A trace formula for nodal domains on quantum graphs?

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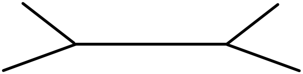
Solomyak meeting, Cardiff

Joint work with Gregory Berkolaiko and Uzy Smilansky

Outline

Introduction to quantum graphs

- Consider a graph $\Gamma = (\mathcal{V}, \mathcal{E})$.
- Assign a length L_e to each edge $e \in \mathcal{E}$.
- Local coordinate on each edge: $x_e \in [0, L_e]$.



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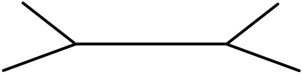

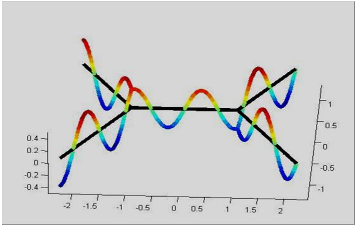

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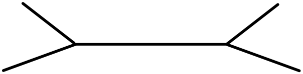

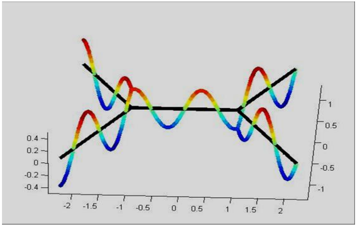

- Functions take values along the edges: $f \in \bigoplus_{e \in \mathcal{E}} H^2(e)$.
- Consider $-\Delta$, which equals $-\frac{\mathrm{d}^2}{\mathrm{d}x_e^2}$ on each of the edges.
- Supplement vertex conditions at each vertex.

Introduction to quantum graphs

Examples of vertex conditions:

Introduction to quantum graphs

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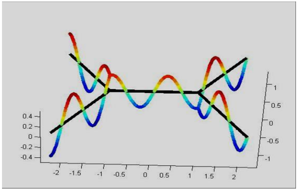

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Spectral trace formula of quantum graphs

Consider the eigenvalue problem $-\Delta f = k^2 f^2$

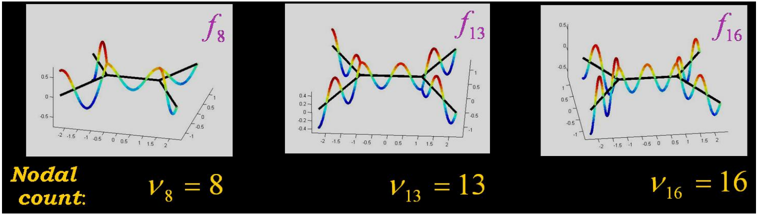
A trace formula for the spectral density

$$d(k) = \sum_j \delta(k - k_j) = \frac{L}{\pi} + C\delta(k) + \frac{1}{\pi}\mathrm{Re}\sum_{p\in P} A_p e^{ikL_p},$$

- L is the sum of all edge lengths.
- $\mathcal{P}$ is the set of periodic orbits p of length L_p.
- A_p are certain explicitly defined factors.

Roth (1983), Kottos, Smilansky (1997)

Nodal domains on quantum graphs



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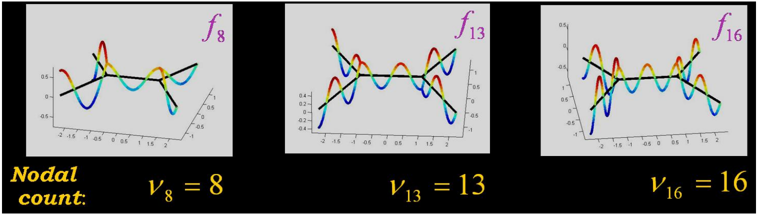

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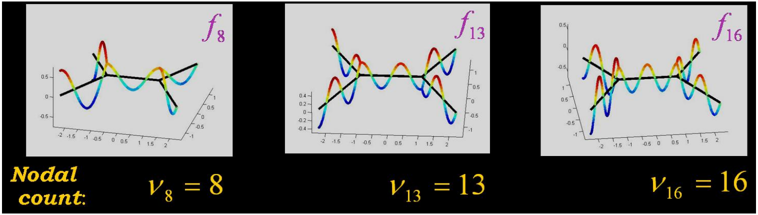

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  Courant (1923) $\rightarrow$ Pleijel (1956) $\rightarrow$ Gnutzmann, Smilansky, Weber (2004).

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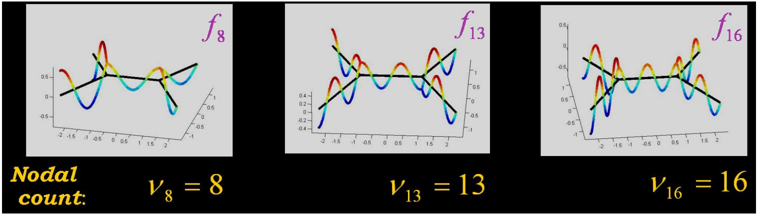

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- The nodal count of a *tree graph* is $\nu_n = n$.
  Al-Obeid, Pokornyi, Pryadiev (1992), Schapotschnikow (2006).

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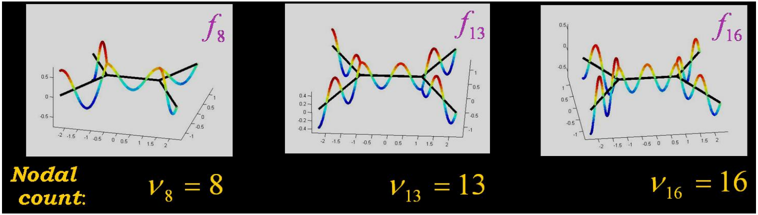

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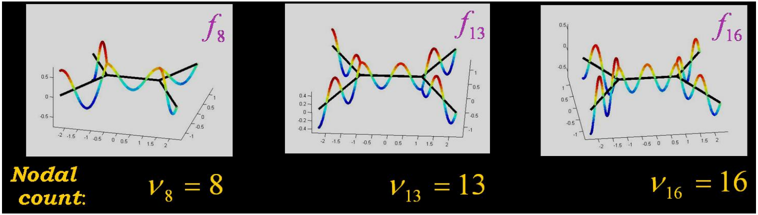

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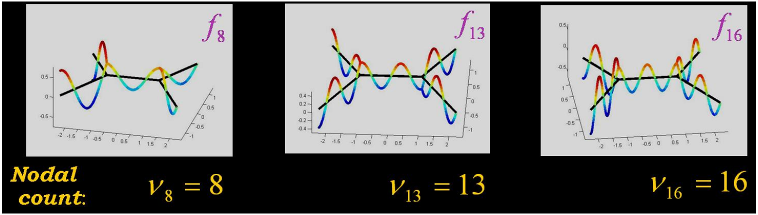

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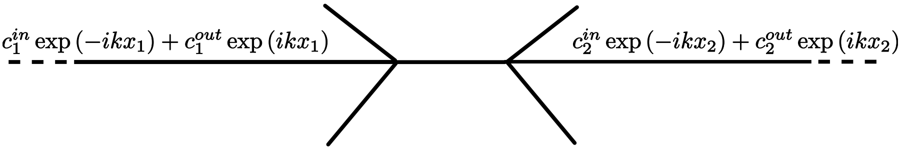
The scattering matrix



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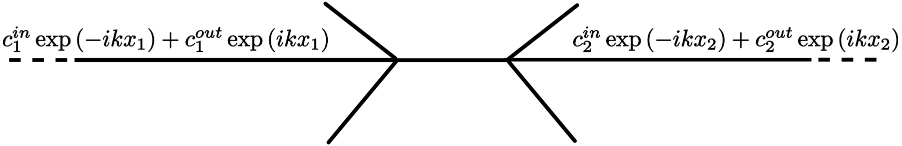

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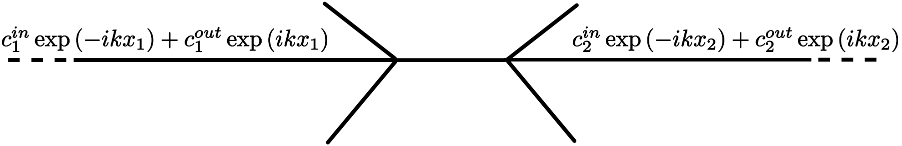

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$$\begin{pmatrix} c_1^{out} \\ c_2^{out} \end{pmatrix} = S(k) \begin{pmatrix} c_1^{in} \\ c_2^{in} \end{pmatrix},$$

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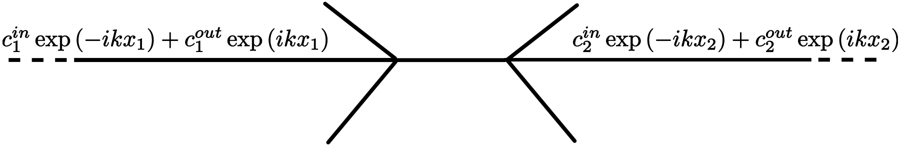

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  where $S(k)$ is the scattering matrix.
- Any number of leads $\mathcal{L}$ can be attached to the graph -
  $S(k)$ is then of dimension $\mathcal{L}$ and it is unitary for every $k \in \mathbb{R}$.

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  k is an eigenvalue of Γ if and only if $\det(\mathbb{I} - S(k)) = 0$.

  Explanation: When $c_l^{out} = c_l^{in}$ we have that $f_l(x_l) \propto \cos(kx_l)$ and Neumann conditions at the vertex are fulfilled even if this lead is removed.

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- When $S(k)$ has an eigenvalue -1, there exists a generalized eigenfunction that vanishes at all vertices.

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  This holds for tree graphs. What complicates the scenario?
- When a nodal point reaches a vertex -
  A split or a merge may change the number of nodal points.

A single lead - changing the technique

- When a nodal point reaches a vertex v, sign$[f(v)]$ is changed.

A single lead - changing the technique

- When a nodal point reaches a vertex v, $\mathrm{sign}\,[f(v)]$ is changed.
- The number of nodal points on an edge $e = (v, u)$ is given by

$$\left\lfloor \frac{kL_e}{\pi} \right\rfloor + \frac{1}{2}\left(1 - (-1)^{\left\lfloor \frac{kL_e}{\pi} \right\rfloor} \mathrm{sign}\,(f(v)f(u))\right).$$

Gnutzmann, Smilansky, Weber (2004)

- Connect a lead to a vertex v. Choose an adjacent vertex u. We wish to count the sign changes of $f(u)$ and $f(v)$. $\mathrm{sign}\,[f(v)]$ is easy, how about $\mathrm{sign}\,[f(u)]$?

Two leads

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Definition

The *sign-weighted counting function* $N_{u,v}(k)$

$$N_{u,v}(k) = \#\{k_n \leq k : f_n(u)f_n(v) > 0\} - \#\{k_n \leq k : f_n(u)f_n(v) < 0\}.$$

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Theorem

Denote by $S(k)$ the 2×2 scattering matrix obtained by attaching leads to the vertices u and v. Let $\mathbb{I}_\epsilon$ be the matrix

$$\mathbb{I}_\epsilon = \begin{pmatrix} 1 & -\epsilon \\ -\epsilon & 1 \end{pmatrix}$$

Then

$$N_{u,v}(k) = \frac{1}{\pi} \lim_{\epsilon \to 0} \arg \frac{\det(\mathbb{I}_\epsilon - S(k))}{\sqrt{\det S(k)}}.$$

Towards a nodal count formula

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The (ordinary) spectral counting function is

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and can be represented as

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Combining (1) and (2) we obtain another counting function

$$N_{u,v}^{(-)}(k) := \#\{k_n \le k : f_n(u) f_n(v) < 0\} = \frac{1}{2\pi} \lim_{\epsilon \to 0} \arg \frac{\det((1-\epsilon)\,\mathbb{I} - S(k))}{\det(\mathbb{I}_\epsilon - S(k))}.$$

Towards a nodal count formula

- Recall that the number of nodal points on an edge $e = (v, u)$ is given by

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  Sum over all the edges of the graph to get the total number of nodal points.

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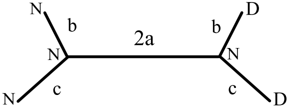
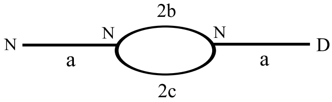
- Future work:
  - Expanding the above to a trace formula for $\nu(k)$.
  - Using an inversion of $N(k)$ to express ν_n,
    or adopting another approach for attacking it directly.

An explicit formula

- The two graphs below are isospectral





- The nodal count of the tree graph is trivial.
  The nodal count of the second graph is

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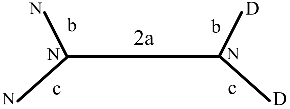

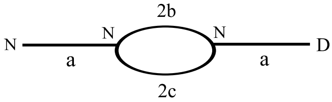

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- Denote $\alpha := \frac{b+c}{a+b+c}$, the normalized length of the loop.
- We may get the following orbit expansion of the formula

$$\nu_n = n - \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{(2k+1)\pi} \sin\left((2k+1)\,\pi\alpha n\right).$$

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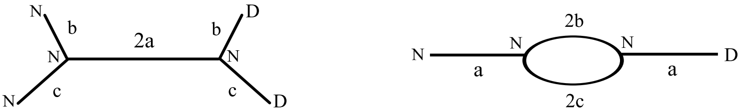

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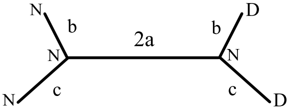

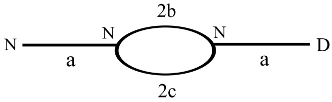

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