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Resolving the isospectrality of the dihedral graphs by counting nodal domains

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Abstract. We discuss isospectral quantum graphs which are not isometric. These graphs are the analogues of the isospectral domains in R^2 which were introduced recently in [1–5] all based on Sunada's construction of isospectral domains [6]. After discussing some of the properties of these graphs, we present an example which support the conjecture that by counting the nodal domains of the corresponding eigenfunctions one can resolve the isospectral ambiguity.

1 Introduction

M. Kac's classical paper "Can one hear the shape of a drum" [7], triggered intensive research in two complementary aspects of this problem. On the one hand, a search for systems for which Kac's question is answered in the affirmative was conducted, and, on the other hand, various examples of pairs of systems which are isospectral but not isometric were identified. In the present paper we shall focus our attention to quantum graphs and in the following lines will review the subject of isospectrality in this limited context. The interested reader is referred to [1–11] for a broader view of the field where spectral inversion and its uniqueness are discussed.

Shortly after the appearance of Kac's paper, M. E. Fisher published his work "On hearing the shape of a drum" [12], where he addresses isospectrality for the discrete version of the Laplacian. Since then, the study of isospectral combinatorial graphs made very impressive progress. In particular, several methods to construct isospectral yet different graphs were proposed. A review of this problem can be found in [14]. In particular, a method which was originally put forward by Sunada [6] to construct isospectral Laplace-Beltrami operators on Riemann manifolds, was adapted for the corresponding problem in the context of combinatorial graphs. Here we shall go one step further, and show that it can be also adapted for quantum graphs.

The conditions under which the spectral inversion of quantum graphs is unique were studied previously. In [15,16] it was shown that in general, the spectrum does not determine uniquely the length of the bonds and their connectivity. However, it was shown in [13] that quantum graphs whose bond lengths are *rationally independent* "can be heard" – that is – their spectra determine uniquely their connectivity matrices and their bond lengths. This fact follows from the existence of an exact trace formula for quantum graphs [17,18]. Thus, isospectral pairs of non congruent graphs, must have rationally dependent bond lengths. The Sunada method, which is based on constructing the isospectral domains by concatenating several copies of a given building block, automatically provides us with graphs with rationally dependent lengths. An example of a pair of metrically distinct graphs which share the same spectrum was already discussed in [13]. In a previous report we have shown that all the known isospectral domains in R^2 [3,4] have corresponding isospectral pairs of quantum graphs [20]. Here, we shall take the subject one step further, and propose that isospectral graphs can be resolved by counting

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their nodal domains. That is, the nodal counts of eigenfunctions belonging to the same spectral value are not the same. The idea that nodal counts resolve isospectrality was suggested in [19] for a family of isospectral flat tori in 4-d, and was tested numerically. The present work offers a rigorous proof of this conjecture for a certain example of isospectral graphs.

1.1 A short introduction to quantum graphs

We consider finite graphs consisting of V *vertices* connected by B *bonds*. The $V \times V$ *connectivity matrix* will be denoted by $C_{i,j} : C_{i,j} = r$ when the vertices i and j are connected by r bonds, and it vanishes otherwise. The group of bonds which emerge from the vertex i form a "star" which will be denoted by $S^{(i)}$. The *valency* v_i of a vertex is defined as the cardinality of the star $S^{(i)}$ (the total number of vertices connected to the vertex i) and $v_i = \sum_j C_{i,j}$. Vertices with $v_i = 1$ belong to the graph *boundary*. The vertices with $v_i > 1$ belong to the graph *interior*. The bonds are endowed with the standard metric, and the coordinates along the bonds b are denoted by x_b. The length of the bonds will be denoted by L_b, and the total length of the graph is $\mathcal{L} = \sum_b L_b$.

The domain of the Schrödinger operator on the graph is the space of functions which belong to Sobolev space $H^2(b)$ on each bond b and at the vertices they are continuous and obey boundary conditions as is mentioned in (1). The operator is constructed in the following way. On the bonds, it is identified as the Laplacian in 1-d $-\mathrm{d}^2/\mathrm{d}x^2$. It is supplemented by boundary conditions on the vertices which ensure that the resulting operator is self adjoint. We shall consider in this paper the Neumann and Dirichlet boundary conditions:

$$\text{Neumann} \ \ \forall i : \quad \sum_{b \in S^{(i)}} \frac{\mathrm{d}}{\mathrm{d}x_b} \psi_b(x_b) \bigg|_{x_b=0} = 0, \qquad (1)$$
$$\text{Dirichlet} \ \ \forall i : \psi_b(x_b)\big|_{x_b=0} = 0.$$

The derivatives in (1) are directed out of the vertex i. *Comment*: The Neumann boundary conditions will be assumed throughout, unless otherwise stated. A wave function with a wave number k can be written as

$$\psi_b(x_b) = \frac{1}{\sin kL_b} \left(\phi_i \sin k(L_b - x_b) + \phi_j \sin kx_b\right) \qquad (2)$$

where b connects the vertices i and j, where the wave function ψ_b takes the values ϕ_i and ϕ_j respectively. The form (2) ensures continuity. The spectrum $\{k_n\}$ and the corresponding eigenfunctions are determined by substituting (2) in (1). The resulting homogeneous linear equations for the ϕ_i are written as

$$\forall\ 1 \le i \le V : \quad \sum_{j=1}^{B} A_{i,j}(L_1, \ldots, L_B; k)\phi_j = 0, \qquad (3)$$

and a non trivial solution exists when

$$f(L_1, \ldots, L_B; k) \doteq \det A(L_1, \ldots, L_B; k) = 0. \qquad (4)$$

The spectrum $\{k_n\}$, which is a discrete, positive and unbounded sequence is the zero set of the *secular function* $f(L_1, \ldots, L_B; k)$. The secular functions of the type (4) have poles on the real k axis, which renders them rather inconvenient for numerical studies. The secular function can be easily regularized in various ways (see e.g., [18,20]).

It is easy to show that the complete wave function can be written down in terms of the vertex wave functions at the interior vertices with $v_i \geq 3$ only. In the sequel we shall denote their number by V_{int}. This reduces the dimension of the matrix A above from V to V_{int}.

1.2 The nodal domains of quantum graphs

The nodal domains of the eigenfunctions (the connected domains where the wave function is of constant sign), are of two types. The ones that are confined to a single bond are rather trivial. Their length is exactly half a wavelength and their number is on average $k\mathcal{L}/\pi$. The nodal domains which extend over several bonds emanating from a single vertex vary in length and their existence is the reason why counting nodal domains on graphs is not a trivial task. The number of nodal domains in a general graph can be written as

$$\nu_n = \frac{1}{2}\sum_i \sum_{b\in S^{(i)}} \left\{ \left\lfloor \frac{k_n L_b}{\pi} \right\rfloor + \frac{1}{2}\left(1-(-1)^{\lfloor \frac{k_n L_b}{\pi} \rfloor}\mathrm{sign}[\phi_i]\mathrm{sign}[\phi_j]\right)\right\} - B + V. \qquad (5)$$

where $\lfloor x \rfloor$ stands for the largest integer which is smaller than x, and ϕ_i, ϕ_j are the values of the eigenfunction at the vertices connected by the bond b [21]. (5) holds for the case of an eigenfunction which does not vanish on any vertex: $\forall i\, \phi_i \neq 0$.

Sturm's oscillation theorem for systems in 1-d, and its extension by Courant to any dimension, establish the connection between the number of nodal domains and the spectrum: The number of nodal domains ν_n of the n'th eigenfunction is bounded by n. (the eigenfunctions are arranged by increasing value of their eigenvalues). Courant's theorem for combinatorial graphs and for quantum graphs was proved in [21,24], respectively. Recently Schapotschnikow [22] proved that Sturm's Oscillation Theorem extends to finite tree (loop-less) graphs: the number of nodal domains of the n'th eigenfunction (ordered by increasing eigenvalues) is n. Berkolaiko [23] has shown that the number of nodal domains is bounded to the interval $[n-l, n]$ where l is the minimal number of bonds which should be cut so that the resulting graph is a tree.

Nodal domains can be also defined and counted in an alternative way which makes use of the vertex wave functions $\{\phi_i\}$ (see (2)) exclusively:

A nodal domain consists of a maximal set of connected interior vertices ($v_i \geq 3$) *where the vertex wave functions have the same sign.*

This definition has to be modified if any of the ϕ_i vanishes. Then, the sign attributed to it is chosen to maximize the number of nodal domains [24].

We thus have two independent ways to define and count nodal domains. To distinguish between them we shall refer to the first as *metric* nodal domains, and the number of metric domains in the n'th eigenfunction will be denoted by ν_n. Berkolaiko's theorem states that $(n-l) \leq \nu_n \leq n$. The domains defined in terms of the vertex wave functions will be referred to as the *discrete* nodal domains. The number of discrete nodal domains of the n'th vertex wave function will be denoted by μ_n. The sequences of metric and discrete nodal domains counts $\{\nu_n\}$ and $\{\mu_n\}$ are the main objects of study of the present paper.

2 Isospectral quantum graphs

The first pair of isospectral planar domains which was introduced by Gordon, Web and Wolpert [1] is a member of a much larger set which was discussed in [3,4]. This was extended in [5] to include domains which differ in the distribution of boundary conditions (Dirichlet or Neumann) along their boundaries. The common feature of these sets of pairs of isospectral domains is that they are constructed using the Sunada method [6].

The most simple pair of isospectral graphs can be built by applying the Sunada method to the dihedral group D_4 [27]. The resulting graphs are shown in figure 1, where the letters D or N specify the boundary conditions at the boundary vertices. (The boundary conditions at the interior vertices are always Neumann).

The resulting dihedral graphs have secular determinants which can be shown to be identical functions of k. This is an explicit demonstration that the graphs are indeed isospectral (another indication of the isospectrality is through the transplantation to be presented soon). The secular equation is obtained from

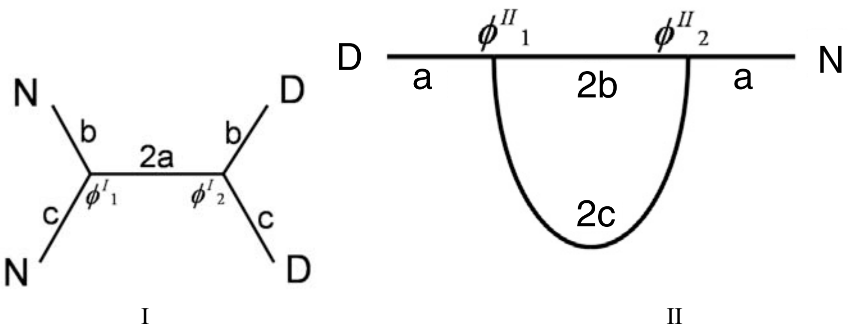

Fig. 1. The isospectral pair with boundary conditions. D stands for Dirichlet and N for Neumann.

$$A(a,b,c;k) = \begin{pmatrix} \eta - (\beta+\gamma) & -\alpha \\ -\alpha & \eta + (\beta+\gamma) \end{pmatrix}$$
$$\alpha(a;k) = \frac{1}{\sin(2ka)}; \qquad \beta(b;k) = \frac{1}{\sin(2kb)}; \qquad \gamma(a;k) = \frac{1}{\sin(2kc)} \tag{6}$$
$$\eta(a,b,c;k) = \cot(2ka) + \cot(2kb) + \cot(2kc)$$

The corresponding vertex eigenfunction $\boldsymbol{\phi}^I = (\phi_1^I, \phi_2^I)$ is the eigenvector of $A(a,b,c;k_n)$ with a vanishing eigenvalue. As can be easily seen, $\det A$ has poles, and to get a regular secular function we have to multiply it by $\sin(2ka)\sin(2kb)\sin(2kc)$. It takes the form

$$f(a,b,c;k) = \sin(2ka)\left(-2 + 2\cos(2kb)\cos(2kc) - 3\sin(2kb)\sin(2kc)\right) + 2\cos(2ka)\sin(2kb+2kc) \tag{7}$$

The eigenfunctions corresponding to the same eigenvalue are related to each other by a *transplantation.* That is, the eigenfunction in a subgraph of one graph can be expressed as a linear combination of the eigenfunction in several subgraphs of the other graph. The transplantation matrix is independent of the considered eigenvalue. The transplantation of the dihedral pair is described in figure 2. We exploit the fact that the eigenfunctions are defined up to a multiplication by a scalar to write the transplantation matrix as:

$$T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad \text{so that } \boldsymbol{\phi}^{II} = T\boldsymbol{\phi}^I. \tag{8}$$

The eigenvectors $\boldsymbol{\phi}^{I,II}$ point at the directions $\theta^{I,II}$, with $\tan(\theta^{I,II}) = {\phi_2}^{I,II}/{\phi_1}^{I,II}$. The transplantation (8) implies that $\boldsymbol{\phi}^{II}$ is obtained from $\boldsymbol{\phi}^{\,I}$ by a rotation of $\pi/4$ counterclockwise. Direct substitutions shows that

$$\tan[\theta^I(k_n)] = g(a,b,c;k_n), \tag{9}$$

where,

$$g(a,b,c;k) = \cos(2ka) + \sin(2ka)\left[\frac{\cos(2kb)-1}{\sin(2kb)} + \frac{\cos(2kc)-1}{\sin(2kc)}\right]. \tag{10}$$

The explicit form of the functions $f(a,b,c;k)$ and $g(a,b,c;k)$ will be used in the discussion of nodal counting in the next section.

3 Nodal counts and the resolution of isospectrality

Given a pair of isospectral domains. Are the sequences of nodal counts $\{\nu_n\}_{n=1}^{\infty}$ identical? In other words, can one use the information stored in the nodal sequences to resolve isospectrality? This question was recently discussed in the context of isospectral flat tori in $d \geq 4$ [19], and

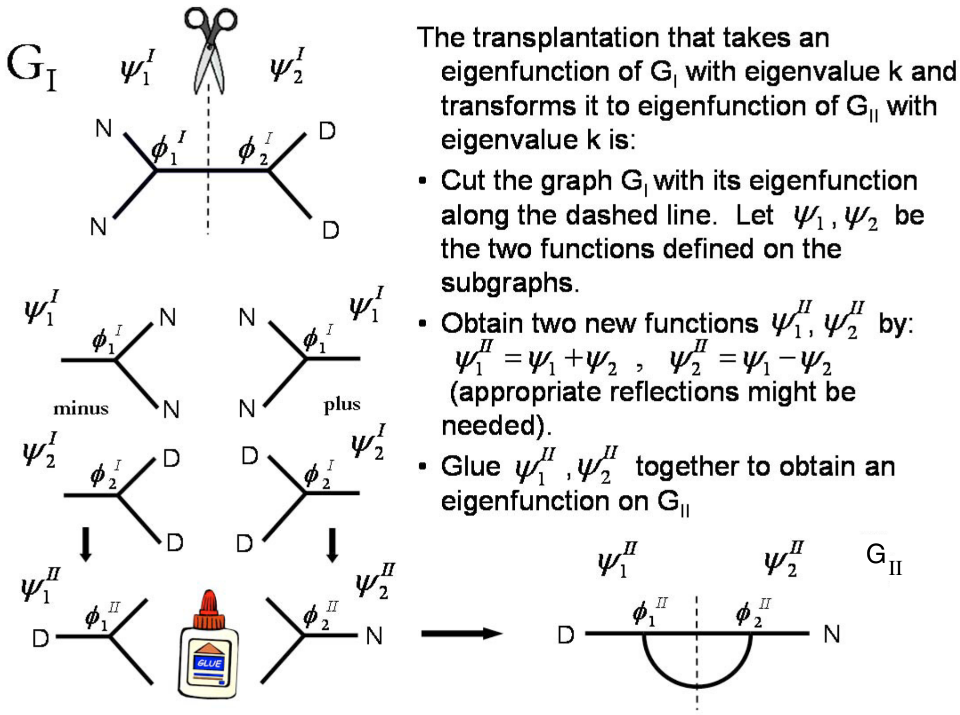

Fig. 2. The transplantation of the dihedral graphs.

numerical as well as analytical evidence was brought to substantiate the conjecture that the nodal sequences resolve isospectrality. In this section we show that the same is true for isospectral quantum graphs of the types discussed above. We shall start by discussing the *discrete* nodal sequences and then proceed to the *metric* nodal sequences.

3.1 The discrete nodal sequences

The *discrete* nodal domains were defined as the maximally connected sets of interior vertices with vertex wave functions of equal sign. Their number is denoted by μ_n, and the nodal sequence is $\{\mu_n\}_{n=1}^{\infty}$. We shall prove here that for the dihedral graphs half of the entries in the sequences of discrete nodal counts are different.

The number of discrete nodal domains for this pair of dihedral graphs (see figure 1) can be either one or two, depending on whether the signs of the components of the (two dimensional) vertex eigenfunctions have the same sign or not. In other words (see (9)), the number of nodal domains depend on the quadrant in the (ϕ_1, ϕ_2) plane where the eigenvectors point: $\mu = 1$ in the first and the third quadrants, and $\mu = 2$ in the second and the fourth quadrants:

$$\mu_n = 1 + \frac{1}{2}\left[1 - \mathrm{sign}(\tan\theta_n)\right]. \tag{11}$$

The transplantation implies that the eigenvectors $\boldsymbol{\phi}_n^{II}$ are obtained by rotating $\boldsymbol{\phi}_n^{I}$ by $\pi/4$ counterclockwise. Therefore, $\mu_n^I \neq \mu_n^{II}$ if the transplantation rotates the vectors across the quadrant borders. In other words,

$$\mu_n^I \neq \mu_n^{II} \Longleftrightarrow \tan(\theta_n) \in \{(-1, 0) \cup (1, \infty)\} \tag{12}$$

This observation is essential to our discussion since it expresses the problem of nodal counting in geometrical terms.

It is convenient to construct finite subsequences $\{\mu_n^I\}$ and $\{\mu_n^{II}\}$ of discrete nodal count of graphs I and II, restricted to the spectral points in the interval $0 \le k_n \le K$. We denote the number of terms by $N(K)$. Define

$$P(K) = \frac{1}{N(K)} \, \#\{ \, n \le N(K) : \; \mu_n^I \ne \mu_n^{II} \} \, . \tag{13}$$

We shall now prove

Theorem 1 *Consider the dihedral graphs I, II discussed above with rationally independent bond lengths a, b, c. Then*

$$\lim_{K\to\infty} P(K) = \frac{1}{2}. \tag{14}$$

Proof of theorem 1: The rational independence of a, b, c implies that the eigenfunctions never vanish on the inner vertices and hence that the spectrum is simple.

In order to study $P(K)$ above, we consider the distribution of the directions of the eigenfunctions $\boldsymbol{\phi}_n^I$ in the spectral interval. Using (7,9, 10) we get

$$h(x;K) = \langle \delta(x - \tan\theta_n)\rangle_K = \frac{1}{N(K)} \int_0^K \mathrm{d}k \; \delta(f(a,b,c;k)) \left| \frac{\mathrm{d}f}{\mathrm{d}k} \right| \delta[x - g(a,b,c;k)] \tag{15}$$

In taking the limit $K \to \infty$ we use $N(K) = \frac{2a+2b+2c}{\pi} K$ [18]. Moreover, since a, b, c are assumed to be rationally independent, k creates an ergodic flow on the 3-torus T_3 spanned by $r = 2ka \mod 2\pi$, $s = 2kb \mod 2\pi$, $t = 2kc \mod 2\pi$ [25]. Ergodicity implies that the integral over k in (15) may be replaced by an integral over T_3 leading to

$$\begin{aligned}
h(x) &= \frac{\pi}{2(a+b+c)} \frac{1}{\pi^3} \int_0^{2\pi} dr \int_0^{2\pi} ds \int_0^{2\pi} dt \; \delta(f(r,s,t)) \left| \frac{df}{dk} \right| \delta[x - g(r,s,t)] \\
&= \frac{1}{2(a+b+c)\pi^2} \int_0^{2\pi} ds \int_0^{2\pi} dt \left\{ \int_0^{\pi} dr \delta(f(r,s,t)) \left| \frac{df}{dk} \right| \delta[x - g(r,s,t)] \right. \\
&\quad + \left. \int_{\pi}^{2\pi} dr \delta(f(r,s,t)) \left| \frac{df}{dk} \right| \delta[x - g(r,s,t)] \right\} \\
&= \frac{1}{2(a+b+c)\pi^2} \int_0^{2\pi} ds \int_0^{2\pi} dt \, [I_1(s,t;x) + I_2(s,t;x)]
\end{aligned} \tag{16}$$

Now we note that under the transformation $r \mapsto r' = (r+\pi) \mod 2\pi$ we have

$$f(r',s,t) = -f(r,s,t) \qquad \frac{\mathrm{d}f}{\mathrm{d}k}(r',s,t) = -\frac{\mathrm{d}f}{\mathrm{d}k}(r,s,t) \qquad g(r',s,t) = -g(r,s,t)$$

Thus, we conclude that

$$\begin{aligned}
I_1(s,t;x) = I_2(s,t;-x) \; &\Rightarrow h(x) = h(-x) \\
&\Rightarrow \int_{-\infty}^{-1} h(x)dx + \int_0^1 h(x)dx = \int_{-1}^0 h(x)dx + \int_1^{\infty} h(x)dx = \frac{1}{2} \\
&\Rightarrow \lim_{K\Rightarrow\infty} P(K) = \frac{1}{2} \qquad \square
\end{aligned}$$

3.2 The metric nodal sequences

Consider the pair of dihedral graphs, and let $\{\nu_n^{I,II}\}$ denote the metric nodal counts of graphs I and and II. Using (5) we express the difference $\nu_n^I - \nu_n^{II}$ as

$$\delta\nu_n = \nu_n^I - \nu_n^{II} = \frac{1}{2} \left\{ 1 - \mathrm{sign}[\phi_{2,n}^I \sin(2k_n a)] + \mathrm{sign}[\phi_{2,n}^{II} \sin(2k_n b)] + \mathrm{sign}[\phi_{2,n}^{II} \sin(2k_n c)] \right\} \tag{17}$$

To obtain (17) a slight modification of (5) was needed: the term in the sum that corresponds to a bond with boundary vertex that has Dirichlet boundary conditions should be modified to be $\lfloor \frac{k_n L_b}{\pi} \rfloor$ instead of $\lfloor \frac{k_n L_b}{\pi} \rfloor + \frac{1}{2}(1-(-1)^{\lfloor \frac{k_n L_b}{\pi} \rfloor}\text{sign}[\phi_i]\text{sign}[\phi_j])$. In addition while deriving (17) we made use of the identity $\lfloor \frac{2x}{\pi} \rfloor - 2\lfloor \frac{x}{\pi} \rfloor = \frac{1}{2}\{1 - \text{sign}[\sin(2x)]\}$ and of the freedom to choose the sign of the first component of the wave-functions to be positive. A theorem by Berkolaiko [23] guarantees that $\delta\nu_n$ can only take the values 0 and 1. We would like to compute the distribution of these values on the spectrum. For this purpose we consider the spectral interval $0 < k_n < K$ and study the function

$$Q(x;K) = \frac{1}{N(K)} \#\{n \leq N(k) : \delta\nu_n = x\} \tag{18}$$

and prove the following theorem.

Theorem 2 *Consider the dihedral graphs I, II discussed above with rationally independent bond lengths a, b, c. Then*

$$\exists \lim_{K\to\infty} Q(x,K) = Q(x) \qquad \text{and} \qquad Q(0) = Q(1) = \frac{1}{2}. \tag{19}$$

Proof of theorem 2: In order to study the $Q(x,K)$ above we consider the distribution of $\delta\nu_n$, which is given by the integral

$$h(x;K) = \langle \delta(x - \delta\nu_n)\rangle_K = \frac{1}{N(K)} \int_0^K \mathrm{d}k \delta(f(a,b,c;k)) \left|\frac{\mathrm{d}f}{\mathrm{d}k}\right| \delta[x - \delta\nu(a,b,c;k)], \tag{20}$$

where the spectral secular function $f(a,b,c;k)$ is defined in (7) above, and $\delta\nu(a,b,c;k)$ coincides with $\delta\nu_n$ for $k = k_n$ and can be written explicitly as

$$\begin{aligned}\delta\nu(a,b,c;k) = \frac{1}{2}\Bigg\{1 &- \text{sign}\left[\cos(2ka)\sin(2ka) + \sin^2(2ka)\left(\cot(2kb) + \cot(2kc) - \frac{1}{\sin(2kb)} - \frac{1}{\sin(2kc)}\right)\right] \\ &+ \text{sign}\left[\frac{\sin^2(2kb)\sin(2kc)}{\sin(2kb)+\sin(2kc)}\left(\frac{1}{\sin(2ka)} + \cot(2ka) + \cot(2kb) + \cot(2kc)\right)\right] \\ &+ \text{sign}\left[\frac{\sin(2kb)\sin^2(2kc)}{\sin(2kb)+\sin(2kc)}\left(\frac{1}{\sin(2ka)} + \cot(2ka) + \cot(2kb) + \cot(2kc)\right)\right]\Bigg\}. \end{aligned} \tag{21}$$

Following the same route as in the proof of theorem (1) we take the limit $K \to \infty$ while making use of the ergodic theorem, and replace the k integration by an integration over the 3 - torus with coordinates $r = 2ka \bmod 2\pi$, $s = 2kb \bmod 2\pi$, and $t = 2kc \bmod 2\pi$.

$$\begin{aligned} h(x) &= \frac{\pi}{2a+2b+2c}\frac{1}{\pi^3} \int_0^{2\pi} \mathrm{d}r \int_0^{2\pi} \mathrm{d}s \int_0^{2\pi} \mathrm{d}t \delta(f(r,s,t)) \left|\frac{\mathrm{d}f}{\mathrm{d}k}\right| \delta[x - \delta\nu(r,s,t)] \\ &= Q(-1)\delta(x-(-1)) + Q(0)\delta(x) + Q(1)\delta(x-1) + Q(2)\delta(x-2) \end{aligned} \tag{22}$$

The last line follows from the fact that x is an integer which is written as half the sum of four unimodular numbers. Next we note that under the transformation $r \mapsto r' = (-r) \bmod 2\pi$, $s \mapsto s' = (-s)\bmod 2\pi, t \mapsto t' = (-t)\bmod 2\pi$ we have

$$f(r',s',t') = -f(r,s,t) \qquad \frac{\mathrm{d}f}{\mathrm{d}k}(r',s',t') = \frac{\mathrm{d}f}{\mathrm{d}k}(r,s,t) \qquad \delta\nu(r',s',t') = 1 - \delta\nu(r,s,t)$$

Thus, we conclude that $Q(-1) = Q(2) \wedge Q(0) = Q(1)$. But due to Berkolaiko's theorem [23] $Q(-1) = Q(2) = 0$. Therefore,

$$Q(0) = Q(1) = \frac{1}{2} \qquad \square$$

4 Discussion and summary

There is now a growing amount of evidence that nodal-count sequences store information on the geometry of the system under study, which is similar but not equivalent to the information stored in the spectral sequence. In a recent paper [26] it was shown that the nodal count sequence for separable Laplace - Beltrami operators can be expressed in terms of a trace formula which consists of a smooth (Weyl - like) part, and an oscillatory part. The smooth part depends on constants which can be derived from the geometry of the domain, and the oscillatory part depends on the classical periodic orbits, much in the same way as the spectral trace formula. However, what we have shown in the present paper is that the information stored in the two sequences is not identical: For the isospectral pairs of graphs considered here, the nodal-count sequences are different in a substantial way. In this respect, the nodal sequence resolves isospectrality.

The methods presented here together with numerical studies on other types of graphs form a manuscript which was recently submitted for publication [27]. Till this work was done, the conjecture that isospectrality is resolved by counting nodal domains was substantiated by numerical studies only. Here we presented for the first time a system where this fact is proved rigorously. The main breakthrough which enabled the proof was by formulating the counting problem in a geometrical setting. We hope that this way will pave the way to further analytical studies, where more complex systems will be dealt with.

Finally, we would like to mention a set of open problems which naturally arise in the present context: Can one find metrically different domains where the Laplacians have different spectra but the nodal counting sequences are the same? A positive answer is provided for domains in 1 dimension (Sturm) or for tree graphs (Schapotschnikow). Are there other less trivial examples?

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