How to count ?

nodal domains of quantum graphs

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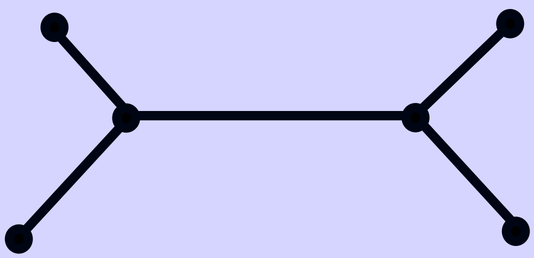
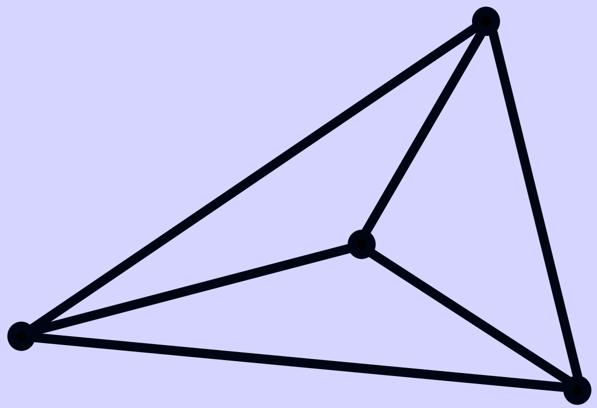
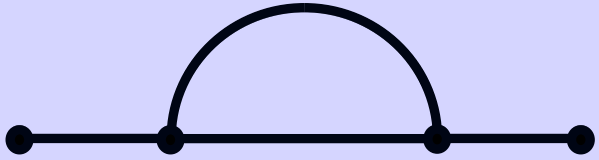
- **1, 1, 2, 3, 5, 8, 13, 21, ?**
  - What is the next number?
  - Which formula generates this sequence?

- **1, 2, 2, 4, 5, 5, 7, 7, 8, ?**
  - What is the next number?
  - Which formula generates this sequence?



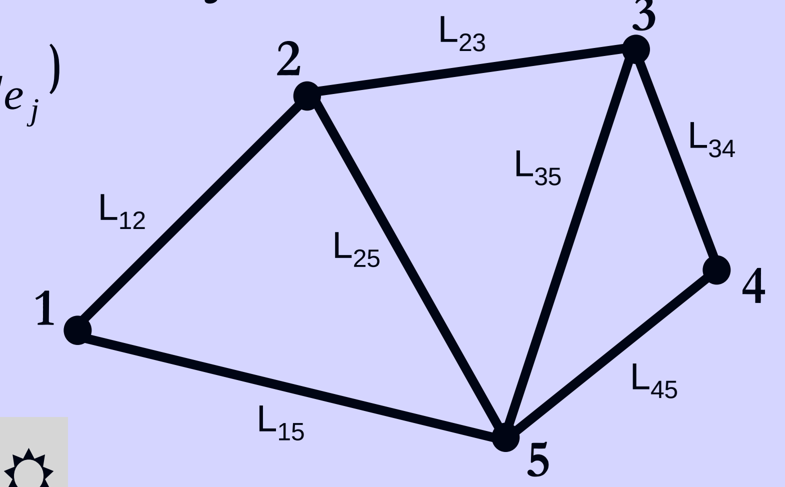




- This sequence is the ***nodal count***, $\{\nu_n\}_{n=1}^{\infty}$, of one of the graphs above.

Metric Graphs - Introduction

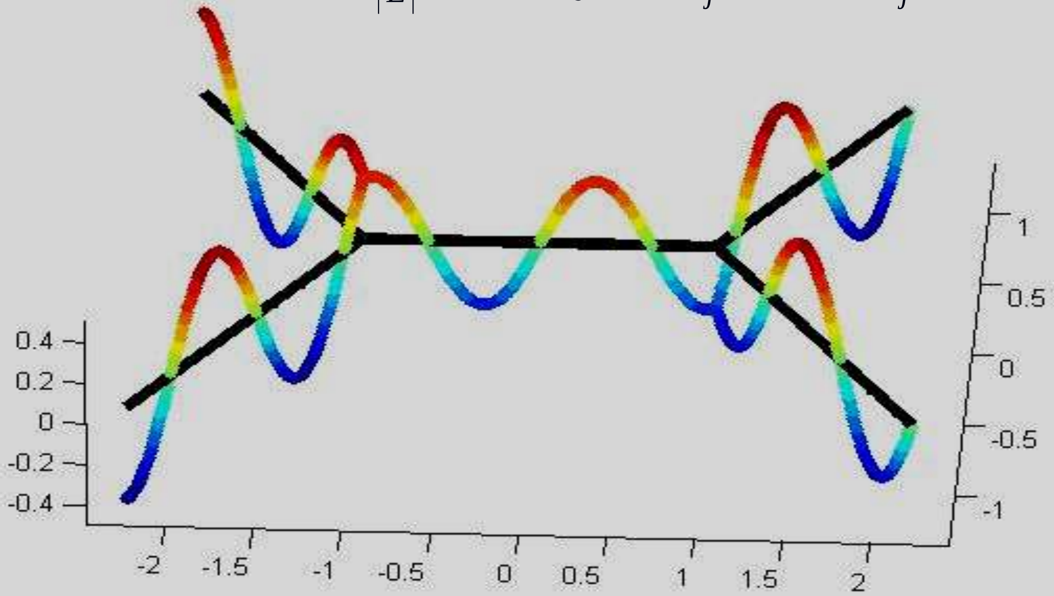
- A ***graph Γ*** consists of a finite set of vertices ***V={vᵢ}***
  and a finite set of edges ***E={eⱼ}***.

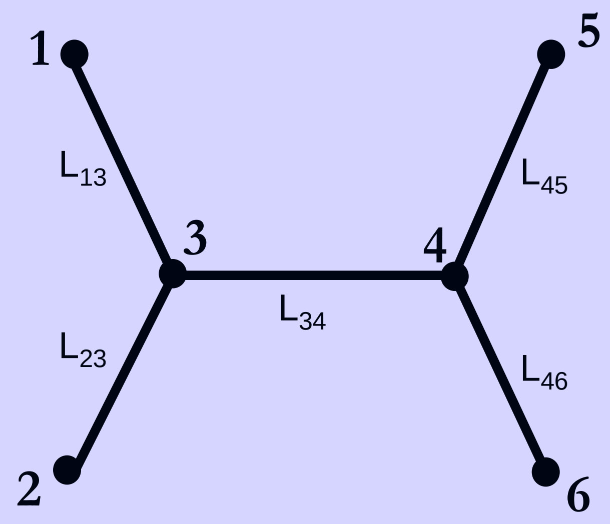


- A ***metric graph*** has a finite length (L_{e_j}) assigned to each edge.

- A ***function*** on the graph is a vector of functions on the edges:

$$f = (f_{e_1}, \ldots, f_{e_{|E|}}) \quad \forall e_j, \;\; f_{e_j} : [0, L_{e_j}] \to$$ ☼



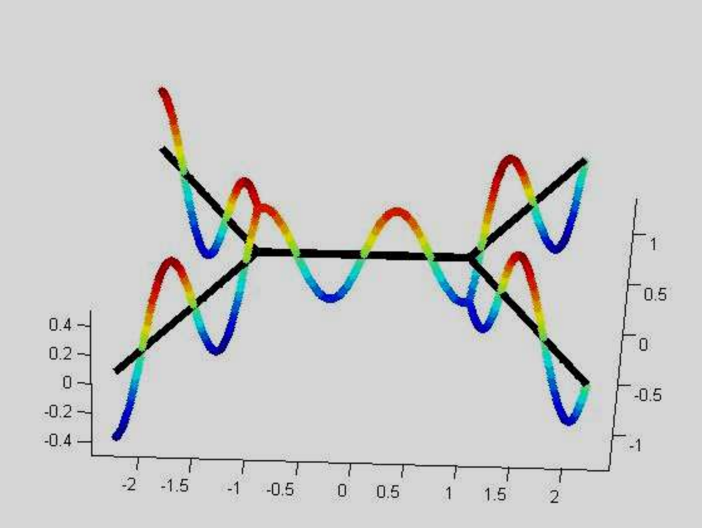


Quantum Graphs - Introduction

- A ***quantum graph*** is a metric graph equipped with an operator, such as the negative ***Laplacian***: $-\Delta f = (-f''|_{e_1}, \ldots, -f''|_{e_{|E|}})$

- For each vertex v, we impose ***boundary conditions***, such as: (The ***Neumann*** boundary conditions)
  - Continuity $\forall e_1, e_2 \in E_v \quad f|_{e_1}(v) = f|_{e_2}(v)$
  - Zero sum of derivatives $\sum_{e \in E_v} f'|_e(v) = 0$

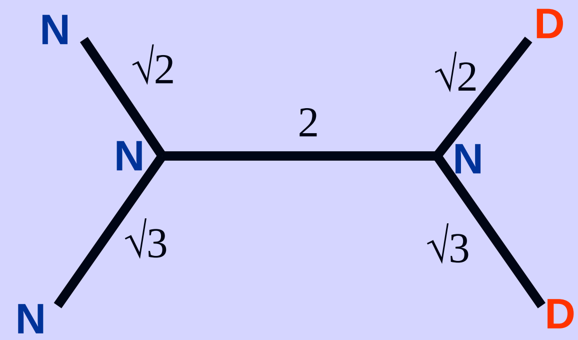
- For vertices of degree one, there are two special cases of boundary conditions, denoted
  - ***Dirichlet***: $f(v) = 0$
  - ***Neumann***: $f'(v) = 0$



The eigenfunctions of Quantum Graphs

- A quantum graph is defined by specifying:
  - Metric graph
  - Operator
  - Boundary conditions for each vertex

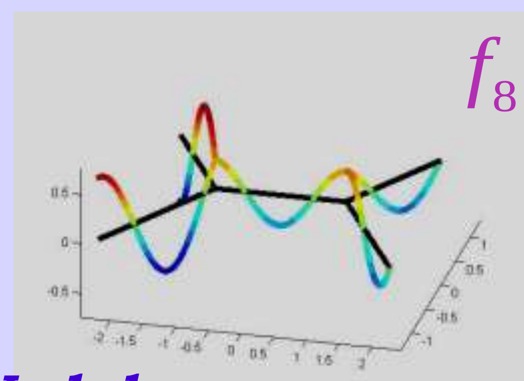
$$-\Delta = -\left(\left.\frac{d^2}{dx^2}\right|_{e_1}, \ldots, \left.\frac{d^2}{dx^2}\right|_{e_{|E|}} \right)$$

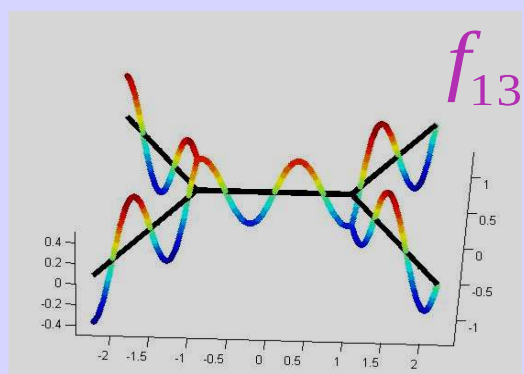


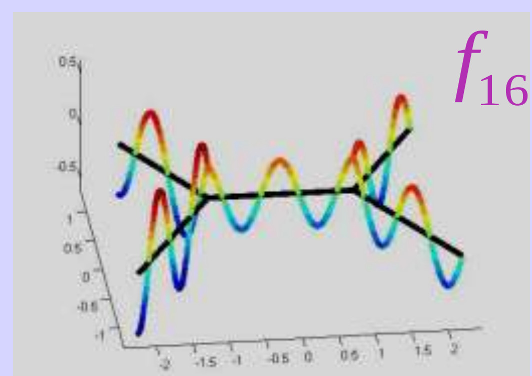
We are interested in the ***eigenfunctions*** of the Laplacian:

$$-\Delta f = k^2 f \quad \Rightarrow \quad \left(-f''\big|_{e_1}, \ldots, -f''\big|_{e_{|E|}} \right) = \left(k^2 f\big|_{e_1}, \ldots, k^2 f\big|_{e_{|E|}} \right)$$

Examples of several eigenfunctions of the Laplacian on the graph above:





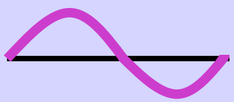


Nodal count: $\nu_8 = 8$ $\nu_{13} = 13$ $\nu_{16} = 16$

The nodal count of Quantum Graphs

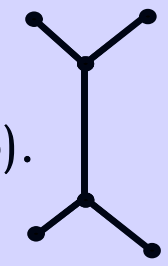
- We denote the nodal count sequence by $\{\nu_n\}_{n=1}^{\infty}$.

- The nodal count of a vibrating string is $\nu_n = n$.
  Sturm’s oscillation theorem (1836).



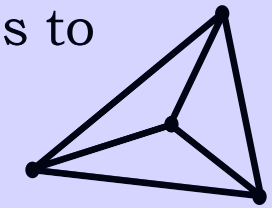
- A general bound of Courant (1923) is $\nu_n \le n$.
  Adapted to quantum graphs by Gnutzmann, Smilansky, Weber (2004) following a method of Pleijel (1956).

- The nodal count of a tree graph is $\nu_n = n$.
  Al-Obeid, Pokornyi, Pryadiev (1992), Schapotschnikow (2006).



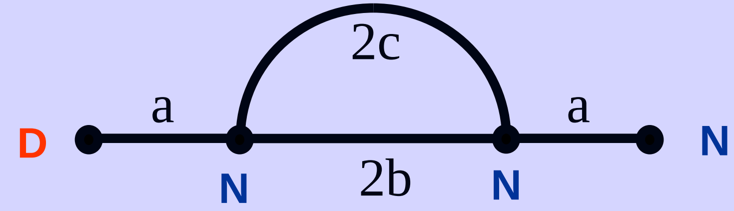
- $\nu_n \ge n - r$ is a bound given by Berkolaiko (2006),
  - where $r = |E| - |V| + 1$ is the minimal number of edges to remove so that the graph turns into a tree.

$r = 6 - 4 + 1 = 3$

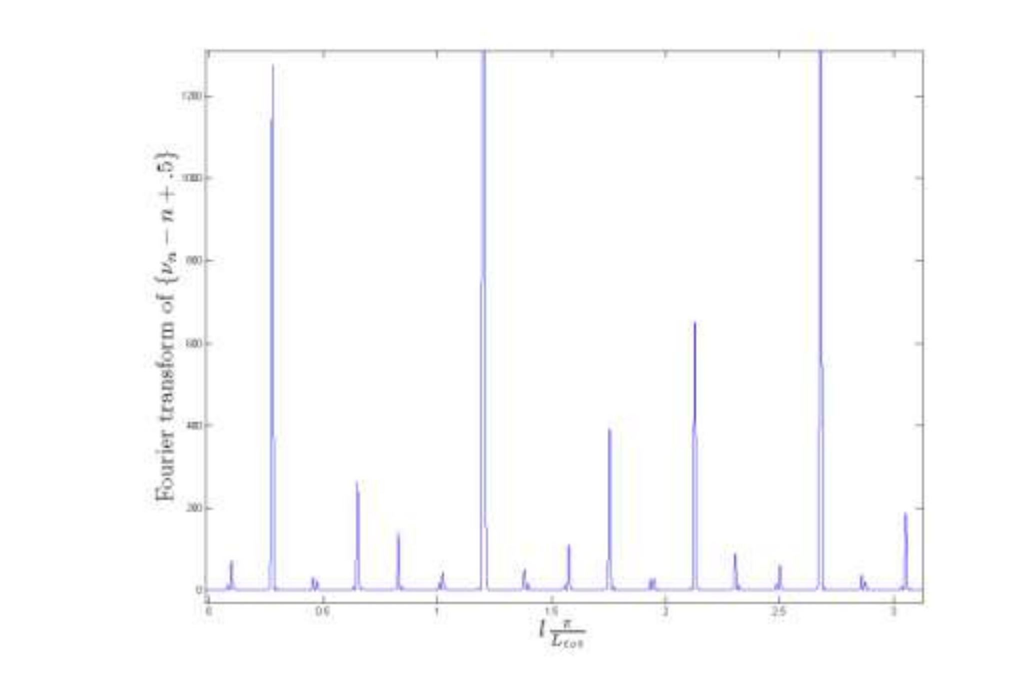


The nodal count of Quantum Graphs

- <u>Example</u> - We try to deal with the nodal count of the graph



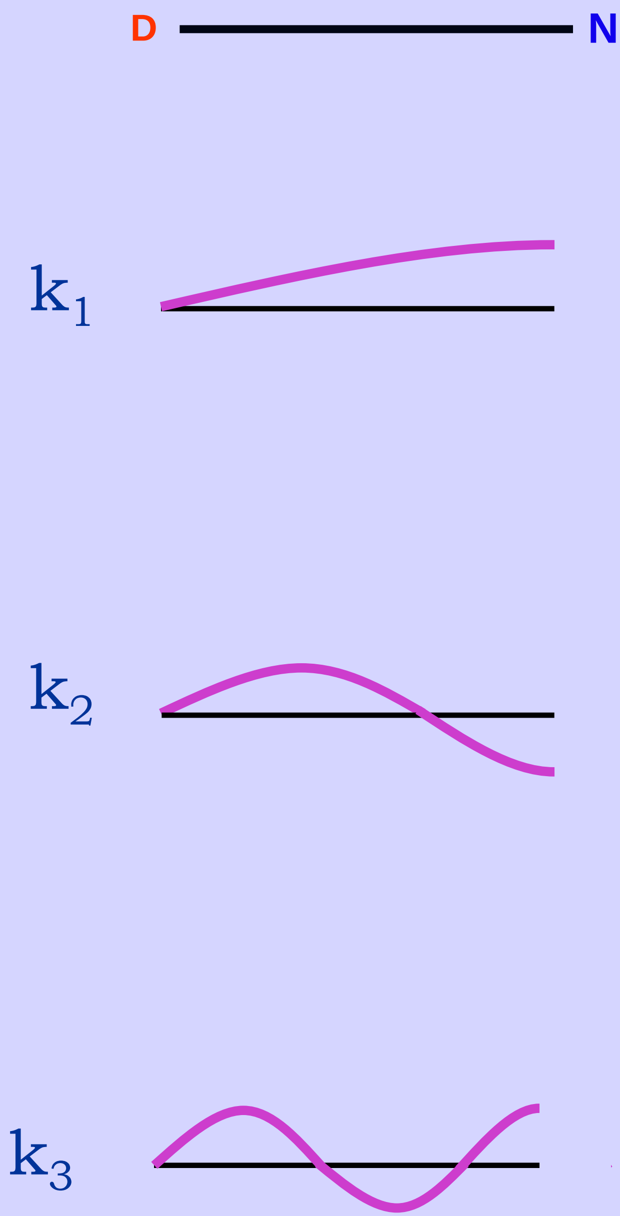
- The known bounds give $n-1 \leq \nu_n \leq n$ (r=1).
- We have the oscillating sequence $\nu_n - n + 0.5 = \pm 0.5$.
  - We calculate this sequence ***numerically*** (for some a,b,c values).
  - We Fourier transform it and obtain the following:



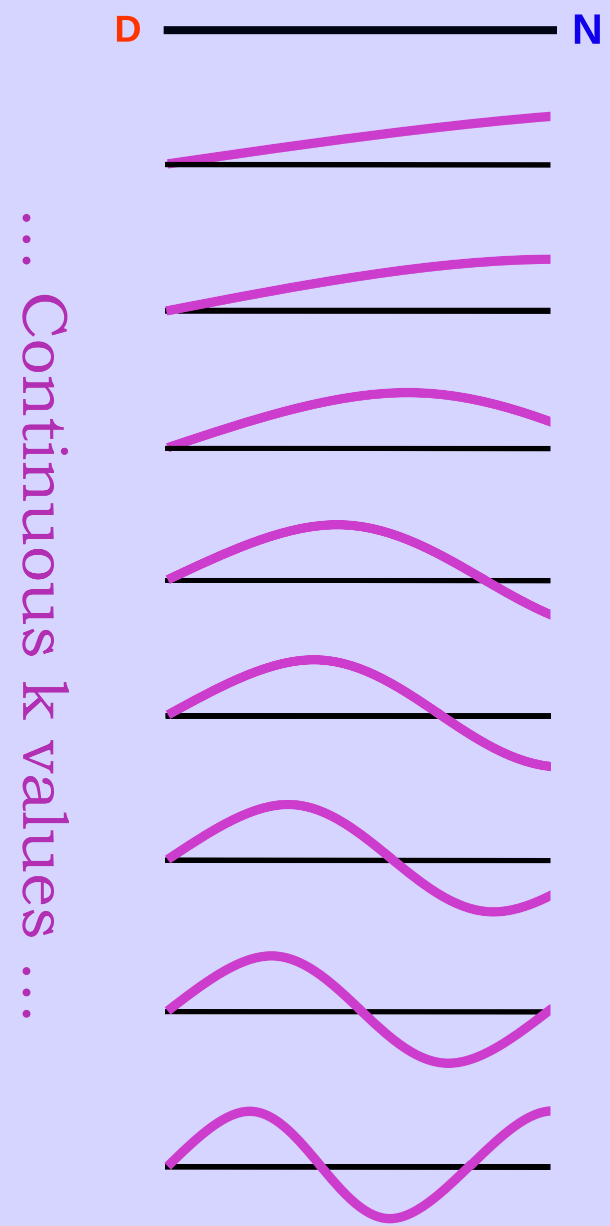
- Is there a formula for the nodal count of this graph ?

The nodal count of a string

The eigenfunctions of

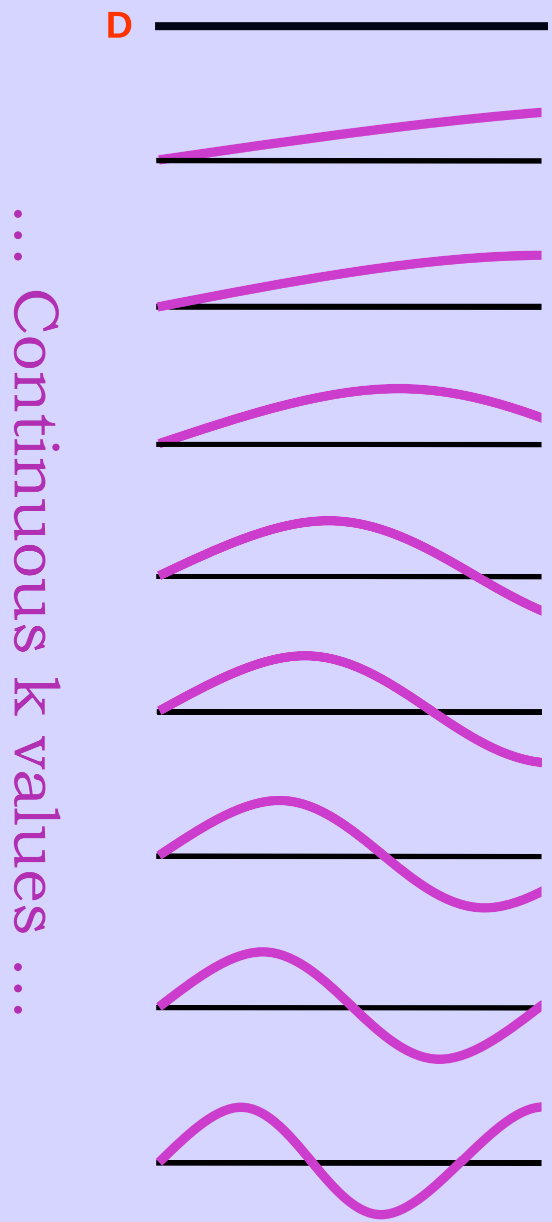


The eigenfunctions of



Removing a boundary condition

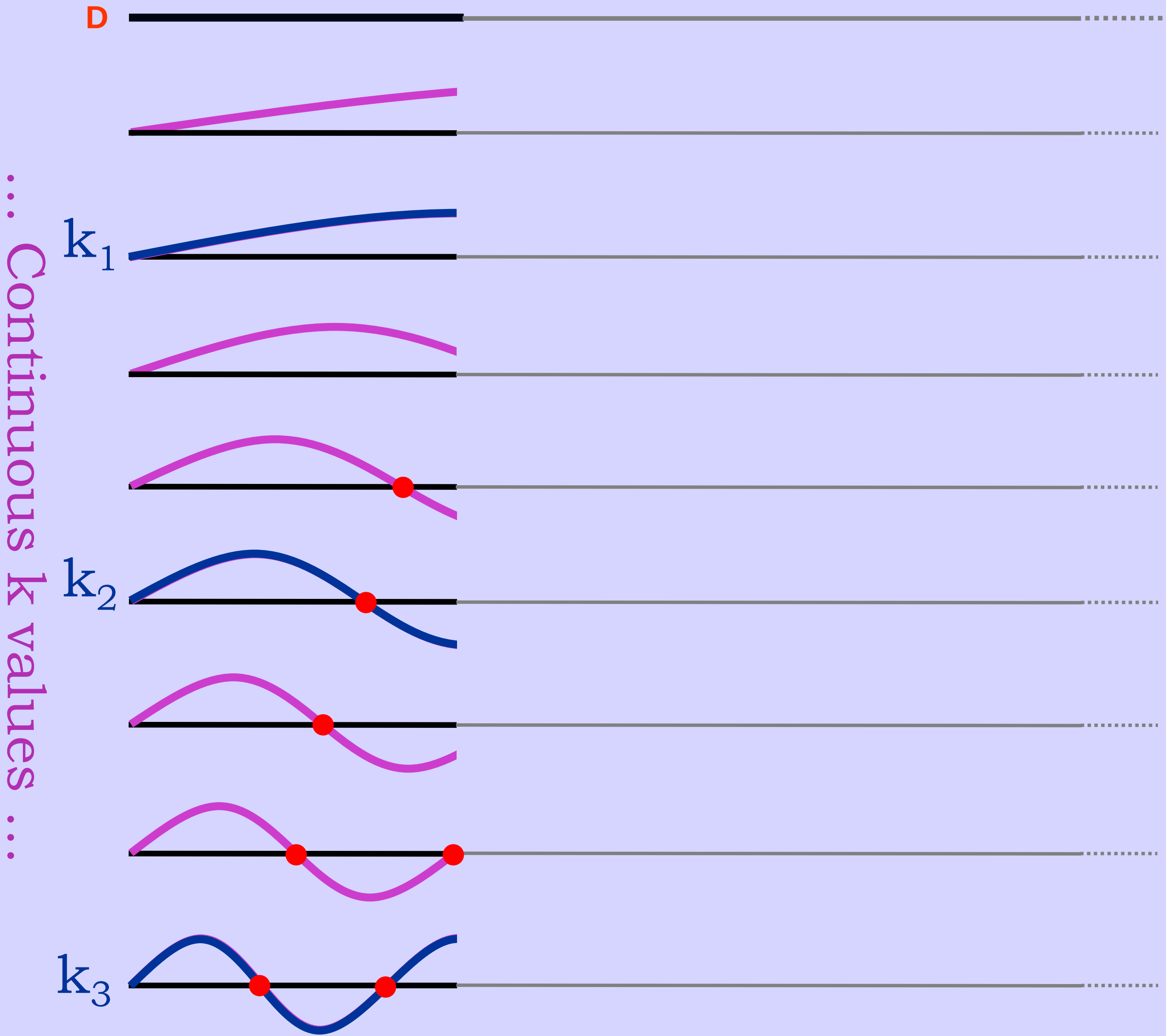
The eigenfunctions of



Removing a boundary condition

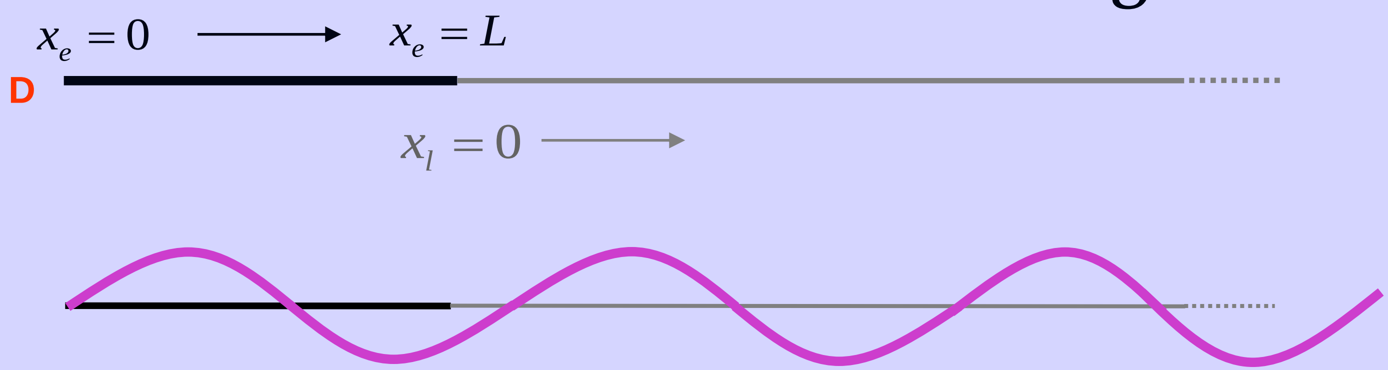
– attaching an infinite lead

The eigenfunctions of



Removing a boundary condition

– attaching an infinite lead



$$C\sin(kx_e) \qquad\qquad a^{in}\exp(-ikx_l)+a^{out}\exp(ikx_l)$$

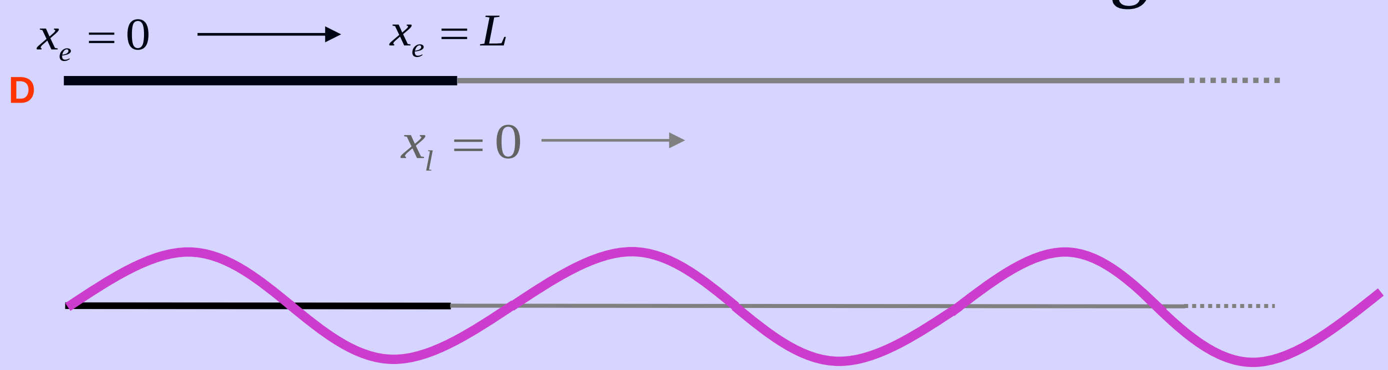
$$\begin{cases} C\sin(kL)=a^{in}+a^{out} & \text{(Continuity)} \\ C\cos(kL)=ik\left(-a^{in}+a^{out}\right) & \text{(Zero sum of derivatives - differentiability)} \end{cases}$$

$$a^{out}=\exp\left(i(2kl+\pi)\right)a^{in}=S(k)\,a^{in} \qquad S(k)=\exp\left(i\varphi(k)\right) \text{ is unitary.}$$

When $\varphi(k)\equiv 0 \pmod{2\pi}$ we have an eigenfunction of the original graph.

When $\varphi(k)\equiv \pi \pmod{2\pi}$ a nodal point enters the original graph.

Removing a boundary condition
– attaching an infinite lead



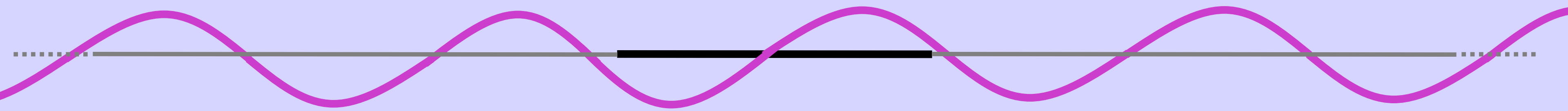
$$C \sin\left(k x_e\right) \qquad a^{in} \exp\left(-ikx_l\right) + a^{out} \exp\left(ikx_l\right)$$

$$a^{out} = \exp\left(i\left(2kl + \pi\right)\right) a^{in} = S\left(k\right) a^{in} \qquad S\left(k\right) = \exp\left(i\varphi\left(k\right)\right) \text{ is unitary.}$$

When $\varphi(k) = 0 \pmod{2\pi}$ we have an eigenfunction of the original graph.
When $\varphi(k) = \pi \pmod{2\pi}$ a nodal point enters the original graph.

- We have that $\varphi'(k) > 0$.
Therefore these events interlace and we have $\nu_n = n$

Removing two boundary conditions – attaching two infinite leads

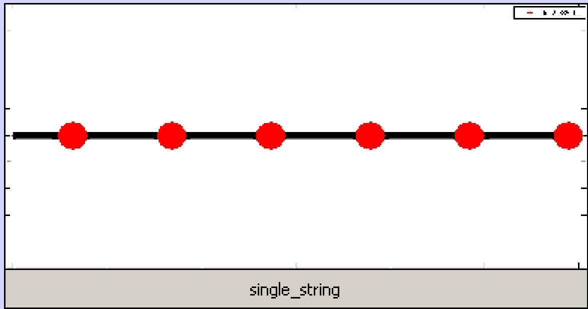


$$a_1^{in}\exp\left(-ikx_{l_1}\right)+a_1^{out}\exp\left(ikx_{l_1}\right) = C_1\sin\left(kx_{l_1}+\theta_1(k)\right)$$

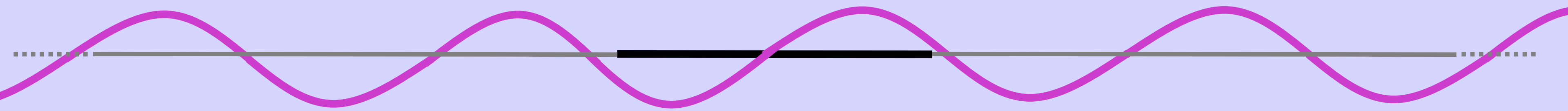
$$a_2^{in}\exp\left(-ikx_{l_2}\right)+a_2^{out}\exp\left(ikx_{l_2}\right) = C_2\sin\left(kx_{l_2}+\theta_2(k)\right)$$

$$\begin{pmatrix} a_1^{out}(k) \\ a_2^{out}(k) \end{pmatrix} = S(k)\cdot\begin{pmatrix} \mathbf{a_1}^{in}(k) \\ \mathbf{a_2}^{in}(k) \end{pmatrix}$$

- For each k we have a two-dimensional space of possible functions.
  - In order to have a single valued movie we need to add a restriction.
  - We choose the restriction $\theta_1(k)=\pi/2+\theta_2(k)$ and get the movie:



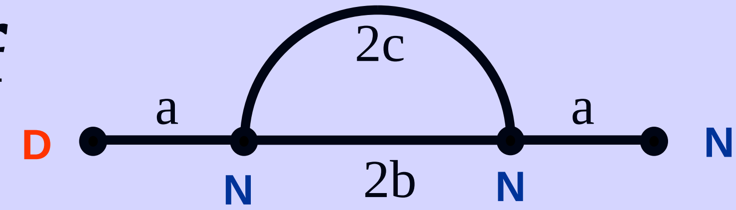
Removing two boundary conditions
– attaching two infinite leads

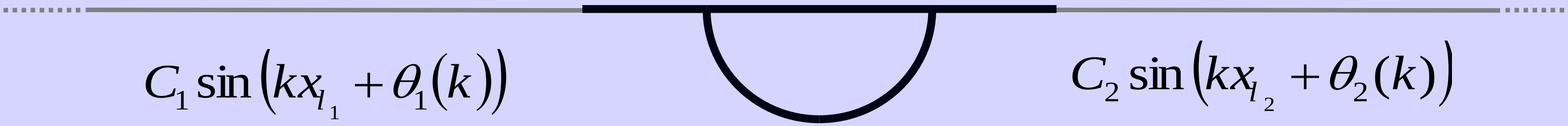


$$= C_1 \sin\left(k x_{l_1} + \theta_1(k)\right) \qquad\qquad = C_2 \sin\left(k x_{l_2} + \theta_2(k)\right)$$

- The interesting events are
  - $\theta_1(k) = 0\ ;\ \theta_2(k) = \pi/2 \pmod{\pi}$ Eigenfunction of the original graph & nodal point enters from the left.
  - $\theta_1(k) = \pi/2\ ;\ \theta_2(k) = 0 \pmod{\pi}$ Eigenfunction of the original graph & nodal point enters from the right.
- For each k we have a two-dimensional space of possible functions.
  - In order to have a single valued movie we need to add a restriction.
  - We choose the restriction $\theta_1(k) = \pi/2 + \theta_2(k)$
- We can check that $\theta_1'(k) > 0$ and $\theta_2'(k) > 0$ and deduce that $\nu_n = n$.

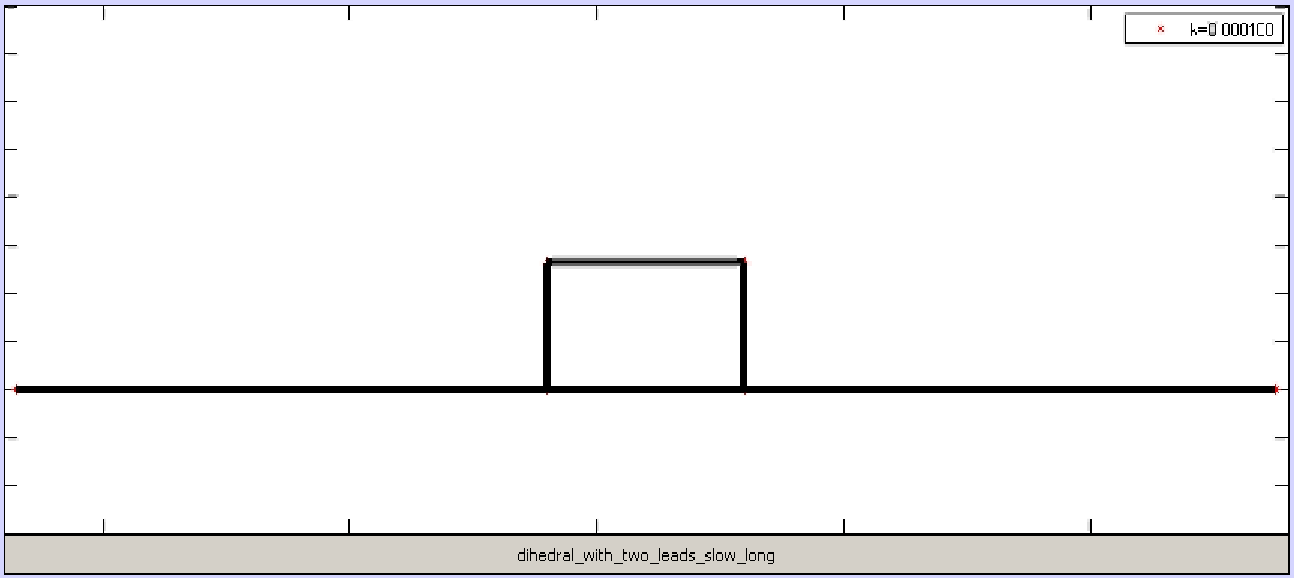
Back to the nodal count of





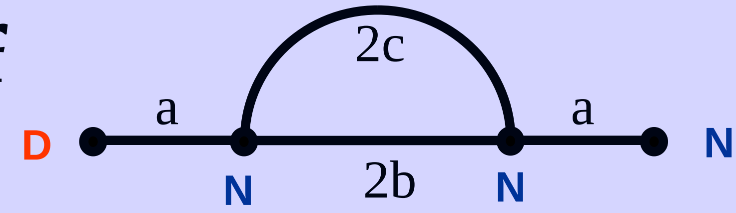
$$C_1 \sin\left(kx_{l_1} + \theta_1(k)\right) \qquad C_2 \sin\left(kx_{l_2} + \theta_2(k)\right)$$

- The same restriction, $\theta_1(k) = \pi/2 + \theta_2(k)$, gives the following movie:

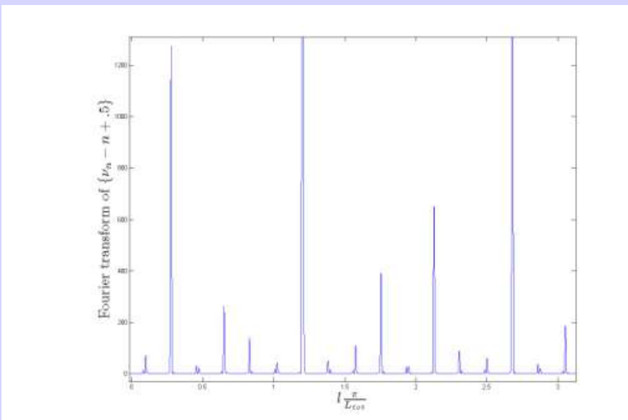


- Conclusions from the movie:
  - Nodal points enter the graph from the leads (as before).
  - An eigenvalue occur when a nodal point enters the graph (as before).
  - However, this time we also have splits and merges of nodal points.

Back to the nodal count of



- Analyzing the movie gives the formula:



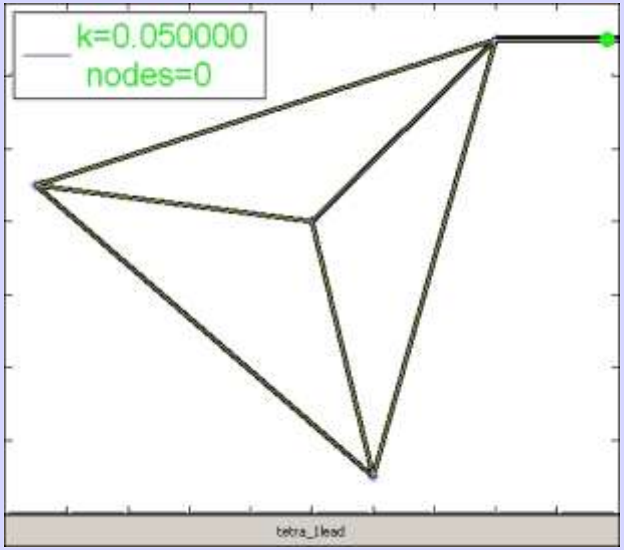
$$\nu_n = n - 0.5 - 0.5 \cdot (-1)^{\left\lfloor \frac{b+c}{a+b+c} n \right\rfloor}$$

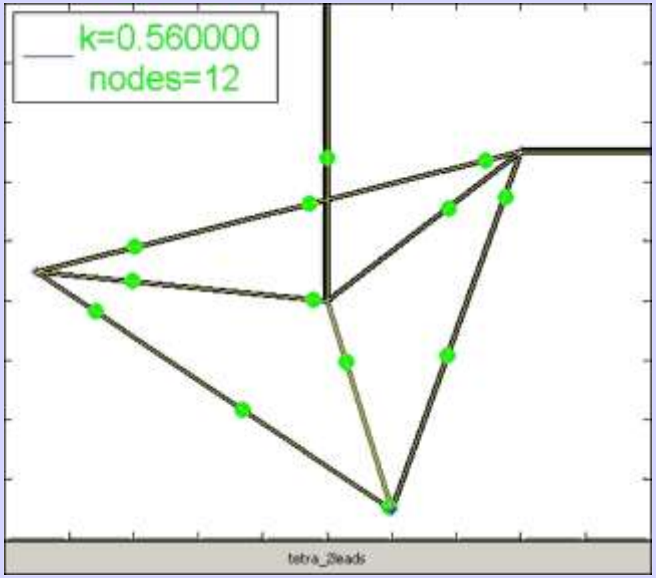
- The nice Fourier transform is due to $(-1)^{\lfloor x \rfloor} = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} \sin((2k+1)\pi x)$

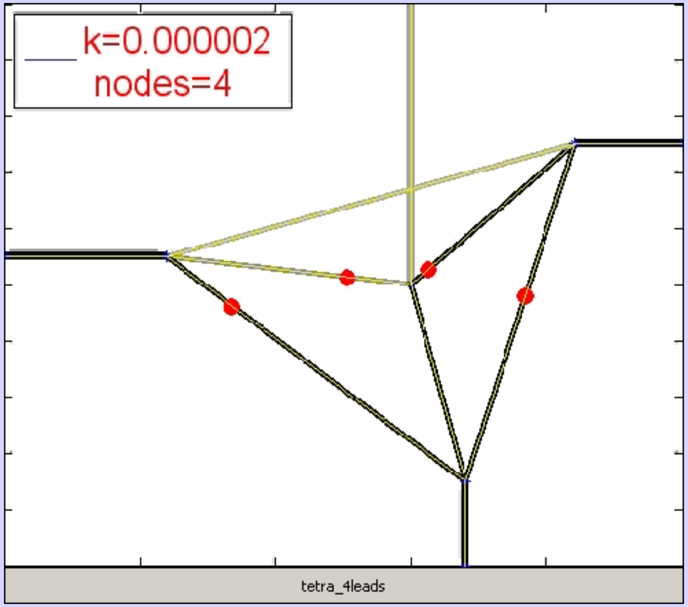
which gives

$$\nu_n = n - 0.5 - \sum_{k=0}^{\infty} \frac{2}{(2k+1)\pi} \sin\left((2k+1) \frac{b+c}{a+b+c} \pi n \right)$$

What's next ?


k=0.050000
nodes=0
tetra_1lead


k=0.560000
nodes=12


k=0.000002
nodes=4
tetra_4leads

- **1, 1, 2, 3, 5, 8, 13, 21, ?**
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- **1, 2, 2, 4, 5, 5, 7, 7, 8, ?**
  - What is the next number?
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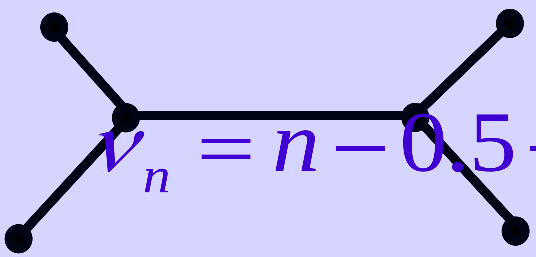
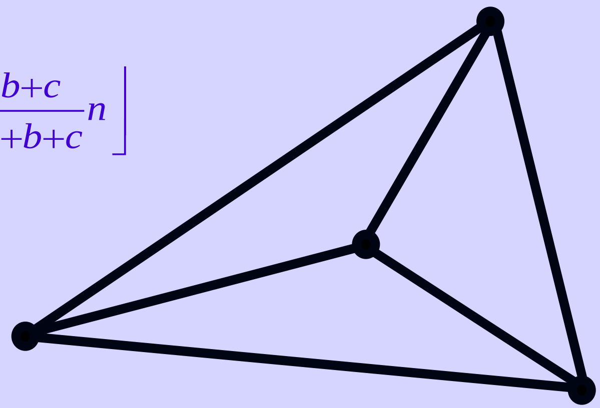
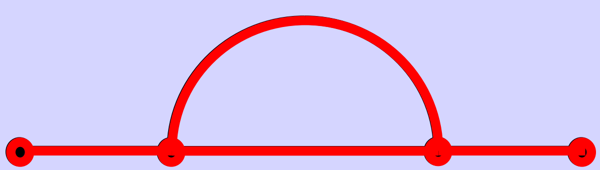


$$\nu_n = n - 0.5 - 0.5\cdot(-1)^{\left\lfloor \frac{b+c}{a+b+c} n \right\rfloor}$$





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