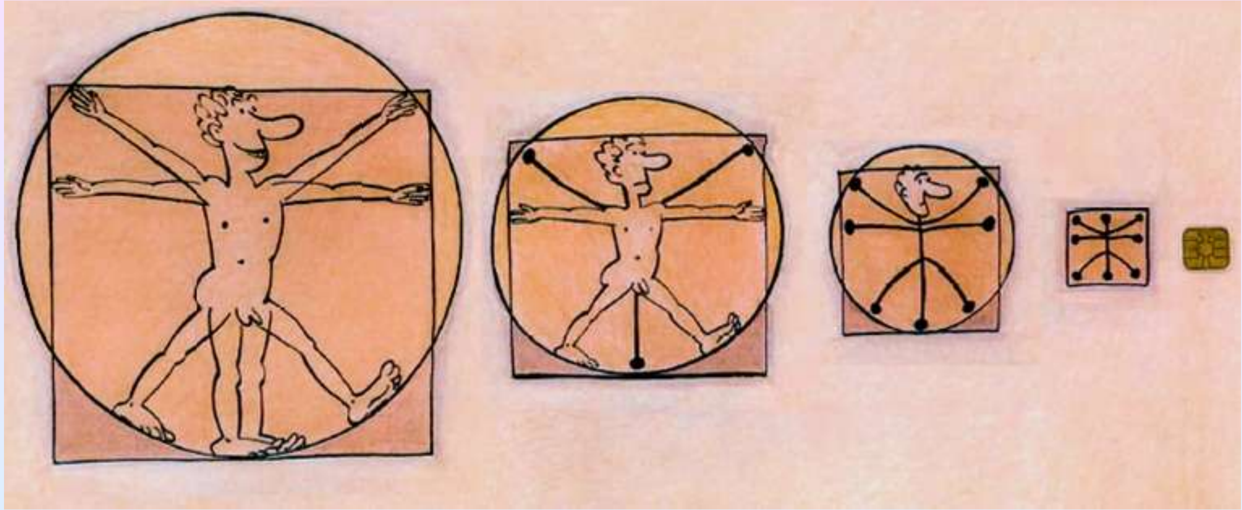
What one cannot hear?
Quantum graphs which sound the same

Rami Band, Ori Parzanchevski, Gilad Ben-Shach



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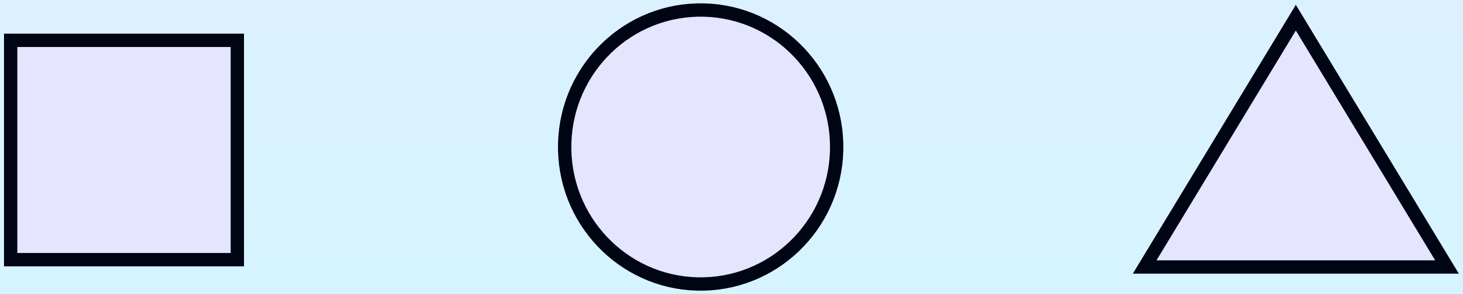
‘Can one hear the shape of a drum ?’

- This question was asked by Marc Kac (1966).

Marc Kac (1914-1984)

- Is it possible to have two different drums with the same spectrum (***isospectral drums***) ?

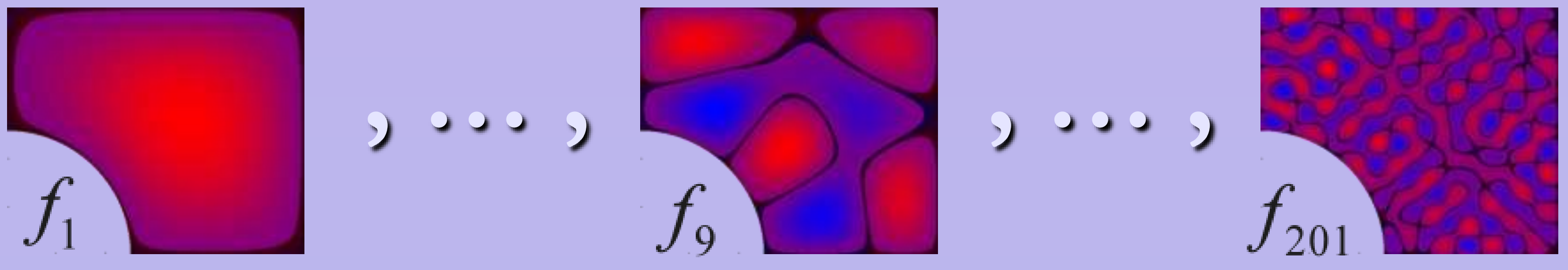


The spectrum of a drum

- A **Drum** is an elastic membrane which is attached to a solid planar frame.
- The spectrum is the set of the Laplacian's eigenvalues, $\{\lambda_n\}_{n=1}^{\infty}$, (usually with Dirichlet boundary conditions):

$$-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)f=\lambda f \qquad f\big|_{boundary}=0$$

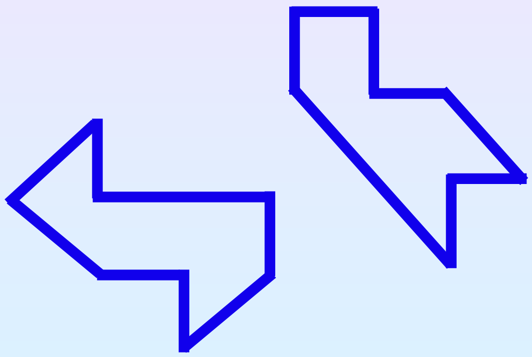
- A few eigenfunctions of the Sinai 'drum':



Isospectral drums

Gordon, Webb and Wolpert (1992):

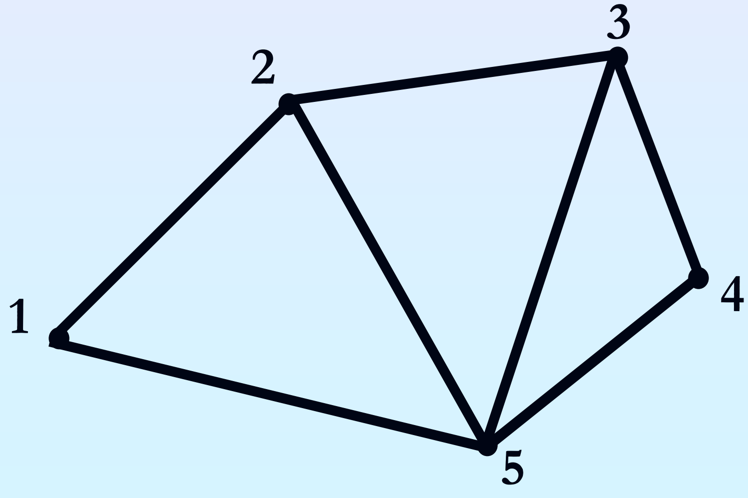
‘One cannot hear the shape of a drum’

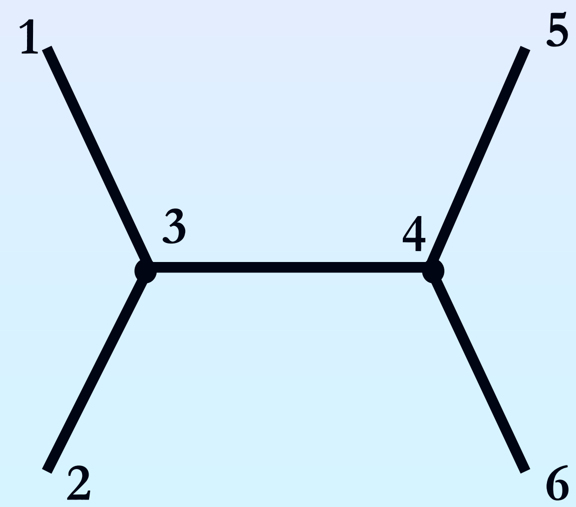


Using Sunada’s construction (1985)

‘Can one hear the shape of [illegible]?’

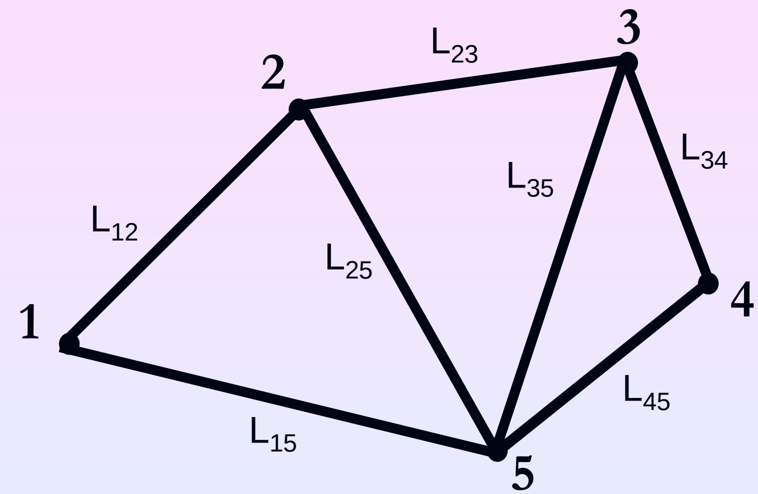
- How do we produce isospectral examples?
- What geometrical \ topological properties we can hear ?





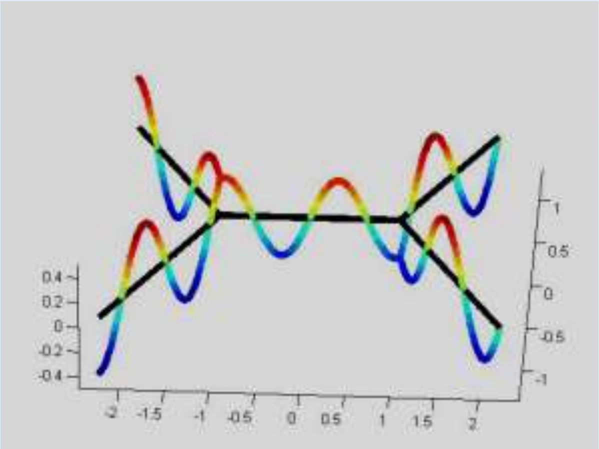
Metric Graphs - Introduction

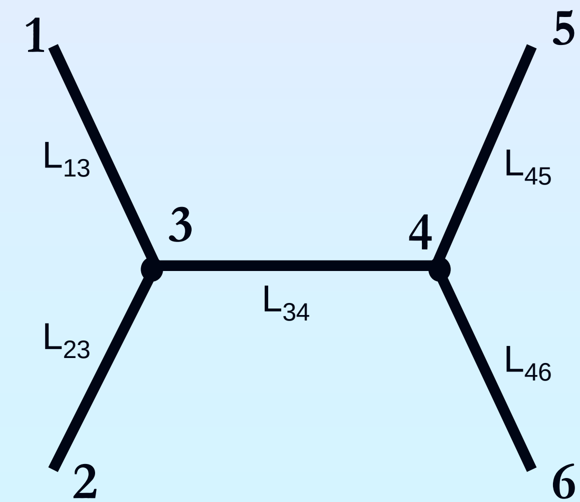
- A *graph* Γ consists of a finite set of vertices $V=\{v_i\}$ and a finite set of edges $E=\{e_j\}$.
- A *metric graph* has a finite length ($L_e>0$) assigned to each edge.



- A function on the graph is a vector of functions on the edges:

$$f=(f_{e_1},\dots,f_{e_{|E|}}) \qquad f_{e_j}:[0,L_{e_j}]\to \text{☼}$$

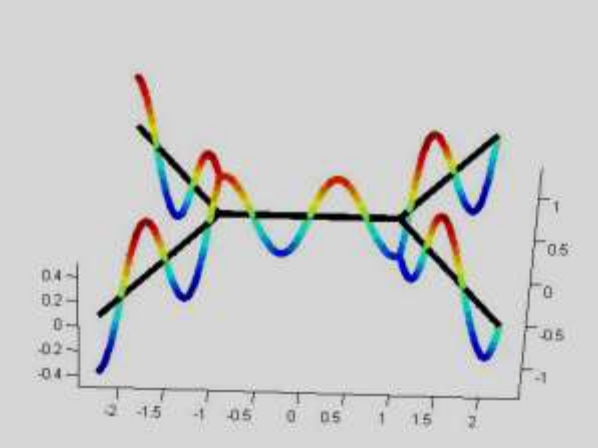




Quantum Graphs - Introduction

- A *quantum graph* is a metric graph equipped with an operator, such as the negative Laplacian: $$-\Delta f = (-f''|_{e_1}, \dots, -f''|_{e_{|E|}})$$
- For each vertex v, we impose vertex conditions, such as
  - *Neumann*
    - Continuity $\forall e_1, e_2 \in E_v \quad f|_{e_1}(v) = f|_{e_2}(v)$
    - Zero sum of derivatives $\sum_{e \in E_v} f'|_e(v) = 0$
  - *Dirichlet*
    - Zero value at the vertex $\forall e \in E_v \quad f|_e(v) = 0$

- A quantum graph is defined by specifying:
  - Metric graph
  - Operator
  - Vertex conditions for each vertex

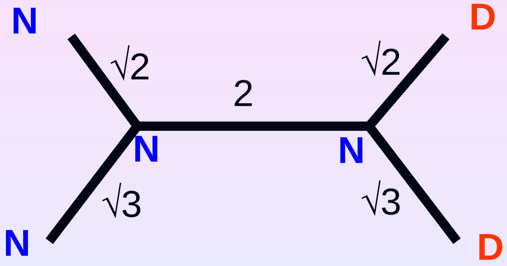


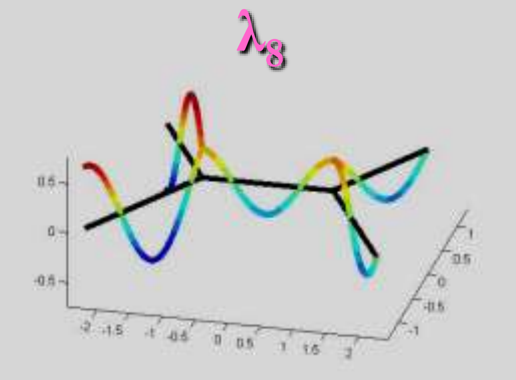
The Spectrum of Quantum Graphs

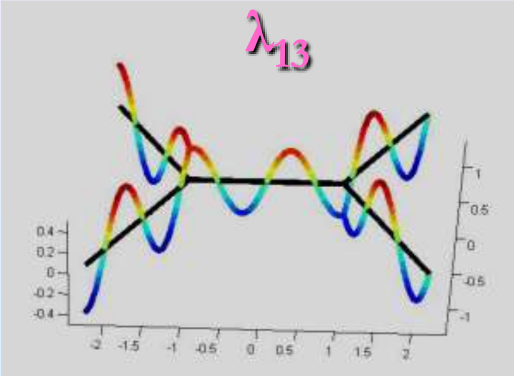
We are interested in the *eigenvalues* of the Laplacian:

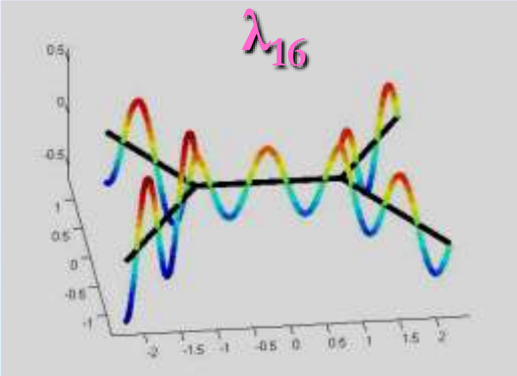
$$-\Delta f = \lambda f \quad \Rightarrow \quad (-f''|_{e_1},\ldots,-f''|_{e_{|E|}}) = (\lambda f|_{e_1},\ldots,\lambda f|_{e_{|E|}})$$

Examples of several eigenfunctions of the Laplacian on the graph:







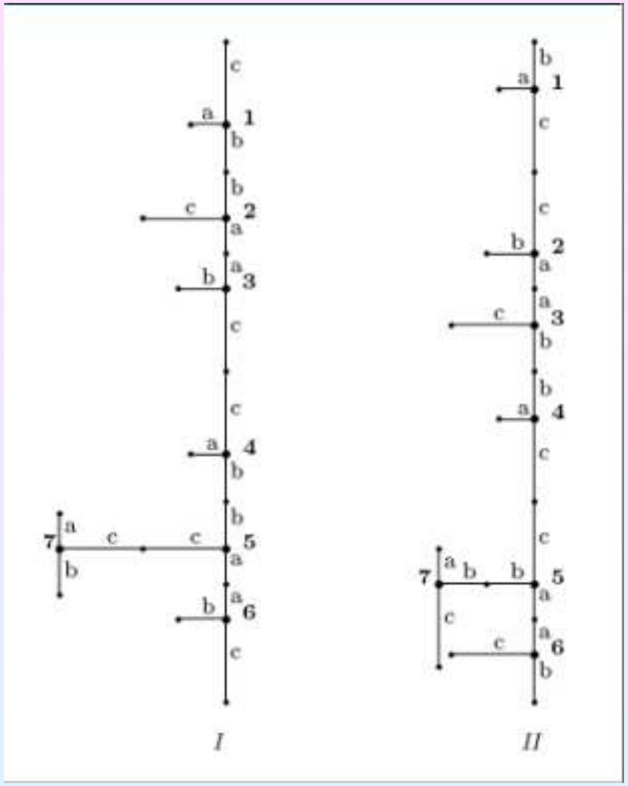


So…

‘Can one hear the shape of a graph?’

‘Can one hear the shape of a graph ?’

- One can hear the shape of a simple graph if the lengths are incommensurate (Gutkin, Smilansky 2001)



- Otherwise, we do have isospectral graphs:
  - Roth (1984)
  - VonBelow (2001)
  - Band, Shapira, Smilansky (2006)
  - Kurasov, enerback (2010)

- There are several methods for construction of isospectrality – the main is due to Sunada (1985).

- We present a method based on representation theory arguments which generalizes Sunada’s method.

Isospectral theorem

Theorem (R.B., Ori Parzanchevski, Gilad Ben-Shach)

Let Γ be a graph which obeys a symmetry group G.
Let H_1, H_2 be two subgroups of G with representations R_1, R_2 that satisfy $\mathrm{Ind}_{H_1}^{G} R_1 \cong \mathrm{Ind}_{H_2}^{G} R_2$

then the graphs Γ/R_1, Γ/R_2 are isospectral.

Constructing Quotient Graphs

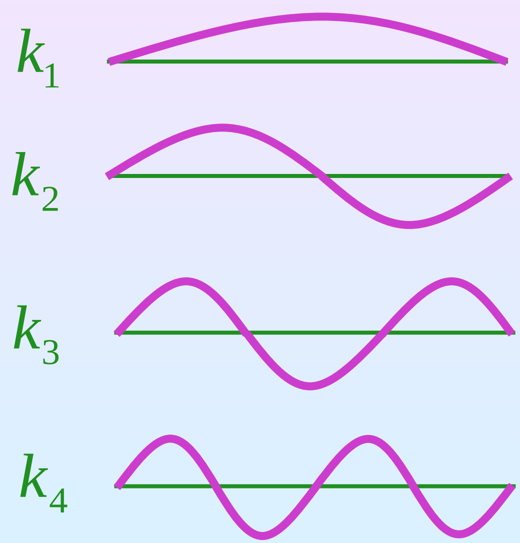
- Example - A string with Dirichlet vertex conditions.

$$-\Delta f = k^2 f$$

- It obeys the symmetry group $Z_2 = \{id, r\}$.
- Two representations of Z_2 are:

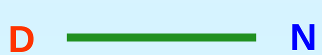
$R_1 : \{id \rightarrow (1), r \rightarrow (1)\}$

$R_2 : \{id \rightarrow (1), r \rightarrow (-1)\}$



We may encode these functions by the following *quotient graphs:*

Γ / R_1



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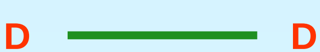
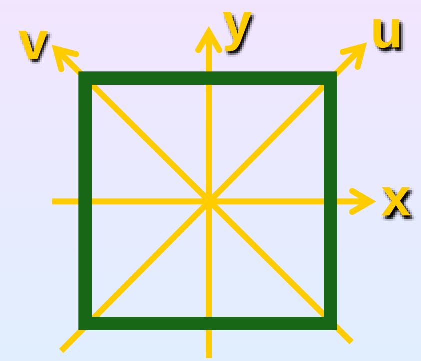


Γ / R_2

Groups & Graphs

- Example: The Dihedral group –
  the symmetry group of the square
  $G = \{ id , a , a^2 , a^3 , r_x , r_y , r_u , r_v \}$

How does the Dihedral group act on a square ?



- Two subgroups of the Dihedral group:
  $H_1 = \{ id , a^2 , r_x , r_y \}$
  $H_2 = \{ id , a^2 , r_u , r_v \}$

Groups - Representations

- Representation – Given a group G, a representation R is an assignment of a matrix $\rho_R(g)$ to each group element g ♏ G, such that: ✂ g_1, g_2 ♏ G $\rho_R(g_1) \cdot \rho_R(g_2) = \rho_R(g_1 g_2)$.

- Example 1 - G has the following 1-dimensional representation

$$\mathrm{id} \to (1) \quad a \to (-1) \quad a^2 \to (1) \quad a^3 \to (-1) \quad r_x \to (-1) \quad r_y \to (-1) \quad r_u \to (1) \quad r_v \to (1)$$

- Example 2 - G has the following 2-dimensional representation

$$\mathrm{id} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad a \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad a^2 \to \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad a^3 \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad r_x \to \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad r_y \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad r_u \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad r_v \to \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

- Induction: take a representation of H_1...

$$\mathrm{id} \to (1) \quad a^2 \to (1) \quad r_x \to (-1) \quad r_y \to (-1)$$

...And turn it into a representation of G (which we denote $\mathrm{Ind}_{H_1}^{G} R$)

$$\mathrm{id} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad a \to \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad a^2 \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad a^3 \to \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad r_x \to \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad r_y \to \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad r_u \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad r_v \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Isospectral theorem

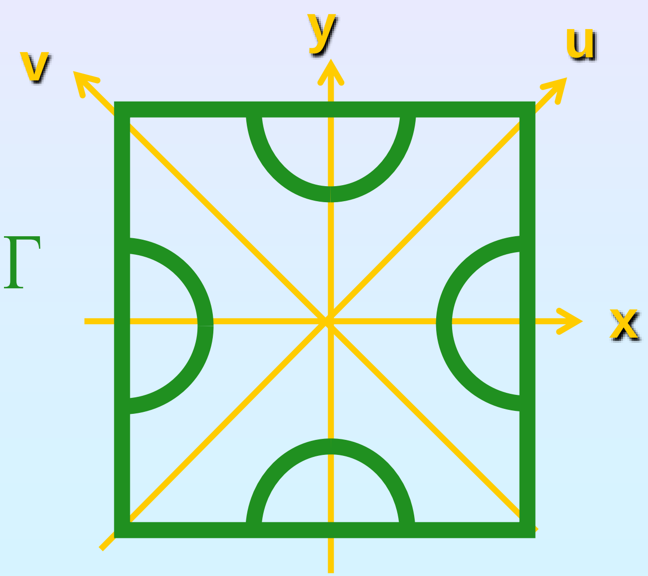
<u>Theorem</u> (R.B., Ori Parzanchevski, Gilad Ben-Shach)

Let Γ be a graph which obeys a symmetry group G.
Let H_1, H_2 be two subgroups of G with representations R_1, R_2
that satisfy $\mathrm{Ind}_{H_1}^{G} R_1 \cong \mathrm{Ind}_{H_2}^{G} R_2$

then the graphs Γ/R_1 , Γ/R_2 are isospectral.

- An application of the theorem with: $G = \{\mathrm{id}, a, a^2, a^3, r_x, r_y, r_u, r_v\}$

Γ



Two subgroups of G: $H_1 = \{\mathrm{id}, \ a^2, r_x, r_y\}$

$H_2 = \{\mathrm{id}, \ a^2, r_u, r_v\}$

We choose representations
R_1 of H_1 and R_2 of H_2

$R_1 : \{\mathrm{id} \to (1), \ a^2 \to (-1), \ r_x \to (-1), \ r_y \to (1)\}$

$R_2 : \{\mathrm{id} \to (1), \ a^2 \to (-1), \ a_u \to (1), \ a_v \to (-1)\}$

such that $\mathrm{Ind}_{H_1}^{G} R_1 \cong \mathrm{Ind}_{H_2}^{G} R_2$

Constructing Quotient Graphs

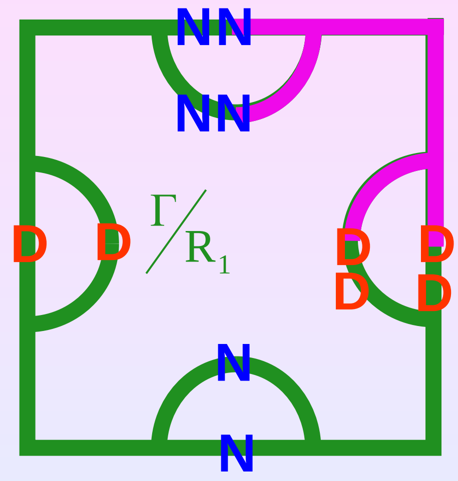
- Consider the following rep. R_1 of the subgroup H_1:

$$R_1 : \left\{ \mathrm{id} \to (1) \quad a^2 \to (-1) \quad r_x \to (-1) \quad r_y \to (1) \right\}$$

We construct Γ/R_1 by inquiring what do we know about a function f on Γ which transforms according to R_1.

$$r_x f = -f \qquad r_y f = f$$

Dirichlet **Neumann**



The construction of a *quotient graph* is motivated by an *encoding scheme*.

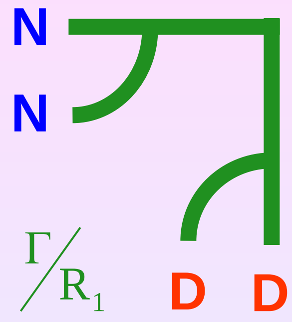
Constructing Quotient Graphs

- Consider the following rep. R_1 of the subgroup H_1:

$$R_1 : \{ \mathrm{id} \to (1) \quad a^2 \to (\text{-}1) \quad r_x \to (\text{-}1) \quad r_y \to (1) \}$$

We construct $\Gamma/_{R_1}$ by inquiring what do we know about a function f on Γ which transforms according to R_1.

$$r_x\, f = -f \qquad\qquad r_y\, f = f$$



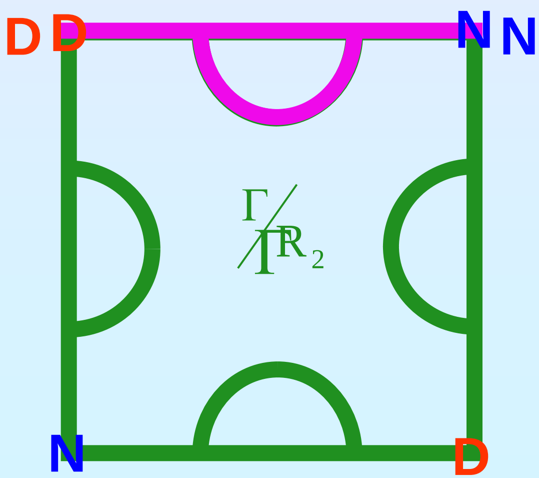
- Consider the following rep. R_2 of the subgroup H_2:

$$R_2 : \{ \mathrm{id} \to (1) \quad a^2 \to (\text{-}1) \quad r_u \to (1) \quad r_v \to (\text{-}1) \}$$

We construct $\Gamma/_{R_2}$ by inquiring what do we know about a function g on Γ which transforms according to R_2.

$$r_u\, g = g \qquad\qquad r_v\, g = -g$$

Neumann **Dirichlet**



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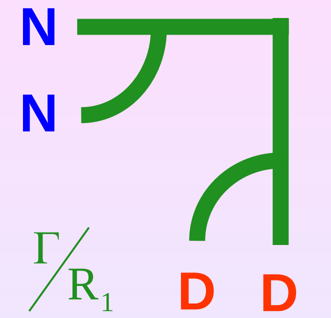
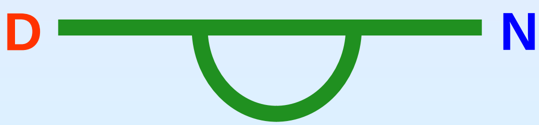


Γ/R_1

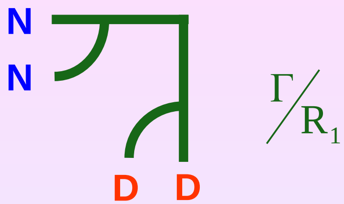


Γ/R_2

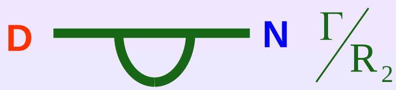
Extending the Isospectral pair

Extending our example: $\mathrm{Ind}_{H_1}^{G} R_1 \cong \mathrm{Ind}_{H_2}^{G} R_2 \cong \mathrm{Ind}_{H_3}^{G} R_3$

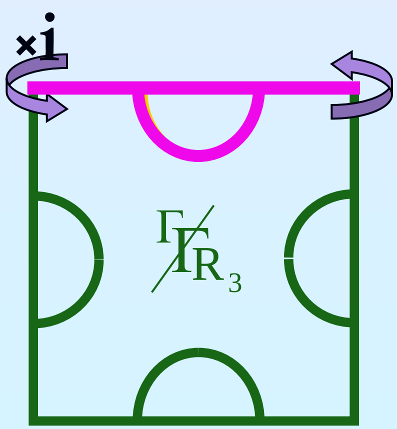
$H_1 = \{ e , a^2, r_x , r_y\}$ R_1: $e \to (1)$ $a^2 \to (-1)$ $r_x \to (-1)$ $r_y \to (1)$



$H_2 = \{ e , a^2, r_u , r_v\}$ R_2: $e \to (1)$ $a^2 \to (-1)$ $r_u \to (1)$ $r_v \to (-1)$



$H_3 = \{ e , a, a^2, a^3\}$ R_3: $e \to (1)$ $a \to (i)$ $a^2 \to (-1)$ $a^3 \to (-i)$



$$a f = i f$$

Extending the Isospectral pair

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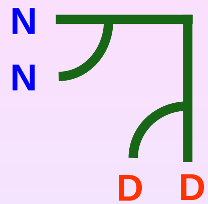

Γ / R_1

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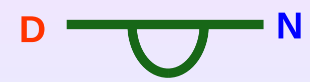

Γ / R_2

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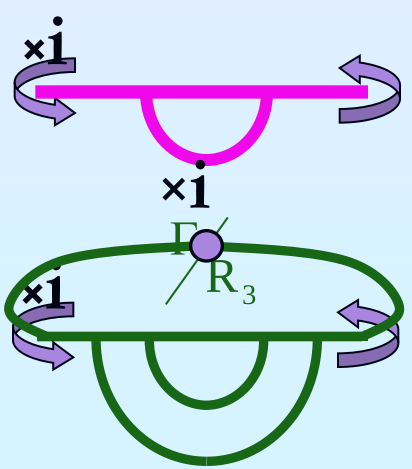

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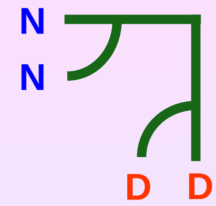

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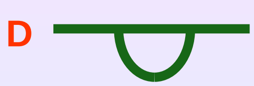

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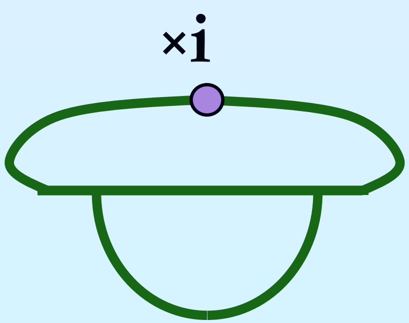
Γ / R_3



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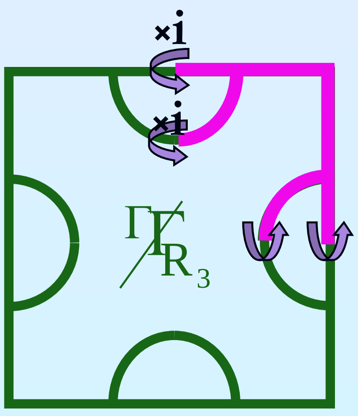

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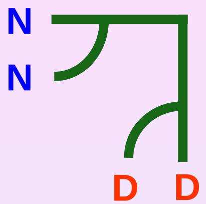

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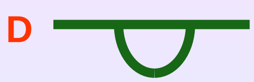

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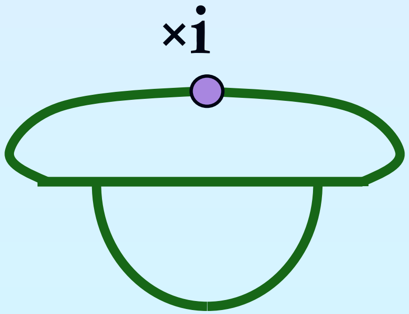
Γ / R_3



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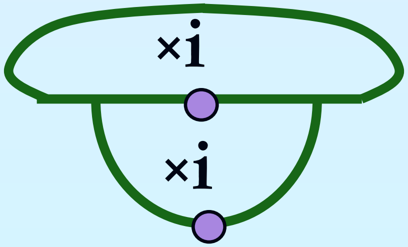
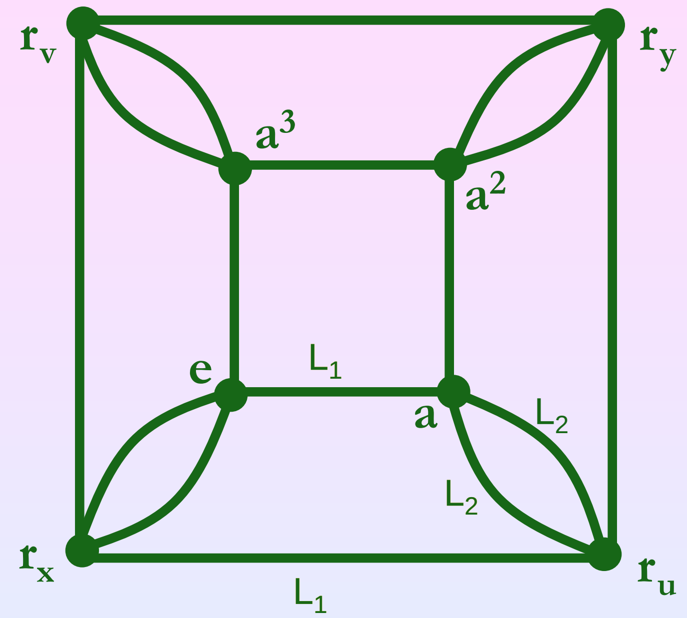
Γ / R_3



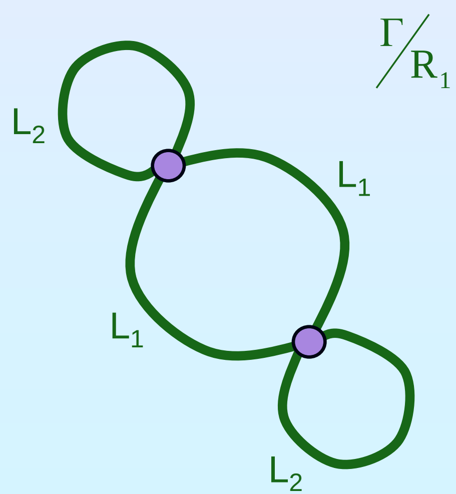
Arsenal of isospectral examples

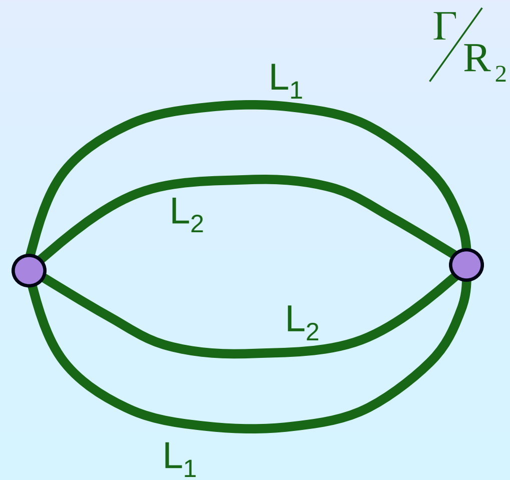
Γ is the Cayley graph of $G=D_4$
(with respect to the generators a, r_x):

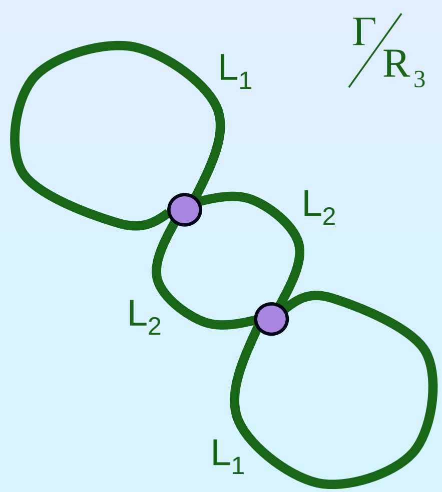


Take the same group and the subgroups:
$H_1 = \{ e , a^2, r_x , r_y\}$ with the rep. R_1
$H_2 = \{ e , a^2, r_u , r_v\}$ with the rep. R_2
$H_3 = \{ e , a , a^2 , a^3\}$ with the rep. R_3

The resulting quotient graphs are:







Arsenal of isospectral examples

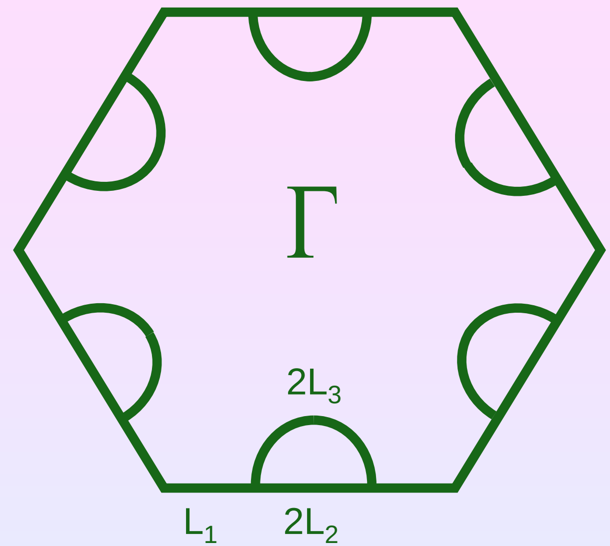
$G = D_6 = \{e, a, a^2, a^3, a^4, a^5, r_x, r_y, r_z, r_u, r_v, r_w\}$

with the subgroups:

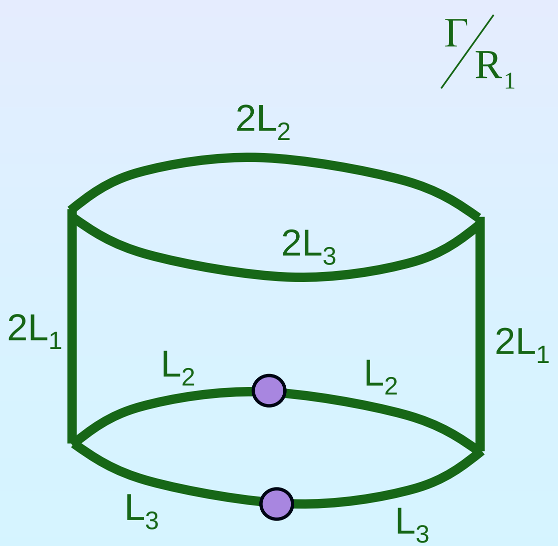
$H_1 = \{ e, a^2, a^4, r_x, r_y, r_z \}$ with the rep. R_1

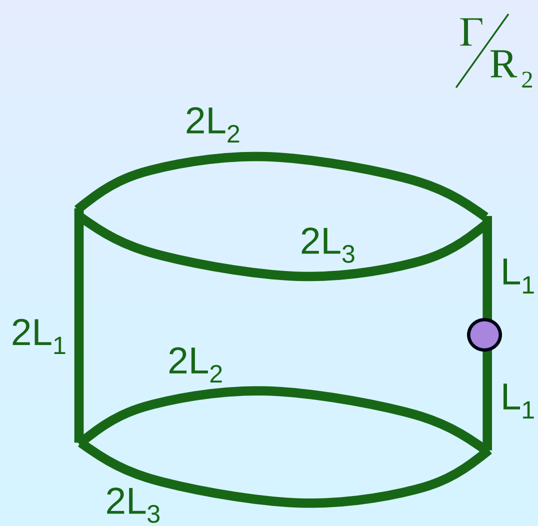
$H_2 = \{ e, a^2, a^4, r_u, r_v, r_w \}$ with the rep. R_2

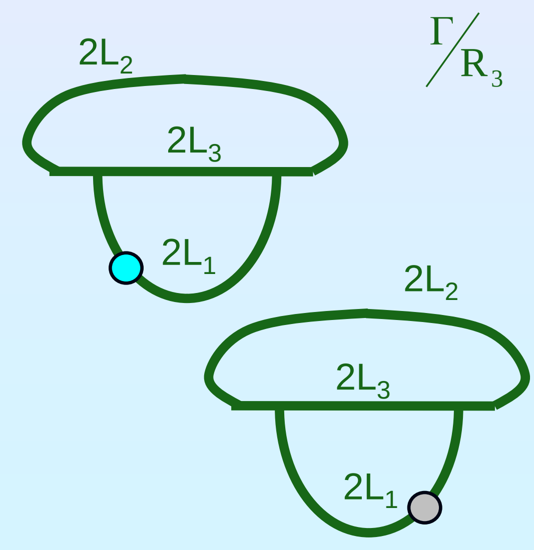
$H_3 = \{ e, a, a^2, a^3, a^4, a^5 \}$ with the rep. R_3



The resulting quotient graphs are:



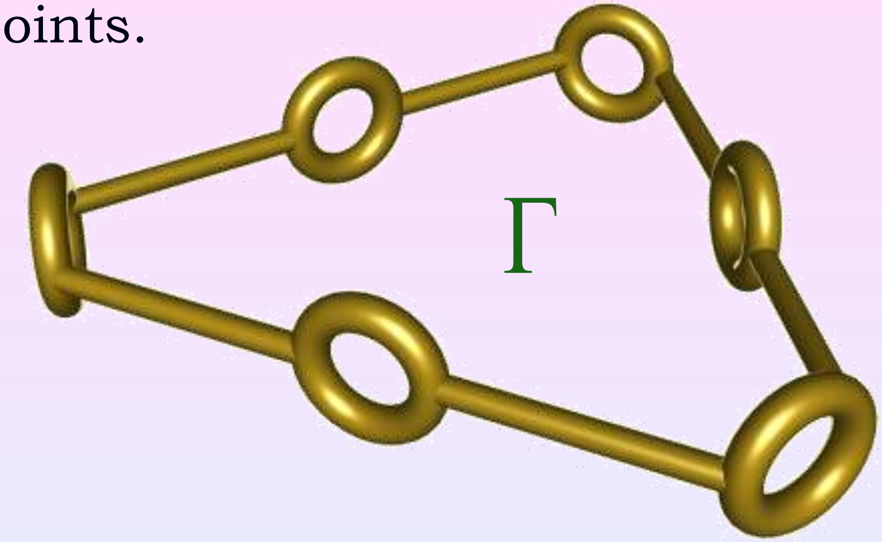




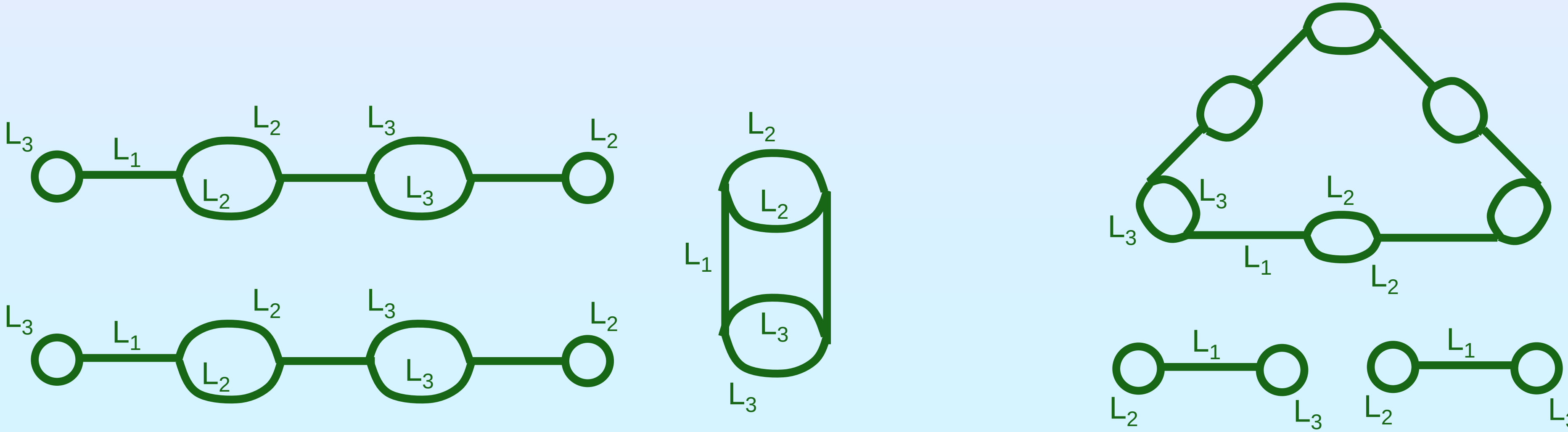
Arsenal of isospectral examples

$G = S_3$ (D_3) acts on Γ with no fixed points.

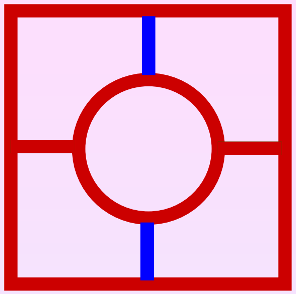
To construct the quotient graph,
we take the same rep. of G,
but use two different bases
for the matrix representation.

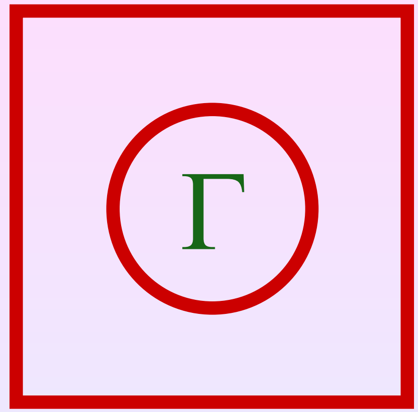


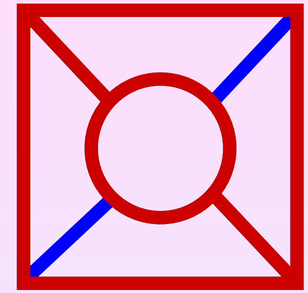
The resulting quotient graphs are:

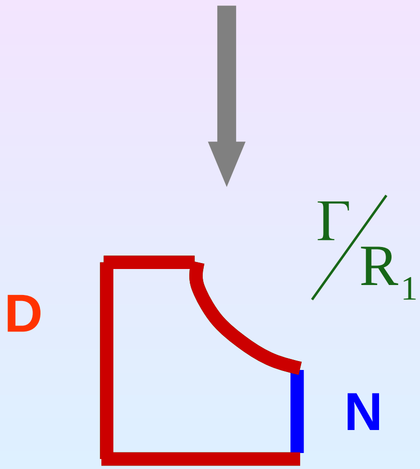


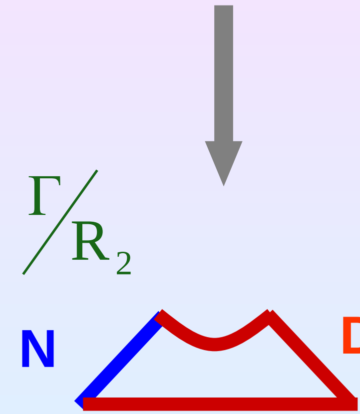
Why quantum graphs? Why not drums?





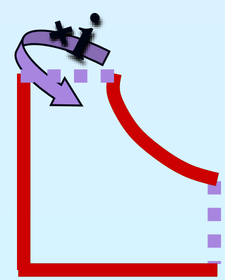






Following Martin Sieber

However, Γ/R_3 is not a planar drum:

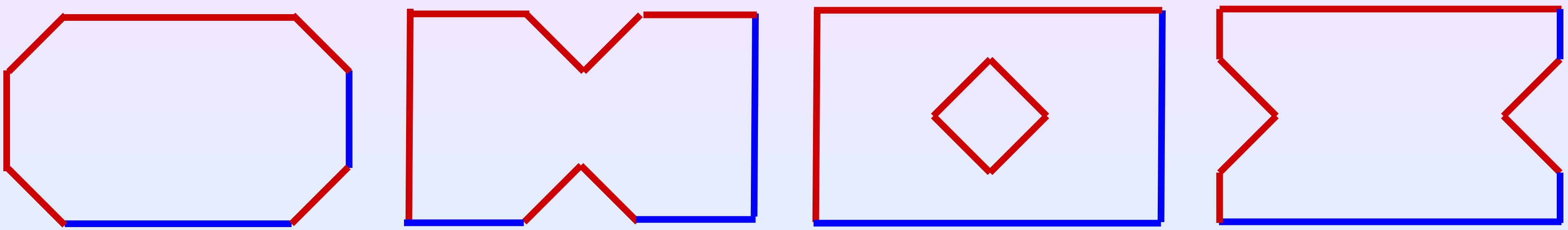


Arsenal of isospectral examples

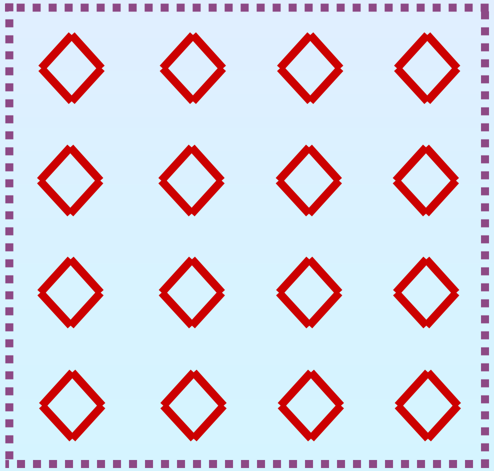
Isospectral drums

'Spectral problems with mixed Dirichlet-Neumann boundary conditions: isospectrality and beyond' D. Jacobson, M. Levitin, N. Nadirashvili, I. Polterovich (2004)

'Isospectral domains with mixed boundary conditions' M. Levitin, L. Parnovski, I. Polterovich (2005)



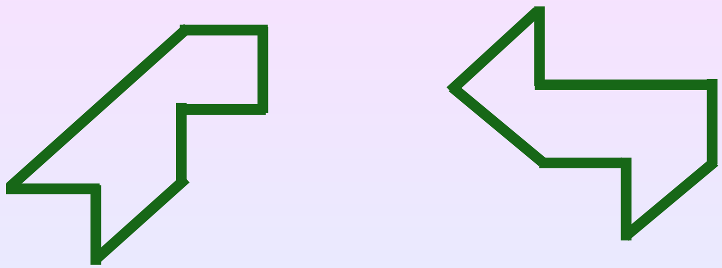
This isospectral quartet can be obtained when acting with the group $D_4 x D_4$ on the following torus:



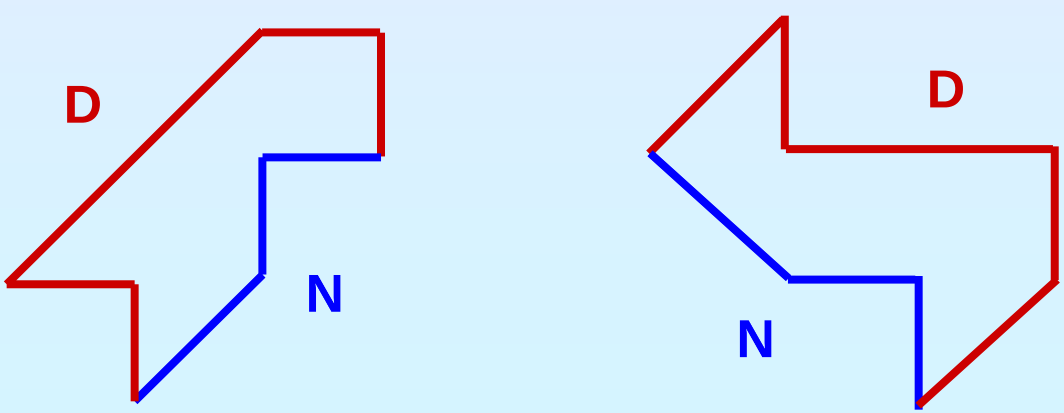
Arsenal of isospectral examples

Isospectral drums

‘One **cannot** hear the shape of a drum’
Gordon, Webb and Wolpert (1992)



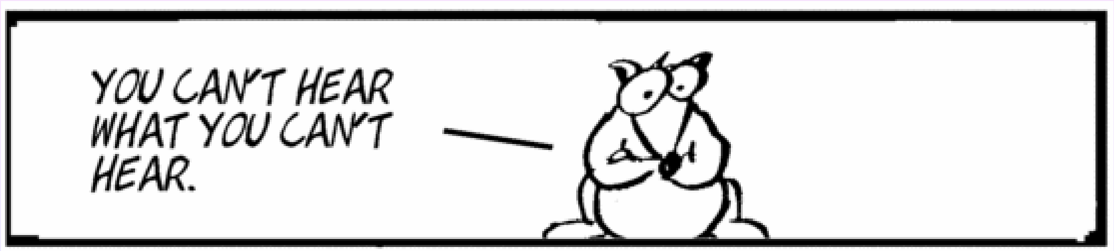
We construct the known isospectral drums of Gordon *et al.* but with new boundary conditions:



What one cannot hear?

On drums\graphs which sound the same

Rami Band, Ori Parzanchevski, Gilad Ben-Shach



R. Band, O. Parzanchevski and G. Ben-Shach,
"*The Isospectral Fruits of Representation Theory: Quantum Graphs and Drums*",
J. Phys. A (2009).

O. Parzanchevski and R. Band,
"*Linear Representations and Isospectrality with Boundary Conditions*",
Journal of Geometric Analysis (2010).

University of BRISTOL

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