free. This causes the disappearance of one of the leads in the quotient $\frac{\tilde{\Gamma}}{R_2}$ (figure 5(b)). The surviving lead is attached to a vertex with the following vertex conditions:

$$f_e(\alpha) = f_l(\alpha) \qquad 2f_e'(\alpha) = f_l'(\alpha),$$

where e stands for edge and l stands for lead. The disappearance of the lead and the modified vertex conditions are typical for a quotient derived from a non-free action (see [1, 2] for a more detailed description). The other quotient $\frac{\tilde{\Gamma}}{R_1}$ has the Neumann conditions at the vertex where the lead is attached since it is a quotient with respect to a representation of a group which acts freely on $\tilde{\Gamma}$.

4.2. *The lack of transplantability*

Not all isospectral graph pairs are constructed by the Sunada method and its generalizations. Here we discuss a specific example and consider its scattering analogue. The isospectral quantum graphs shown in figure 6 were constructed out of weighted discrete graphs. The

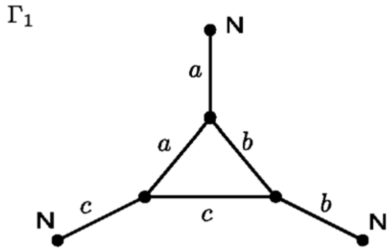

Figure 6. The isospectral graphs Γ_1 and Γ_2. The lengths of the edges are indicated. All vertices obey the Neumann vertex conditions.

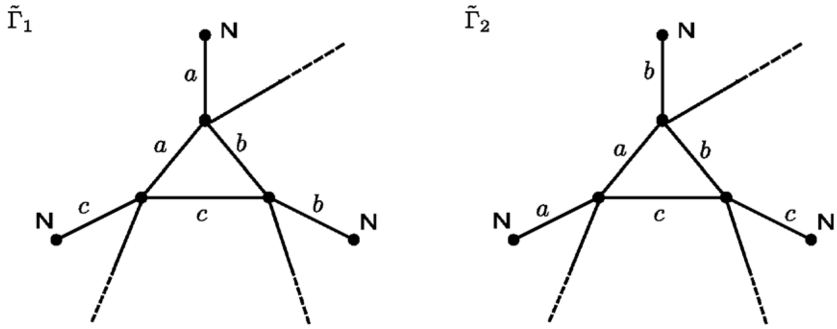

Figure 7. The quantum graphs $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ which are the extensions of $\Gamma_1$and Γ_2.

corresponding isospectral discrete graphs appeared in [30] and were turned into quantum graphs in [17], where their isospectrality was proved by explicitly calculating the spectral determinants and showing their equality.

These graphs can be extended to form scattering systems by attaching leads to their non-trivial vertices (figure 7).

The corresponding scattering matrices were calculated and their poles were numerically computed and are shown in figure 8. The different poles distributions indicate that the scattering matrices are not conjugate, as opposed to the result obtained in section 3.3. We therefore conclude that there is no transplantation which relates the values of the non-trivial vertices of these graphs.

5. Discussion and open questions

This paper discusses the linkage between isospectrality and isoscattering of quantum graphs. We have described how to produce isoscattering graphs using the recently developed isospectral construction method [1, 2]. Isoscattering graphs can be produced in two ways. One can start from a certain graph with leads which forms a scattering system, and construct out of it two graphs with leads that are isoscattering. Another approach is to start from two graphs that are known to be isospectral and discover all the possible ways in which leads can be attached to the graphs to turn them into isoscattering. Both ways are applications of the isospectral construction method and as such they indicate that turning a graph into a scattering system does not reveal more information on the graph then is already given by its spectrum. This

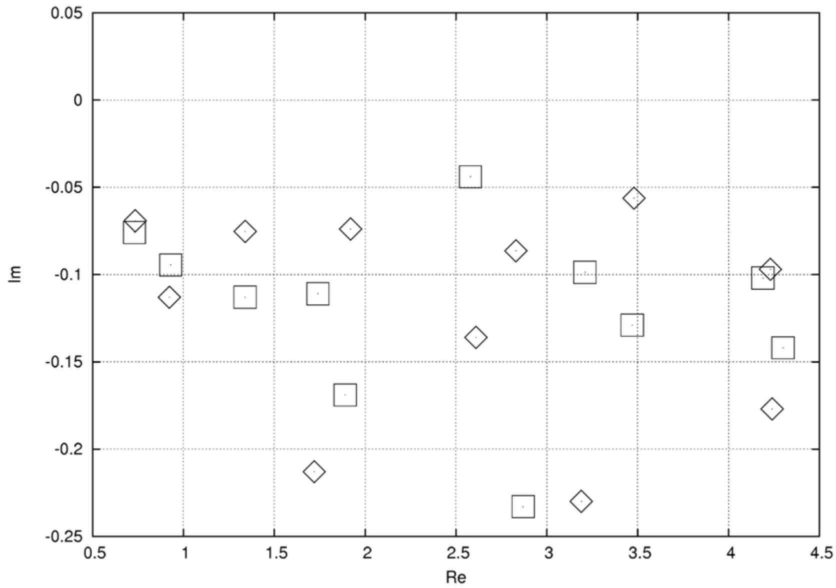

Figure 8. Scattering resonances for graphs $\tilde{\Gamma}_1 - \diamond$ and $\tilde{\Gamma}_2 - \Box$.

is compatible with the exterior–interior duality that relates the spectral information with the scattering data. It is interesting to compare this result with the conjecture brought in a recent work by Okada *et al* [25] in which the scattering from the exterior of isospectral domains in $\mathbb{R}^2$ is discussed. The authors conjecture that the distributions of the poles of the two scattering matrices distinguish between the two domains. In other words, in spite of the fact that the two domains are isospectral, Okada *et al* suggest that they are not isoscattering. This proposition is not proved, but it is ushered by heuristic arguments based on the calculation of the Fredholm determinant, and augmented by numerical simulations. The simulations are performed on the isospectral domains in $\mathbb{R}^2$ that appear in [31]. We have shown that two isospectral graphs can be turned into isoscattering systems in more than one possible way. However, not every way of attaching leads to the isospectral graphs would make them isoscattering. The leads should be connected in a way which reflects the underlying symmetry that was used for the construction of the isospectral pair. This may suggest that the result in [25] is a consequence of the fact that the outside scattering problem of the isospectral drums breaks the symmetry that was used in the isospectral drums construction. In [31] these drums are obtained from a group of isometries of the hyperbolic plane. The construction there yields two quotients which are composed of seven copies of a hyperbolic triangle assembled in different configurations. The planar isospectral drums are then obtained by replacing the fundamental hyperbolic triangle with a suitable Euclidean one. It should be noted that the hyperbolic drums are isometric and they become non-isometric only after the replacement with the Euclidean triangles. Therefore, the hyperbolic drums which obey the underlying symmetry are trivially isoscattering, being isometric.

The isoscattering property that we have obtained is a direct consequence of the existence of a transplantation. The transplantation is not only guaranteed by the isospectral construction method of [1, 2], but also appears in other isospectral constructions. We therefore get that the ability to turn isospectral graphs into isoscattering is not restricted only to those graphs constructed by the discussed isospectral method, but is possible for others as well, as long as the transplantation exists. This observation supplies us with a technique to check the transplantability property of two graphs that are known to be isospectral. All we need to do is to connect leads to the sets of vertices which we suspect to be transplantable and to check whether the corresponding scattering matrices are conjugate. An example of an isospectral pair which does not possess the transplantation property is given in section 4.2.

The key element of the isospectral theory developed in [1, 2] is the notion of the quotient graph. We have made use of the quotient construction in this paper as well, in order to produce isoscattering pairs. While doing so we obtained a relation between the scattering matrix of a graph and that of its quotient. This not only sheds more light on the properties of the quotient graph, but also gives an alternative description of the quotient. As was already mentioned, the vertex conditions of a quantum graph can be described in two ways. One is by a linear set of equations on the function's values and derivatives at the vertex. The other is by a scattering matrix which connects the ingoing amplitudes to the outgoing ones at the vertex. The current isospectral theory in [1, 2] deals with vertex conditions of the first type and describes the quotient construction in these terms. Therefore, an ability to construct a quotient from a graph whose vertex conditions are of the second type gives a new perspective to the isospectral theory and broadens the class of graphs for which the theory can be applied.

Acknowledgments

We acknowledge Idan Oren for suggesting us the new example of isospectral graphs which is examined in section 4.2. We thank Akira Shudo and Ori Parzanchevski for fruitful discussions. The work was supported by the Minerva Center for nonlinear Physics, the Einstein (Minerva) Center at the Weizmann Institute and the Wales Institute of Mathematical and Computational Sciences (WIMCS). Grants from EPSRC (grant EP/G021287), ISF (grant 166/09) and BSF (grant 2006065) are acknowledged. The support by SFB/TR12 'Symmetries and Universality in Mesoscopic Systems' program of the Deutsche Forschungsgemeischaft and Polish MNiSW grant no DFG-SFB/38/2007 is gratefully acknowledged.

References

[1] Band R, Parzanchevski O and Ben-Shach G 2009 The isospectral fruits of representation theory: quantum graphs and drums *J. Phys. A: Math. Theor.* **42** 175202
[2] Parzanchevski O and Band R 2010 Linear representations and isospectrality with boundary conditions *J. Geom. Anal.* **20** 439–71
[3] Kac M 1966 Can one hear the shape of a drum? *Am. Math. Mon.* **73** 1–23
[4] Sunada T 1985 Riemanninan coverings and isospectral manifolds *Ann. Math.* **121** 169–86
[5] Gordon C, Webb D and Wolpert S 1992 One cannot hear the shape of a drum *Bull. Am. Math. Soc.* **27** 134–38
[6] Gordon C, Webb D and Wolpert S 1992 Isospectral plane domains and surfaces via Riemannian orbifolds *Invent. Math.* **110** 1–22
[7] Berard P 1992 Transplantation et isospectralité i *Math. Ann.* **292** 547–59
[8] Steven Z 1992 Kuznecov sum formulae and szego limit formulae on manifolds *Commun. Partial Diff. Equ.* **17** 221–60
[9] Guillope L and Zworski M 1997 Scattering asymptotics for Riemann surfaces *Ann. Math.* **145** 597–660
[10] Brooks R and Davidovich O 2002 Isoscattering on surfaces *J. Geom. Anal.* **13** 39–53
[11] Gordon C, Perry P and Schueth D 2005 Isospectral and isoscattering manifolds: a survey of techniques and examples *Geometry, Spectral Theory, Groups, and Dynamics (Contemp. Math.* vol 387*)* (Providence, RI: American Mathematical Society) pp 157–79
[12] Gordon C 2009 Sunada's isospectrality technique: Two decades later *Spectral Analysis in Geometry and Number Theory (Contemp. Math.* vol 484*)* (Providence, RI: American Mathematical Society) pp 45–58

[13] Brooks R 1999 The Sunada method *Tel Aviv Topology Conference: Rothenberg Festschrift (1998) (Contemp. Math.* vol 231*)* (Providence, RI: American Mathematical Society) pp 25–35
[14] Gutkin B and Smilansky U 2001 Can one hear the shape of a graph? *J. Phys. A: Math. Gen.* **34** 6061
[15] Roth J P 1983 *Le spectre du Laplacien sur un graphe Théorie du Potentiel (Lecture Notes in Mathematics* vol 1096*)* (Berlin: Springer) pp 521–39
[16] vonBelow J 2000 *Can One Hear the Shape of a Network? (Partial Differential Equations on Multistructeres Lecture Notes in Pure and Applied Mathematics* vol 219*)* (New York: Dekker) pp 19–36
[17] Oren I 2008 Private communication
[18] Shapira T and Smilansky U 2005 Quantum graphs which sound the same in *NATO Science Series II: Mathematics, Physics and Chemistry (Nonlinear Dynamics and Fundamental Interactions* vol 213*)* pp 17–29
[19] Band R, Shapira T and Smilansky U 2006 Nodal domains on isospectral quantum graphs: the resolution of isospectrality? *J. Phys. A: Math. Gen.* **39** 13999–4014
[20] Gerasimenko N I 1988 The inverse scattering problem on a noncompact graph *Teor. Mat. Fiz.* **75** 187–200
[21] Kostrykin V and Schrader R 2000 Kirchhoff's rule for quantum wires: II. The inverse problem with possible applications to quantum computers *Fortschr. Phys.* **48** 703–16 (arXiv:quant-ph/9910053)
[22] Kostrykin V and Schrader R 2006 The inverse scattering problem for metric graphs and the traveling salesman problem arXiv:math-ph/0603010
[23] Boman J and Kurasov P 2005 Symmetries of quantum graphs and the inverse scattering problem *Adv. Appl. Math.* **35** 58–70
[24] Kurasov P and Stenberg F 2002 On the inverse scattering problem on branching graphs *J. Phys. A: Math. Gen.* **35** 101
[25] Okada Y, Shudo A, Tasaki S and Harayama T 2005 'Can one hear the shape of a drum?': revisited *J. Phys. A: Math. Gen.* **38** L163–L170
[26] Gnutzmann S and Smilansky U 2006 Quantum graphs: applications to quantum chaos and universal spectral statistics *Adv. Phys.* **55** 527–625
[27] Kuchment P 2004 Quantum graphs: I. Some basic structures *Waves Random Media* **14** S107–S128
[28] Band R, Berkolaiko G and Smilansky U Dynamics of nodal points and the nodal count on a family of quantum graphs arXiv:1008.0505v1
[29] Kottos T and Smilansky U 2003 Quantum graphs: a simple model for chaotic scattering *J. Phys. A: Math. Gen.* **36** 3501–24
[30] McDonald P and Meyers R 2003 Isospectral polygons, planar graphs and heat content *Proc. Am. Math. Soc.* **131** 3589–99 (electronic)
[31] Buser P, Conway J, Doyle P and Semmler K-D 1994 Some planar isospectral domains *Int. Math. Res. Not.* **9** 391–400