Finding the nodal points on a quantum graph

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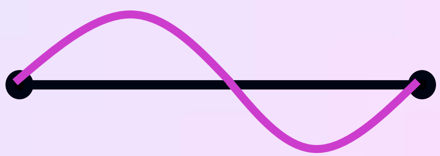
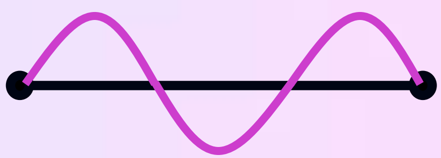
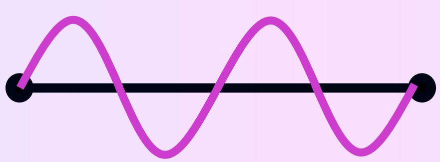
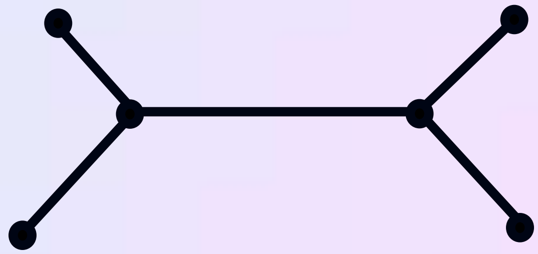
The vibration modes of a string

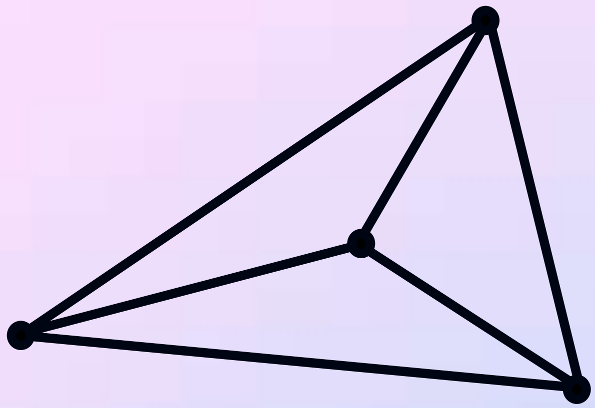
$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} f = \lambda f$$

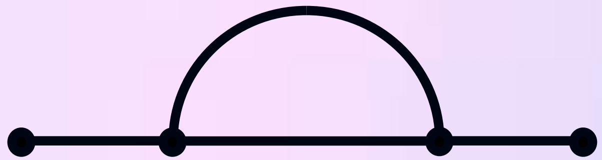
Eigenvalue	Eigenfunction	Nodal Domain Count
λ_1		1
λ_2		2
λ_3		3
λ_4		4
$\vdots$	$\vdots$	$\vdots$

The nodal domain count of graphs

- We are interested in the ***nodal count***, $\{\nu_n\}_{n=1}^{\infty}$, of a quantum graph.







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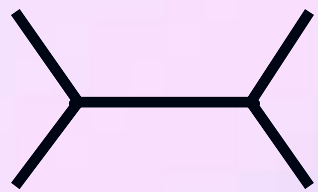
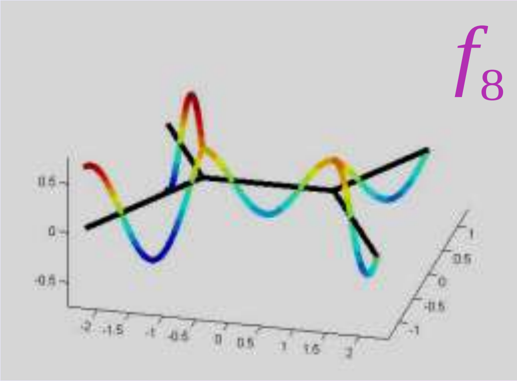
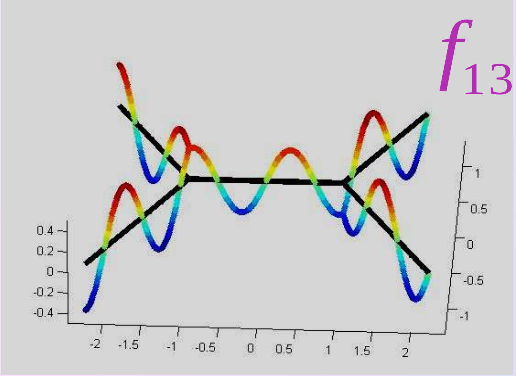
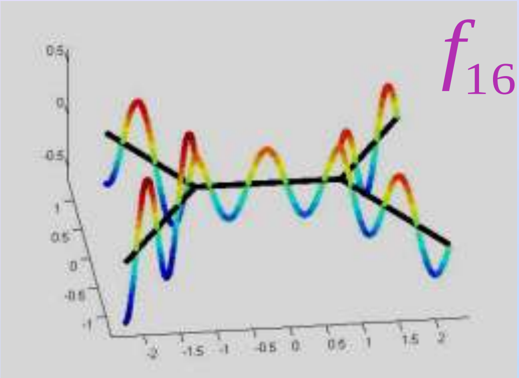
Eigenfunctions of the graph



$$-\frac{d^2}{\partial x^2} f_n = \lambda_n f_n$$







Nodal domains

$\nu_8 = 8$ $\nu_{13} = 13$ $\nu_{16} = 16$

The nodal domain count of graphs

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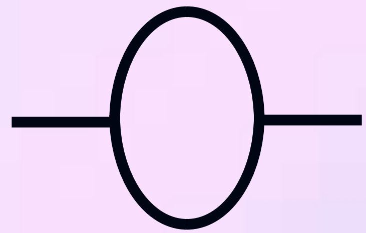
Eigenfunctions of the graph



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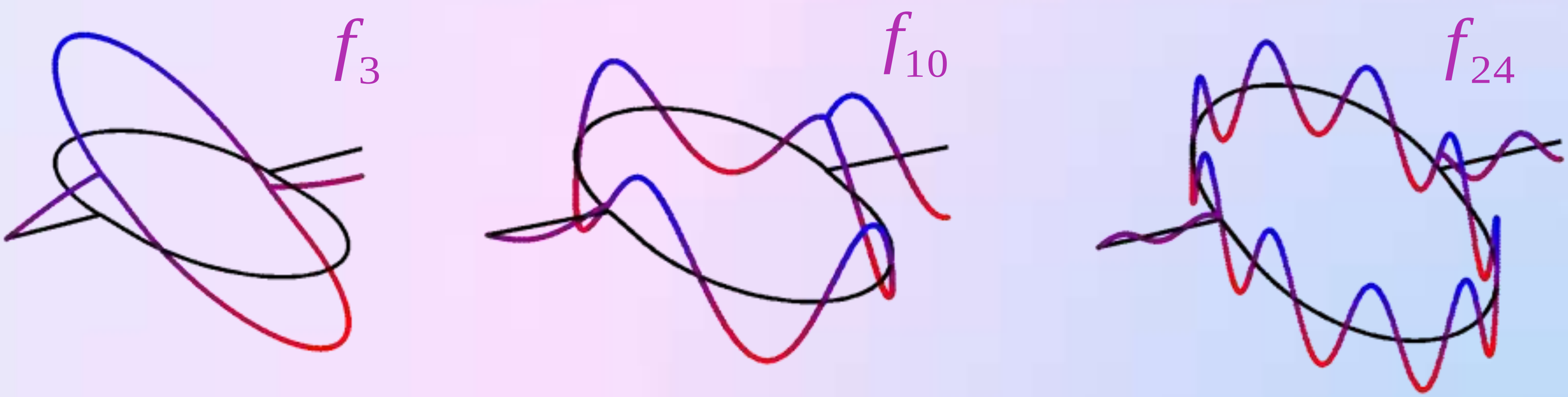
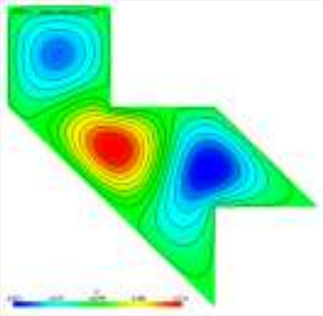


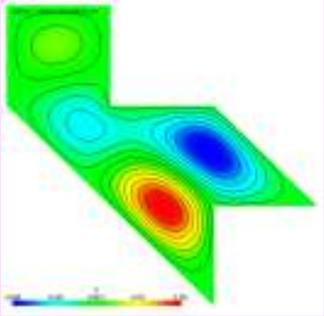
Nodal count

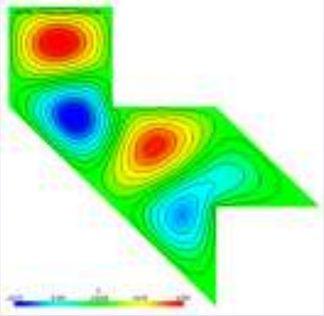
$\nu_3 = 2$ $\nu_{10} = 10$ $\nu_{24} = 23$

The nodal domain count of drums\billiards

$$-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)f_n=\lambda_n f_n \qquad f_n\big|_{boundary}=0$$

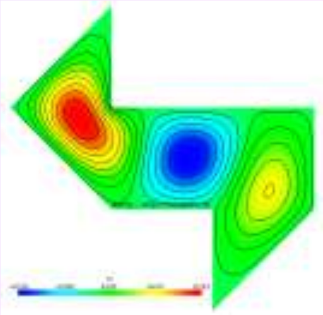


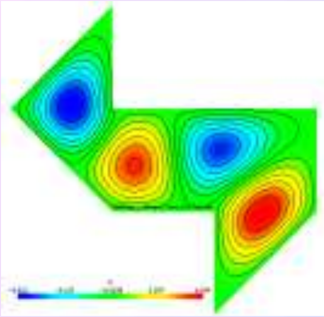


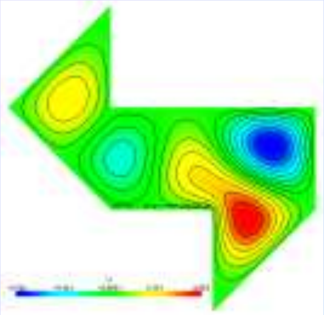


Nodal domains

$\nu_3=3$ $\nu_4=3$ $\nu_5=4$







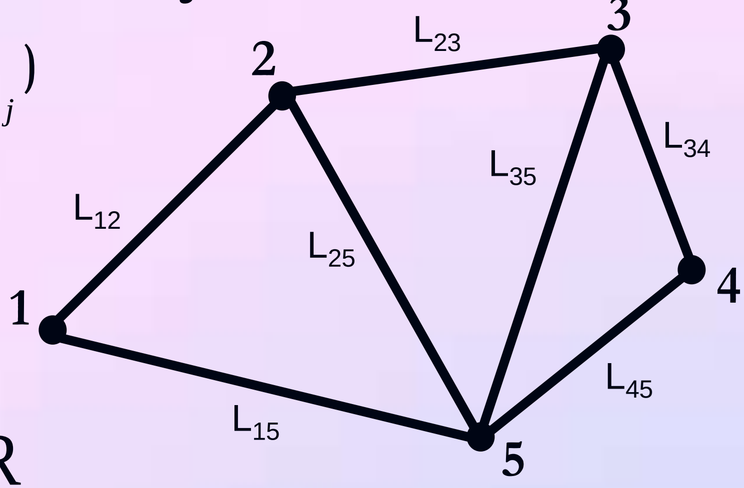
Nodal domains

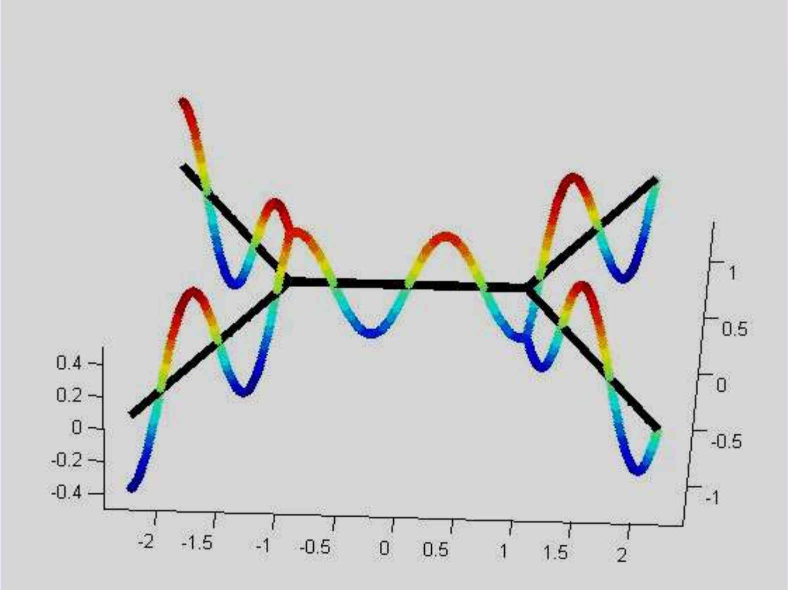
$\nu_3=3$ $\nu_4=4$ $\nu_5=4$

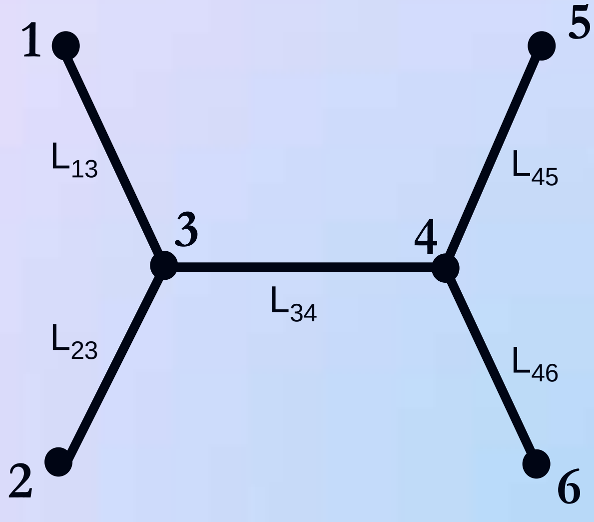
Metric Graphs - Introduction

- A ***graph Γ*** consists of a finite set of vertices ***V={v_i}*** and a finite set of edges ***E={e_j}***.
- A ***metric graph*** has a finite length (L_{e_j}) assigned to each edge.
- A ***function*** on the graph is a vector of functions on the edges:

$$f = (f_{e_1}, \ldots, f_{e_{|E|}}) \quad \forall e_j,\ f_{e_j} : [0, L_{e_j}] \to \mathbb{R}$$







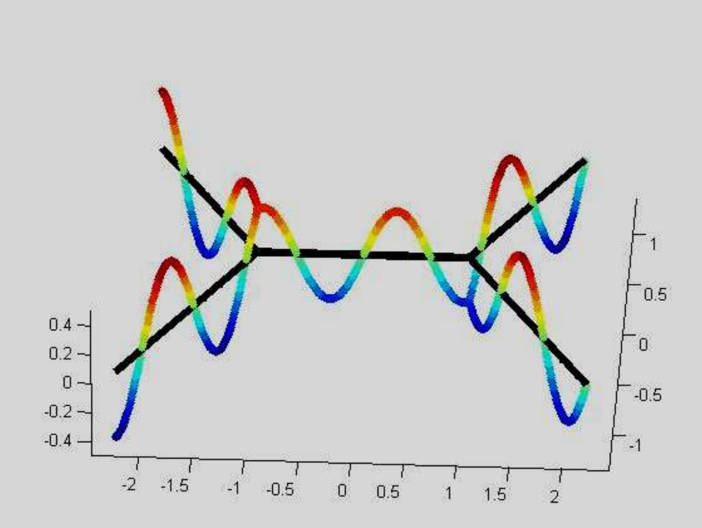
Quantum Graphs - Introduction

- A ***quantum graph*** is a metric graph equipped with an operator, such as the negative ***Laplacian*** with a ***bounded potential***:

$$-\frac{d^2}{dx^2} + v(x)$$

- For each vertex v, we impose ***vertex conditions***, such as: (The ***delta-type*** conditions)
  - Continuity $\quad \forall e_1, e_2 \in E_v \quad f\big|_{e_1}(v) = f\big|_{e_2}(v)$
  - Sum of derivatives $\quad \sum_{e \in E_v} f'\big|_e(v) = \alpha_v f(v)$

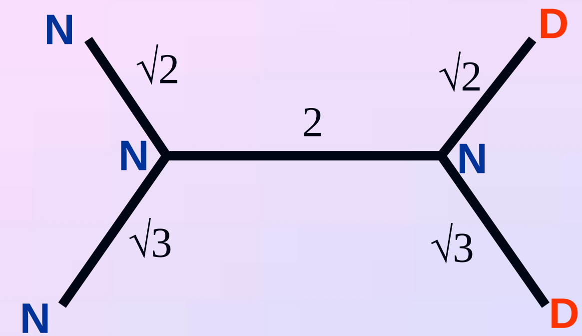
- Two special cases of vertex conditions:
  - ***Neumann***: $\alpha_v = 0$
  - ***Dirichlet***: $\alpha_v = \infty$



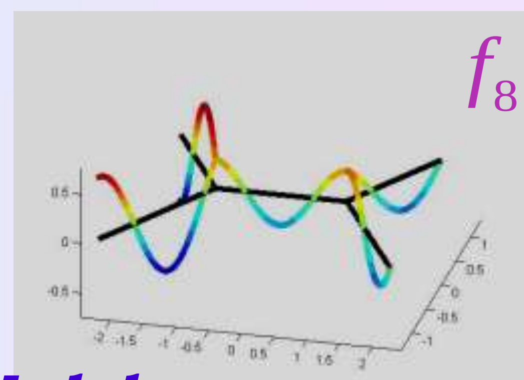
The eigenfunctions of Quantum Graphs

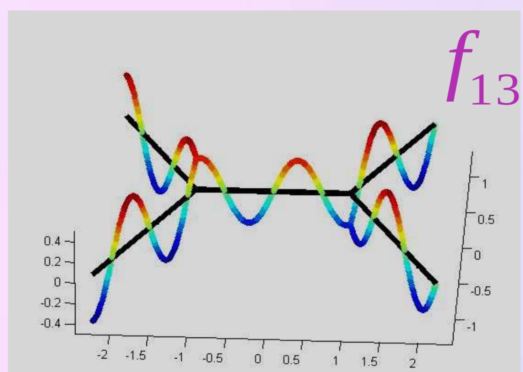
- A quantum graph is defined by specifying:
  - Metric graph
  - Operator
  - Vertex conditions for each vertex

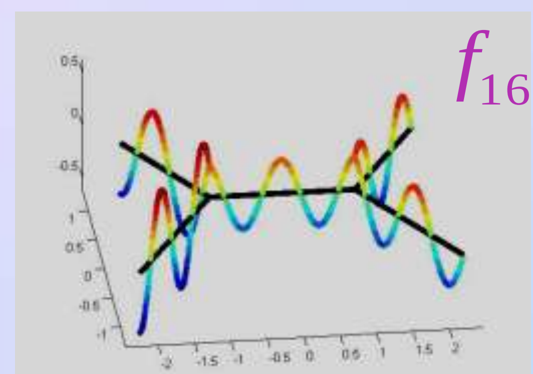
$$-\frac{d^2}{dx^2}+v(x)$$



Examples of several eigenfunctions :





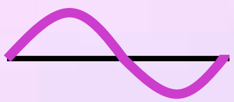


***Nodal count*:** $\nu_8 = 8$ $\nu_{13} = 13$ $\nu_{16} = 16$

The nodal count of Quantum Graphs – Known results

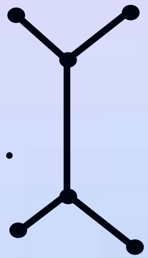
- We denote the nodal count sequence by $\{\nu_n\}_{n=1}^{\infty}$.

- The nodal count of a vibrating string is $\nu_n = n$.
  Sturm's oscillation theorem (1836).



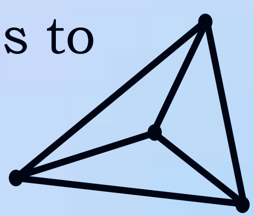
- A general bound of Courant (1923) is $\nu_n \leq n$.
  Adapted to quantum graphs by Gnutzmann, Smilansky, Weber (2004) following a method of Pleijel (1956).

- The nodal count of a tree graph is $\nu_n = n$.
  Al-Obeid, Pokornyi, Pryadiev (1992), Schapotschnikow (2006).



- $\nu_n \geq n - \beta$ is a bound given by Berkolaiko (2006),
  - where $\beta = |E| - |V| + 1$ is the minimal number of edges to remove so that the graph turns into a tree.

$\beta = 6 - 4 + 1 = 3$



The nodal count of Quantum Graphs – Known results

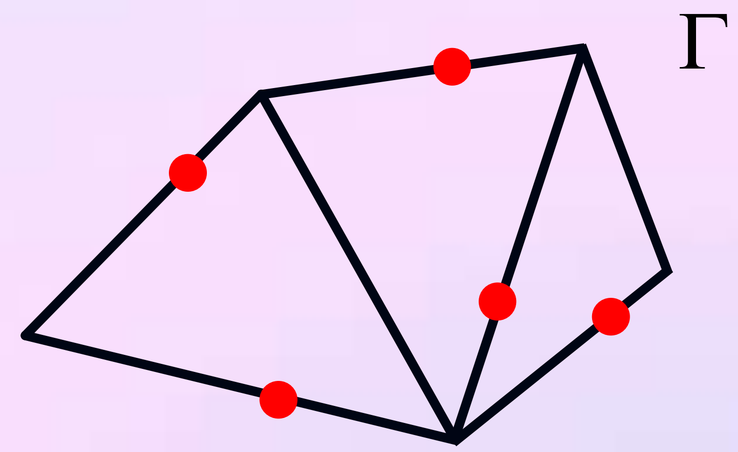
- Definition: the *nodal deficiency* of an eigenfunction is $d_n = n - \nu_n$.
- Summary of the previous results: $0 \le d_n \le \beta$ $\left(\text{where } \beta = |E| - |V| + 1\right)$

The current results in a nutshell

- The *eigenfunctions* of a quantum graph correspond to *critical points* of some energy function.
- The *nodal deficiency* of the eigenfunction equals the *Morse index* of the critical point.

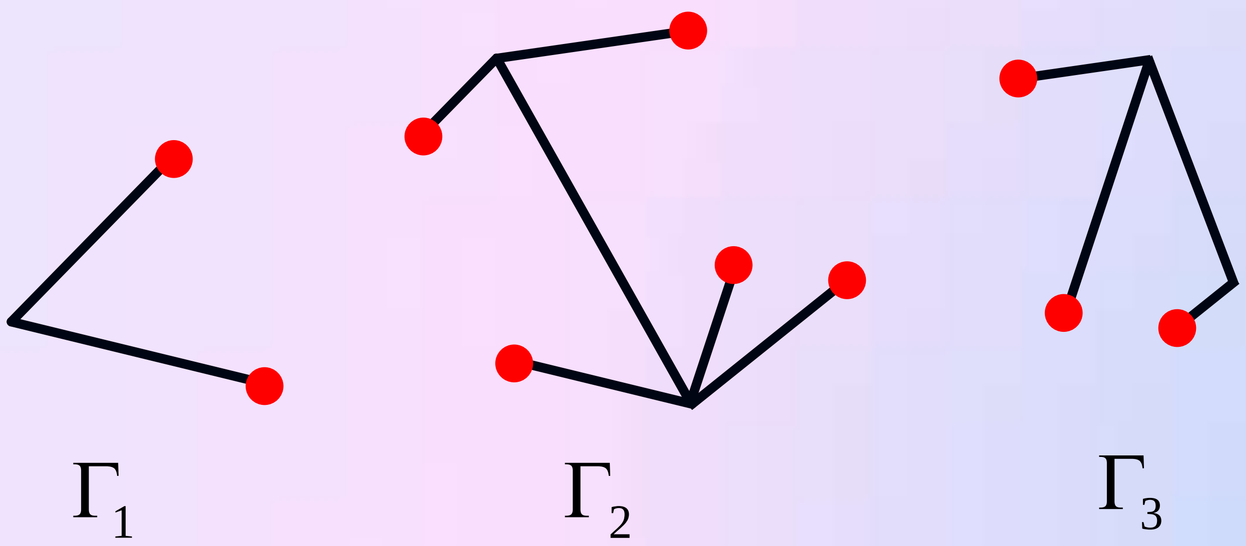
The *Morse index* of a function at a critical point is the number of negative eigenvalues of the Hessian at this point.

Partitions of a graph



- A partition of the graph - a guess for zeros' locations of some eigenfunction.

- The zeros partite the graph into several ***sub-graphs,*** $\{\Gamma_1, \Gamma_2 . \Gamma_3, \dots\}$.

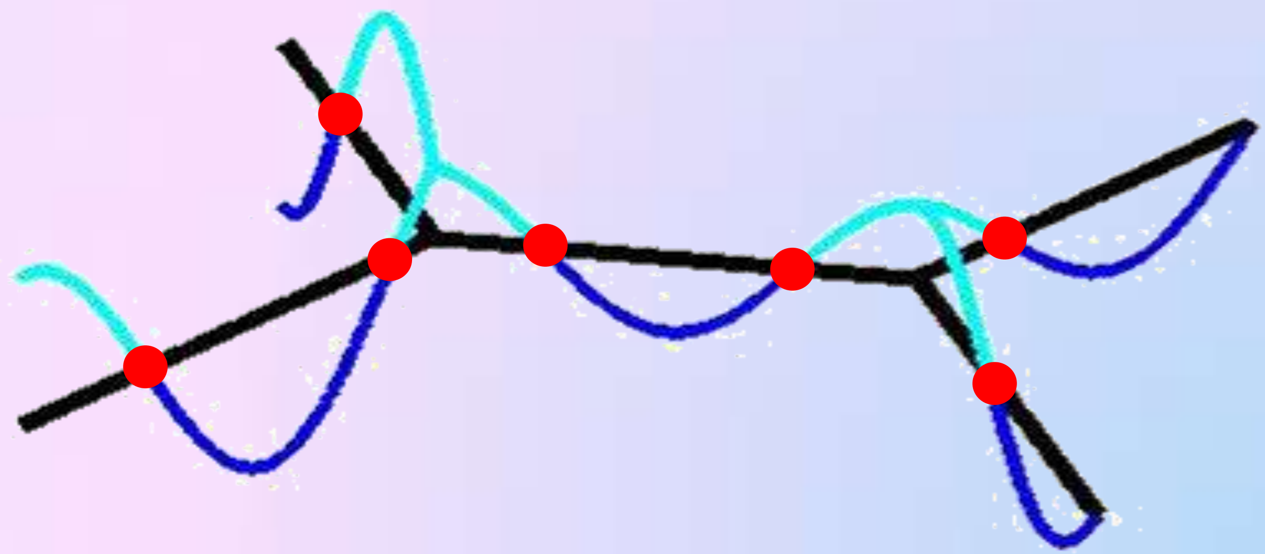


Partitions of a graph

- Properties of a partition which corresponds to an eigenfunction:
  - Each subgraph corresponds to a nodal domain,
  - The partition is ***bipartite***:

    we can assign a sign {−,+} to each subgraph such that neighbouring subgraphs will have different signs.

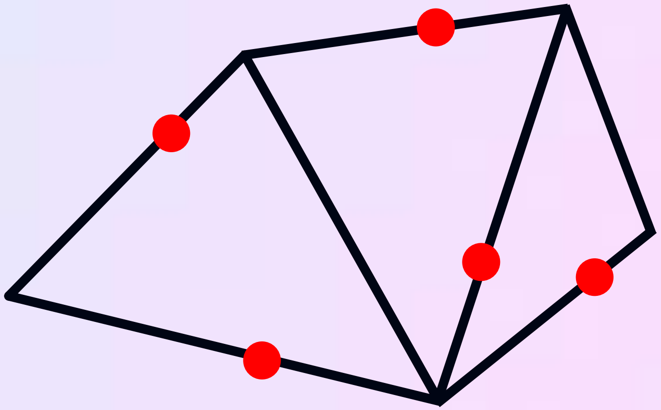
Γ



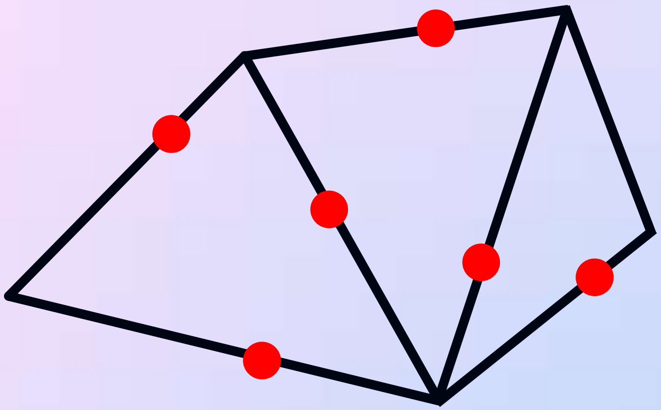
Bipartite & Non-bipartite partitions

A partition is called ***bipartite*** if:

we can assign a sign {−,+} to each subgraph such that neighbouring subgraphs will have different signs.



Bipartite partition



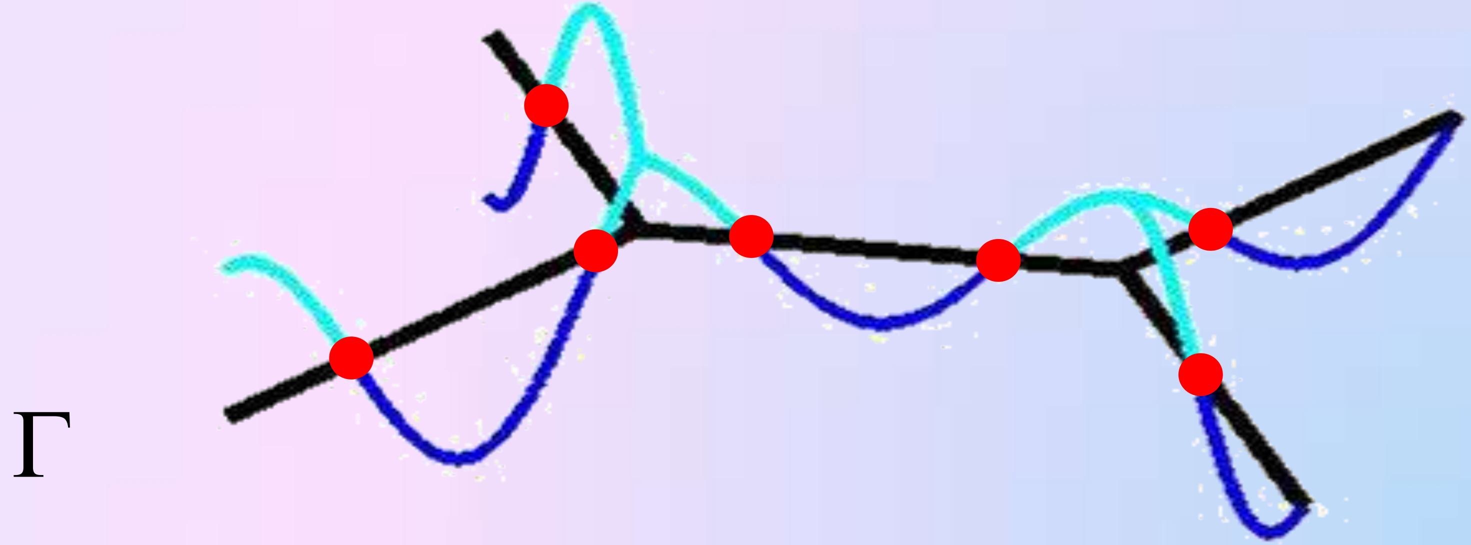
Non-Bipartite partition

Partitions of a graph

- Properties of a partition which corresponds to an eigenfunction:
  - The partition is ***bipartite***.
  - The first eigenvalues of the subgraphs are equal:

$$\lambda_1(\Gamma_1) = \lambda_1(\Gamma_2) = \lambda_1(\Gamma_3) = \dots$$

  Such a partition is called an ***equipartition***.



Partitions of a graph

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  Such a partition is called an ***equipartition***.

Equipartitions

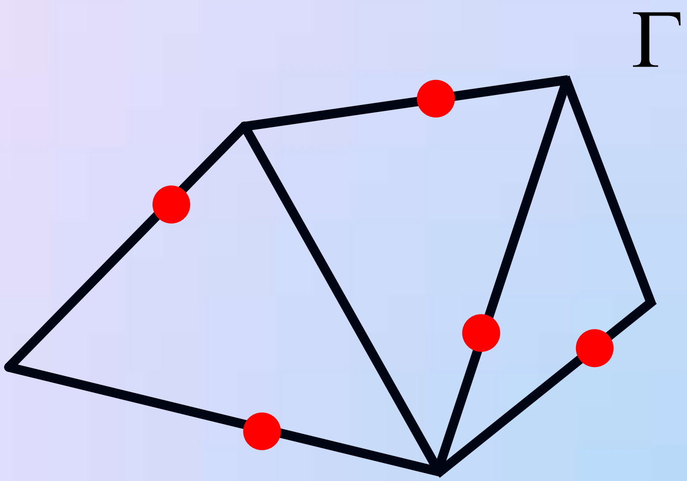
- Let $Q_n(\Gamma)$ denote the space of all equipartitions with n zeros.

- Define the energy function as $\Lambda : Q_n(\Gamma) \rightarrow R$

$$\Lambda(\mathrm{Q}) := \lambda_1(\Gamma_1) = \lambda_1(\Gamma_2) = \lambda_1(\Gamma_3) = \ldots$$

<u>Theorem 1</u>

If Q is a critical point of Λ, and Q is bipartite then Q represents the zeros' location of some eigenfunction on Γ.



The main theorems

Theorem 1

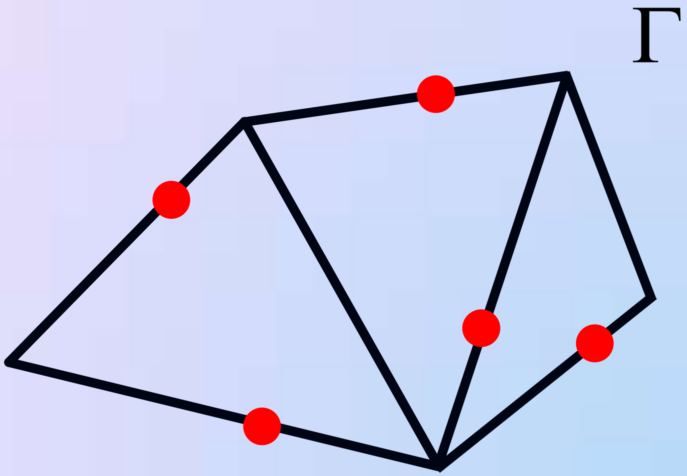
If Q is a critical point of Λ, and Q is bipartite then Q represents the zeros' location of some eigenfunction on Γ.

Theorem 2

Let Q be a non-degenerate critical point of Λ as in Theorem 1,
And let f be the corresponding eigenfunction.
Then the nodal deficiency of f equals the Morse index of Λ at Q.

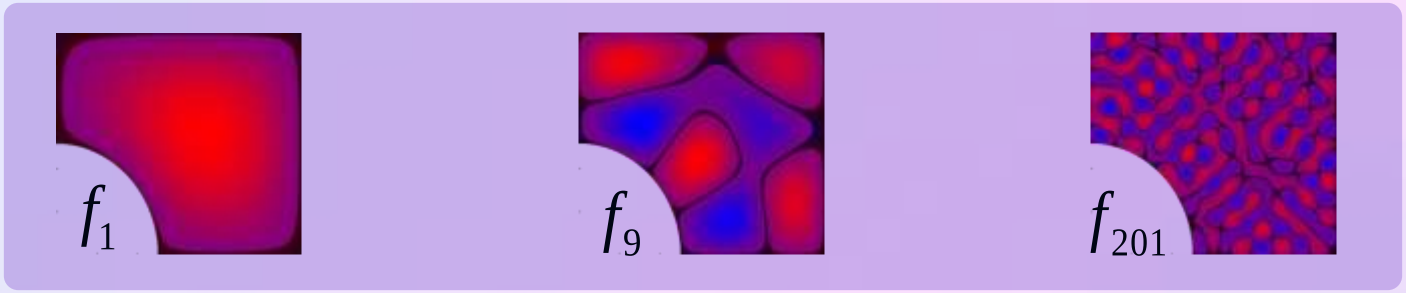
Recall:

- The *nodal deficiency* is $d_n = n - \nu_n$.
- The *Morse index* is the number of negative eigenvalues of the Hessian.



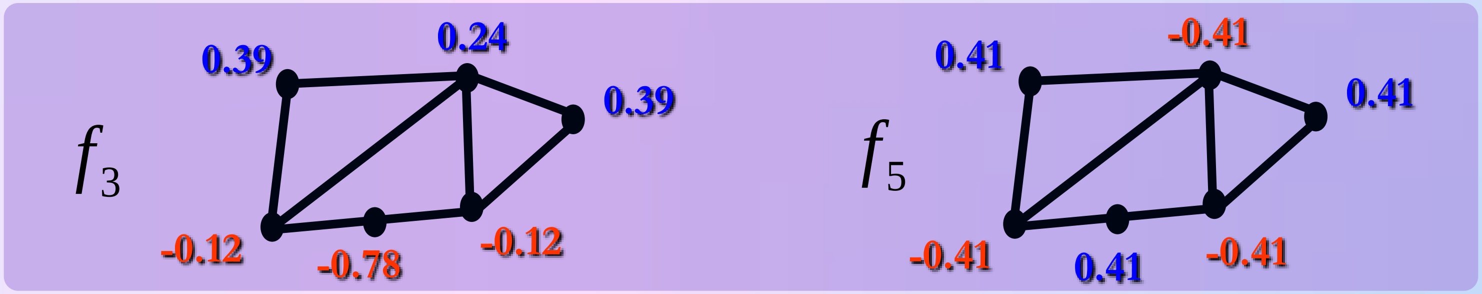
What else?

- Two-dimensional planar domains



  - Helffer, Hoffmann-Ostenhof and Terracini
    a similar result to theorem 2, for nodal deficiency = zero. (2006).
  - Analogue results for two-dimensional domains
    Berkolaiko, Kuchment, Smilansky (2011).

- Analogue results for Combinatorial graphs
  Berkolaiko, Raz, Smilansky (2011).



Finding the nodal points on a quantum graph

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On the connection between the number of nodal domains on quantum graphs and the stability of graph partitions.
Comm. Math. Phys., 2011. preprint arXiv:1103.1423 [math-ph].

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