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Linear Representations and Isospectrality with Boundary Conditions

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Abstract We present a method for constructing families of isospectral systems, using linear representations of finite groups. We focus on quantum graphs, for which we give a complete treatment. However, the method presented can be applied to other systems such as manifolds and two-dimensional drums. This is demonstrated by reproducing some known isospectral drums, and new examples are obtained as well. In particular, Sunada's method (Ann. Math. 121, 169–186, 1985) is a special case of the one presented.

Keywords Isospectrality · Symmetry · Hear the shape of a drum · Linear representations · Boundary value problem · Quantum graphs

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1 Introduction

"Can one hear the shape of a drum?"—This question was posed by Marc Kac in 1966 [2]. In other words, is it possible to determine the shape of a planar Euclidean domain from the spectrum of the Laplace operator on it? Though, as Kac accounts in his paper, related questions were raised before, this explicit formulation of the problem inspired a fertile research, investigating it from various aspects. Two main

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approaches were, on the one hand, attempts to deal with the inverse question of reconstructing the shape from the spectrum, and on the other hand, trying to find systems whose shapes are different, yet have the same spectrum. Such examples are called isospectral. Although Kac's original problem regards two dimensional planar drums, the research on isospectrality expanded with time to encompass many types of systems. We will not go into detail here but refer the interested reader to [1–10] for a broader view of the field. However, we will mention here two milestones in the field of isospectrality. A theorem by Sunada gave an important machinery for the construction of isospectral Riemannian manifolds [1]. Later, this method was used by Gordon, Webb and Wolpert to construct the first pair of isospectral planar Euclidean domains [3, 4] thus negatively answering Kac's original question.

This paper starts with a presentation of the basic theory of quantum graphs and existing results on quantum graph isospectrality. We then present the algebraic part of our theory and its main theorem. This is followed by a section which explains the construction of the so called quotient graphs that lie in the heart of the theory. After the theory is fully presented, we apply it to obtain various examples of isospectral quantum graphs. We then demonstrate how to apply the method to other systems, explaining some known results, as well as obtaining new ones. In particular we discuss the relation to Sunada's method. We conclude by pointing out key elements of the theory that are to be investigated further and by presenting open questions.

2 Quantum Graphs

A *graph* Γ consists of a finite set of vertices $V=\{v_i\}$ and a finite set $E=\{e_j\}$ of edges connecting the vertices. We assume that there are no parallel edges (different edges with the same endpoints) or loops (edges connecting a vertex to itself), but we shall see that this inflicts only a small loss of generality. We denote by E_v the set of all edges incident to the vertex v. The degree (valency) of the vertex v is $d_v=|E_v|$. Γ becomes a *metric graph* if each edge $e\in E$ is assigned a finite length $l_e>0$. It is then possible to identify the edge e with a finite segment $[0,l_e]$ of the real line, having the natural coordinate x_e along it. A function on the graph is a vector $f=(f|_{e_1},\dots,f|_{e_{|E|}})$ of functions $f|_{e_j}:[0,l_{e_j}]\to\mathbb{C}$ on the edges. In general, it is not required that for $v\in V$ and $e,e'\in E_v$ the functions $f|_e$ and $f|_{e'}$ agree on v. Endowing the space of functions on the graph with the inner product $\langle f,g\rangle=\sum_{e\in E}\int_0^{l_e}\overline{f|_e}\cdot g|_e\,dx_e$, we obtain the Hilbert space $L_2(\Gamma)=\bigoplus_{e\in E}L_2(e)$. The additional ingredients needed to define a *quantum graph* are a differential operator, by default the negative Laplacian: $-\Delta f=(-f''|_{e_1},\dots,-f''|_{e_{|E|}})$, and boundary conditions which define the domain of this operator. For each vertex $v\in V$, we consider homogeneous boundary conditions involving the values and derivatives of the function at the vertex, of the form $A_v\cdot f|_v+B_v\cdot f'|_v=0$. Here A_v and B_v are $d_v\times d_v$ complex matrices, $f|_v$ is the vector $(f|_{e_{n_1}}(v)\dots f|_{e_{n_{d_v}}}(v))^T$ of the values of f on the edges in E_v at v, and $f'|_v=(f'|_{e_{n_1}}(v)\dots f'|_{e_{n_{d_v}}}(v))^T$ is the vector of outgoing derivatives of f taken at the vertex. The domain of the Laplacian consists of all functions in the Sobolev space $\bigoplus_{e\in E}H^2(e)$ which in addition obey the chosen boundary conditions. We shall denote this space, as is customary, by $H^2(\Gamma)$. We remark that upon moving to generalized functions one must use Sobolev traces in order to consider the values and derivatives at the vertices (see [13]). The reader interested in more information about quantum graphs is referred to the reviews [11–13].

A common choice of boundary conditions which we use is the so called *Neumann* boundary condition[1]:

- f agrees on the vertices: $\forall v \in V\ \forall e, e' \in E_v\ :\ f|_e(v) = f|_{e'}(v)$.
- The sum of outgoing derivatives at each vertex is zero: $\forall v \in V : \sum_{e \in E_v} f'|_e(v) = 0$.

The Neumann boundary condition can thus be represented by the matrices

$$A_v = \begin{pmatrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \\ 0 & \cdots & 0 & 0 \end{pmatrix}, \qquad B_v = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

For a vertex of degree one the Neumann condition is expressed by the matrices $A_v = (0)$, $B_v = (1)$, and means that the derivative of the function is zero at the leaf v. Another natural boundary condition for leaves is the *Dirichlet* boundary condition: $A_v = (1)$, $B_v = (0)$, which means that the function vanishes at the vertex. Notice that before stating the boundary conditions, the graph is merely a collection of independent edges with functions defined separately on each edge. The connectivity of the graph is manifested through the boundary conditions.

Neumann vertices of degree two deserve a special attention. They can be thought of as inner points along a single edge—the concatenation of the two edges incident to the vertex—and one can add or remove such inner points along Γ's edges without changing $L_2(\Gamma)$ and $H^2(\Gamma)$ (see [12]). For example, loops and parallel edges can be eliminated by the introduction of such "dummy" vertices, so that as mentioned, we shall assume that we are dealing with graphs with no such nuisances.

If for every $v \in V$ the $d_v \times 2d_v$ matrix $(A_v \mid B_v)$ is of full rank, we shall say that the quantum graph is *exact*. Non-exact quantum graphs are not very interesting from the spectral point of view, as their eigenvalue spectrum is all of $\mathbb{C}$. On the other hand, we shall later be led to consider the opposite phenomenon, i.e., vertices at which there are "too many" boundary conditions. In this case we shall admit A_v and B_v to be of size $m \times d_v$, possibly with $m > d_v$, and we shall call the corresponding graphs *generalized quantum graphs*. From the spectral perspective these are much more interesting than non-exact quantum graphs. Consider for example a Y-shaped graph, with a Neumann condition at the center, Dirichlet conditions at two of the leaves, and the condition $A_v = \binom{1}{0}$, $B_v = \binom{0}{1}$ at the third; its eigenvalue spectrum is nonempty if and only if the lengths of the two edges with Dirichlet leaves are commensurable.

Kostrykin and Schrader [14] provide necessary and sufficient conditions for the Laplacian to be self-adjoint. These can be stated in a number of equivalent forms [13], of which we list two:

[1] This condition is also widely encountered under the name of *Kirchhoff* condition.

(1) Γ is exact, and $A_v \cdot B_v^\dagger$ is self-adjoint for every $v \in V$.
(2) For every $v \in V$ there exist a unitary matrix U such that $(A_v \mid B_v)$ is row-equivalent to $(i(U-I) \mid U+I)$.

In particular, Neumann and Dirichlet conditions satisfy these requirements. When the Laplacian is self-adjoint, the spectrum is real, discrete and bounded from below [12]. In particular, it consists entirely of eigenvalues. In the general case (i.e., non self-adjoint Laplacian) one has to make the distinction between the eigenvalue spectrum and the general spectrum of Δ, i.e., the values of λ for which $-\lambda I - \Delta$ is not invertible. In this paper we shall consider only the eigenvalue spectrum, with the exception of Sect. 4.3.2.

There are several known results concerning isospectrality of quantum graphs. Gutkin and Smilansky [16] show that under certain conditions a quantum graph can be heard, meaning that it can be recovered from the spectrum of its Laplacian. On the other hand, constructions of isospectral graphs were also established, by various means: by a trace formula for the heat kernel [17], by turning isospectral discrete graphs into equilateral quantum graphs [6], and weighted discrete graphs into non-equilateral ones [18]; in [16, 19] a wealth of examples is given by an analogy to the isospectral drums obtained by Buser et al. [5], and in [20] is presented an example, whose generalization has led to the theory presented in this paper.

3 Algebra

For a quantum graph Γ and $\lambda \in \mathbb{C}$ we denote by $\Phi_\Gamma(\lambda)$ the λ-eigenspace of Γ's Laplacian, i.e.,

$$\Phi_\Gamma(\lambda) = \ker_{H^2\Gamma}(-\lambda I - \Delta) = \{f \in H^2(\Gamma) \mid -\Delta f = \lambda f\}.$$

The *(eigenvalue, or point) spectrum* of Γ is the function

$$\sigma_\Gamma : \lambda \mapsto \dim_{\mathbb{C}} \Phi_\Gamma(\lambda),$$

which assigns to each eigenvalue of Γ's Laplacian its multiplicity.[2] We say that the quantum graphs Γ and Γ' are *isospectral* when their eigenvalue spectra coincide, that is $\sigma_\Gamma \equiv \sigma_{\Gamma'}$. In Sect. 4.3.2 we shall say a word about isospectrality in the stronger sense (i.e., not only of the eigenvalue spectrum).

A symmetry σ of a quantum graph Γ is a graph automorphism which preserves both the lengths of edges and the boundary conditions at the vertices. Formally, the latter means that for every vertex v, if $E_v = (e_{n_1}, \dots, e_{n_r})$ and $E_{\sigma v} = (e_{m_1}, \dots, e_{m_r})$, then $(A_{\sigma v} \mid B_{\sigma v})$ is row-equivalent to $(A_v \mid B_v)\left(\begin{smallmatrix} P & 0 \\ 0 & P \end{smallmatrix}\right)$, where P is the permutation matrix defined by $P_{ij} = 1 \Leftrightarrow \sigma e_{n_i} = e_{m_j}$. The group of all such symmetries is denoted $\operatorname{Aut}\Gamma$. A left action of a group G on Γ is equivalent to a group homomorphism $G \to \operatorname{Aut}\Gamma$. Such action induces a left action of G on $H^2(\Gamma)$ (by

[2] In effect we have $\sigma_\Gamma : \mathbb{C} \to \{0, \dots, 2|E|\}$, as the eigenvalue of a Laplacian eigenfunction, together with the values $\{f|_e(0), f'|_e(0)\}_{e \in E}$, determine the function.

$(gf)(x) = f(g^{-1}x)$—the inversion accounts for the contravariantness of H^2), which gives $H^2(\Gamma)$ the structure of a $\mathbb{C}G$-module, i.e., a complex representation. As the actions of G and Δ commute, the eigenspaces $\Phi_\Gamma(\lambda)$ are subrepresentations of $H^2(\Gamma)$. Assuming that G is finite, with irreducible complex representations $S_1, \ldots, S_r$, we can decompose each eigenspace to its isotypic components:

$$\Phi_\Gamma(\lambda) = \bigoplus_{i=1}^{r} \Phi_\Gamma^{S_i}(\lambda), \tag{3.1}$$

where $\Phi_\Gamma^{S_i}(\lambda) \cong S_i \oplus \cdots \oplus S_i$ as $\mathbb{C}G$-modules.

We start by counting separately, for each irreducible representation S of G, only the λ-eigenfunctions which reside in $\Phi_\Gamma^S(\lambda)$. This means that we are restricting our attention to functions which under the action of $\mathbb{C}G$ span a space that is isomorphic, as a representation of G, to S. However, since $\dim S$ always divides $\dim \Phi_\Gamma^S(\lambda)$, we can already normalize by it. We thus define *the S-spectrum of* Γ as

$$\sigma_\Gamma^S : \lambda \mapsto \dim_{\mathbb{C}} \Phi_\Gamma^S(\lambda) / \dim_{\mathbb{C}} S. \tag{3.2}$$

By the orthogonality relations of irreducible characters, we can rewrite this as $\sigma_\Gamma^S(\lambda) = \langle \chi_S, \chi_{\Phi_\Gamma(\lambda)} \rangle_G$, and expanding this linearly, we define *the R-spectrum of* Γ, for every representation R of G, to be

$$\sigma_\Gamma^R : \lambda \mapsto \langle \chi_R, \chi_{\Phi_\Gamma(\lambda)} \rangle_G. \tag{3.3}$$

$\sigma_\Gamma^S(\lambda)$ has an algebraic significance: it reflects the size of the S-isotypic part of $\Phi_\Gamma(\lambda)$. Looking for a parallel algebraic interpretation of $\sigma_\Gamma^R(\lambda)$, we find that

$$\sigma_\Gamma^R(\lambda) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}G}(R, \Phi_\Gamma(\lambda)).$$

The Laplacian is naturally defined on $H^2(\Gamma)^S$, the S-isotypic component of $H^2(\Gamma)$. A parallel definition for $\mathrm{Hom}_{\mathbb{C}G}(R, H^2(\Gamma))$, with R a general representation, would be $(\Delta f)(r) = \Delta(f(r))$ ($r \in R$, $f \in \mathrm{Hom}_{\mathbb{C}G}(R, H^2(\Gamma))$. In the language of category theory, this "Laplacian" is simply the functor $\mathrm{Hom}_{\mathbb{C}G}(R, \_)$ applied to Γ's Laplacian, i.e., $\mathrm{Hom}_{\mathbb{C}G}(R, \Delta)$. The spectrum of $\Delta = \mathrm{Hom}_{\mathbb{C}G}(R, \Delta)$ corresponds to the R-spectrum of Γ, in the sense that

$$\ker_{\mathrm{Hom}_{\mathbb{C}G}(R, H^2(\Gamma))}(-\lambda I - \Delta) = \mathrm{Hom}_{\mathbb{C}G}\,(R, \Phi_\Gamma(\lambda))$$

gives $\sigma_\Gamma^R(\lambda) = \dim_{\mathbb{C}} \ker_{\mathrm{Hom}_{\mathbb{C}G}(R, H^2(\Gamma))}(-\lambda I - \Delta)$. This motivates our central definition:

Definition 1 Let R be a representation of a group G which acts on a quantum graph Γ. A Γ/R*-graph* is any quantum graph Γ' such that there is an isomorphism $L_2(\Gamma') \cong \mathrm{Hom}_{\mathbb{C}G}(R, L_2(\Gamma))$ which restricts to an isomorphism of H^2, and commutes with the Laplacian, i.e.:

$$\begin{array}{ccc} H^2\left(\Gamma'\right) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathbb{C}G}(R, H^2(\Gamma)) \\ {\scriptstyle \Delta_{\Gamma'}}\downarrow & & \downarrow{\scriptstyle \Delta=\mathrm{Hom}_{\mathbb{C}G}(R,\Delta_\Gamma)} \\ L_2\left(\Gamma'\right) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathbb{C}G}\left(R, L_2(\Gamma)\right) \end{array} \quad . \tag{3.4}$$

Applying $\ker(-\lambda I - \Delta)$ to (3.4) we obtain $\Phi_{\Gamma'}(\lambda) \cong \mathrm{Hom}_{\mathbb{C}G}(R, \Phi_\Gamma(\lambda))$, which by taking dimensions translates to an equality of spectra:

$$\sigma_{\Gamma'} \equiv \sigma_\Gamma^R. \tag{3.5}$$

Since σ_Γ^R is not a spectrum of a graph, we cannot really call this isospectrality. However we do have from this that all Γ/R-graphs are isospectral to one another, and we will use this to speak non-rigorously about "the spectrum of Γ/R", $\sigma_{\Gamma/R} \equiv \sigma_\Gamma^R$. The following proposition exhibits another manifestation of isospectrality.

Proposition 2 *All $\Gamma/\mathbb{C}G$-graphs are isospectral to Γ.*

Proof By (3.1), (3.2), and linearity, the classical spectrum of Γ coincides with its $\mathbb{C}G$-spectrum:

$$\sigma_\Gamma^{\mathbb{C}G} \equiv \sum_{i=1}^{r} \dim S_i \cdot \sigma_\Gamma^{S_i} \equiv \sigma_\Gamma. \tag{3.6}$$

This can also be deduced from the fact that for every R-module M there is an isomorphism $\mathrm{Hom}_R(R, M) \cong M$, so that we have $\mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}G, \Phi_\Gamma(\lambda)) \cong \Phi_\Gamma(\lambda)$ for every eigenvalue λ. □

We can say even more:

Theorem 3 *Let Γ be a quantum graph equipped with an action of G, H a subgroup of G, and R a representation of H. Then Γ/R is isospectral to $\Gamma/\mathrm{Ind}_H^G R$.*

Proof The Laplacians of Γ/R and $\Gamma/\mathrm{Ind}_H^G R$ are conjugate, by the following diagram:

$$\begin{array}{ccccccc} H^2(\Gamma/R) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathbb{C}H}\left(R, H^2(\Gamma)\right) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathbb{C}G}\left(\mathrm{Ind}_H^G R, H^2(\Gamma)\right) & \xrightarrow{\cong} & H^2\left(\Gamma/\mathrm{Ind}_H^G R\right) \\ {\scriptstyle \Delta_{\Gamma/R}}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \Delta_{\Gamma/\mathrm{Ind}_H^G R}} \\ L_2(\Gamma/R) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathbb{C}H}\left(R, L_2(\Gamma)\right) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathbb{C}G}\left(\mathrm{Ind}_H^G R, L_2(\Gamma)\right) & \xrightarrow{\cong} & L_2\left(\Gamma/\mathrm{Ind}_H^G R\right) \end{array} .$$

The isomorphisms in the middle square are given by the Frobenius Reciprocity theorem, and this square is commutative by the naturality of the Frobenius Isomorphism. Note that from the formal point of view, we have actually shown that Γ/R and $\Gamma/\mathrm{Ind}_H^G R$ are identical (as classes of quantum graphs). □

Remark This gives yet another explanation for the equality of the classical spectrum with that of the regular representation (Proposition 2): for $H = \{id\}$ and $\mathbf{1}_H$ its trivial representation, it is clear by the isotypic component perspective that $H^2(\Gamma)^{\mathbf{1}_H} = H^2(\Gamma)$, so that (3.6) follows from $\mathrm{Ind}_H^G \mathbf{1}_H \cong \mathbb{C}G$.

Corollary 4 *If G acts on Γ and H_1, H_2 are subgroups of G with corresponding representations R_1, R_2, such that $\mathrm{Ind}_{H_1}^G R_1 \cong \mathrm{Ind}_{H_2}^G R_2$, then Γ/R_1 and Γ/R_2 are isospectral.*

Remark This corollary is in fact equivalent to the theorem, which follows by taking $H_2 = G$, $R_2 = \mathrm{Ind}_{H_1}^G R_1$. It is presented for being of practical usefulness (it allows one to work with representations of lower dimension, as can be seen in Sect. 5), but also since it indicates the bridge connecting our method with the classical one of Sunada. In Sect. 6.3, we shall cross it.

The observations made in this section would be mere algebraic tautologies, unless we can show that Γ/R-graphs do exist. The next section is devoted to this purpose.

4 Building Γ/R-Graphs

In this section we prove the existence of the quotient graphs Γ/R. This is done by describing an explicit construction of Γ/R, given a graph Γ, a representation R of some group G acting on the graph, and various choices of bases for this representation. As the lengthy technical details of the construction might encloud the essence of the method, the reader may prefer to go over Sect. 5 first, and obtain an intuition for the construction of the quotient graph from the examples presented there. More intuition for the construction can be gained from the examples in [15].

We summarize the main conclusions of this section in the following theorem:

Theorem 5 *For any representation R of a finite group G, which acts upon a quantum graph Γ, there exists a generalized Γ/R quantum graph. Furthermore, if Γ's Laplacian is self-adjoint, then there exists a proper Γ/R quantum graph, and it is exact.*

4.1 Intuition

A motivation for the construction of our quotient graphs is given by thinking about it as an "encoding scheme".[3] An element $\tilde{f}$ in $\mathrm{Hom}_{\mathbb{C}G}(R, H^2(\Gamma))$ can be thought

[3] In Sect. 6 we show that the same construction and motivation can be applied analogously to other geometric systems.

of as a family of functions on Γ, parametrized by R. To emphasize this view, we shall write $\tilde{f}_r$ instead of $\tilde{f}(r)$ (where $r \in R$). Our goal is to build a new graph, each of whose H^2-functions encodes exactly one such family. The desired map $\Psi : \mathrm{Hom}_{\mathbb{C}G}(R, H^2(\Gamma)) \to H^2(\Gamma/R)$ (see Definition 1) is in fact this encoding. An encoding scheme should always be injective (in order to allow decoding), but we have also required Ψ to be surjective: this can be translated to the idea that the encoding must be "as efficient as possible"[4]—that Γ/R is to be a "minimal" graph allowing such an encoding, since it admits no functions apart from the ones used by the scheme.

First, we reduce the infinite family $\tilde{f}$ to a finite one by choosing a basis $B = \{b_j\}_{j=1}^d$ for R, and restricting our attention to $\{\tilde{f}_{b_j}\}_{j=1}^d$. From these "basis functions" we can reconstruct $\tilde{f}$, since the $\mathbb{C}G$-linearity of $\tilde{f}$ implies in particular $\mathbb{C}$-linearity (i.e., $\tilde{f}_{\sum \alpha_j b_j} = \sum \alpha_j \tilde{f}_{b_j}$). As a first encoding attempt we could take a graph with d times each edge in Γ, and let the jth copy of the edge e carry the jth basis function restricted to e. That is, define $(\Psi \tilde{f})|_{e_j} \equiv \tilde{f}_{b_j}|_e$. However, this encoding is not efficient enough, since we have used only $\mathbb{C}$-linearity. For each $g \in G$, $\mathbb{C}G$-linearity implies that $\{\tilde{f}_r|_e\}_{r\in R}$ determines $\{\tilde{f}_r|_{ge}\}_{r\in R}$, specifically by

$$\tilde{f}_r\big|_{ge} \equiv (g^{-1}\tilde{f}_r)\big|_e \equiv \tilde{f}_{g^{-1}r}\big|_e \tag{4.1}$$

(the inversion occurs since G acts on $H^2(\Gamma)$ by $g \cdot f = f \circ g^{-1}$). Thus, it suffices to encode the basis functions on only **one** edge from each G-orbit of Γ's edges.

It turns out that if the action of G on E is free, then apart from determining the appropriate boundary conditions at the vertices we are done: for $\{e^i\}$, a choice of representatives for E/G, setting $(\Psi \tilde{f})|_{e^i_j} \equiv \tilde{f}_{b_j}|_{e^i}$ (where $1 \le j \le d$) is indeed a "good" encoding (i.e., once the boundary conditions are correctly stated, Ψ is an isomorphism.)

If, however, some edge $e = \{v, v'\}$ has a non-trivial stabilizer $G_e = G_v \cap G_{v'}$, then greater efficiency can (and therefore must) be achieved. For example, assume that $\dim R = 1$ and that for some $g \in G_e$ we have $g \notin \ker \rho_R$, where ρ_R is the structure homomorphism $G \to GL_1(\mathbb{C})$. We then have

$$\tilde{f}_r\big|_e \equiv \tilde{f}_r\big|_{g^{-1}e} \equiv \tilde{f}_{gr}\big|_e \equiv \tilde{f}_{\rho_R(g)\cdot r}\big|_e \equiv \rho_R(g) \cdot \tilde{f}_r\big|_e$$

which implies that $\tilde{f}_r|_e \equiv 0$ for all r, and as a result, the edge e need not have **any** representative in the quotient. We can "decode $\tilde{f}_r|_e$ from thin air", since we know in advance that it can only be the zero function. The generalization of this observation is that for each edge e, the information in $\{\tilde{f}_r|_e\}_{r\in R}$ is encapsulated in R^{G_e} [5]: if r belongs to a nontrivial component of $\mathrm{Res}^G_{G_e} R$, then $\tilde{f}_r|_e \equiv 0$. Therefore, we need only $d_i = \dim R^{G_{e^i}}$ copies of the representative e^i in the quotient.[6] This further "com-

[4]In a suitable sense, since better encoding may exist, but we want the encoding to be by another quantum graph, in a manner which intertwines the corresponding Laplacians.

[5]R^H is the trivial component of $\mathrm{Res}^G_H R$, i.e. $R^H = \{r \in R \mid \forall h \in H : hr = r\}$.

[6]However, we shall later find it convenient to think about $d = \dim R$ copies, where the $d_i + 1 \ldots d$ copies are "dead", meaning that whenever a function on them appears in a formula it is to be understood as zero.

pression" slightly complicates the determination of the boundary conditions. When G acted freely on the edges, we had d functions, $\tilde{f}_{b_j}$, each satisfying the boundary conditions at the vertices of Γ, and we could have translated this quite easily to boundary conditions on the quotient. Now, however, for each edge e^i we need to encode a "function basis" $\{\tilde{f}_{b^i_j}|_{e^i}\}$, where $\{b^i_j\}$ is a basis for $R^{G_{e^i}}$. Since for different e^i's the spaces $R^{G_{e^i}}$ need not even overlap, we now have only function-chunks, indexed by different R-elements for each edge, and no function on the whole of Γ to extract boundary conditions from. Fortunately, algebra is generous and this complication turns out to be solvable.

4.2 Method

We now present the actual construction procedure. Assume we have a representation R of a group G acting on the quantum graph $\Gamma = (E, V)$, and we have chosen representatives $\{\tilde{e}^i\}_{i=1}^{I}$ for the orbits E/G, and likewise $\{\tilde{v}_k\}_{k=1}^{K}$ for V/G. We have also chosen an ordered basis $B = (b_j)_{j=1}^{d}$ for R, and for each $i = 1..I$ another ordered basis for R, $B^i = (b^i_j)_{j=1}^{d}$, such that $\{b^i_j\}_{j=1}^{d_i}$ is a basis for $R^{G_{\tilde{e}^i}}$ and each b^i_j with $j > d_i = \dim R^{G_{\tilde{e}^i}}$ lies in a nontrivial component of $\mathrm{Res}^G_{G_{\tilde{e}^i}} R$.

The quotient graph Γ/R obtained from these choices is defined to have $\{v_k\}_{k=1}^{K}$ as its set of vertices, and $\{e^i_j\}_{j=1..d_i}^{i=1..I}$ for edges, where each e^i_j is of length $l_{\tilde{e}^i}$. If $\tilde{e}^i$ connects $g\tilde{v}_k$ to $g'\tilde{v}_{k'}$ in Γ, then, for all j, e^i_j connects v_k to $v_{k'}$ in Γ/R. We shall assume, by adding "dummy" vertices if needed, that G does not carry any vertex in V to one of its neighbors. This serves two purposes:

(1) It means that Γ/R has no loops; i.e., that $k \neq k'$ in the notation above. This allows us to speak of $f|_{e^i_j}(v)$, the value of e^i_j at v, without confusion regarding which end of e^i_j is meant.
(2) It assures that an edge is not transformed onto itself in the opposite direction, in which case we would have had to take only half of the edge as a representative for its orbit. Differently put, it assures that the fixed points set of each $g \in G$ is a subgraph.

Note that in order that G still act on the graph, the dummy vertices are to be introduced in accordance with its action, i.e., if a vertex is placed at $x \in (0, l_{\tilde{e}})$ along $\tilde{e}$, one should also be placed at x along $g\tilde{e}$, for every $g \in G$.

We can now define Ψ on $\mathrm{Hom}_{\mathbb{C}G}(R, L_2(\Gamma))$:

$$\left(\Psi \tilde{f}\right)\Big|_{e^i_j} \overset{def}{\equiv} \tilde{f}_{b^i_j}\big|_{\tilde{e}^i},$$

and it is clear that Ψ restricts to a homomorphism from $\mathrm{Hom}_{\mathbb{C}G}(R, \bigoplus_{\tilde{e}\in E_\Gamma} H^2(\tilde{e}))$ to $\bigoplus_{e\in E_{\Gamma/R}} H^2(e)$, which intertwines the corresponding Laplacians. We shall later determine vertex conditions for Γ/R that will ensure that Ψ further restricts to an isomorphism $\mathrm{Hom}_{\mathbb{C}G}(R, H^2(\Gamma)) \cong H^2(\Gamma/R)$, but first we establish some properties of Ψ which are independent of the boundary conditions. We start by rephrasing (4.1) basis-wise, and we make the following convention: an expression in bold is to be understood as a row vector of length d, where the #-symbol indicates the place of the index; e.g., $\tilde{\boldsymbol{f}}_{\boldsymbol{b}_\#}|_{\boldsymbol{e}}$ stands for $(\tilde{f}_{b_1}|_e, \ldots, \tilde{f}_{b_d}|_e)$.

Consider $\tilde{f}_r|_{g\tilde{e}^i}$, an arbitrary function in the family $\tilde{f}$ evaluated on an arbitrary edge. Write $r \in R$ as $\boldsymbol{b} \cdot \boldsymbol{\alpha}$, where $\boldsymbol{b} = (b_1, \ldots, b_d) \in M_{1\times d}(R)$ and $\boldsymbol{\alpha} \in M_{d\times 1}(\mathbb{C})$. $r = \boldsymbol{b} \cdot \boldsymbol{\alpha}$ implies $gr = \boldsymbol{b}^{\boldsymbol{i}}_{\#} \cdot [\rho_R(g)]^B_{B^i} \cdot \boldsymbol{\alpha}$, and therefore, by (4.1) we have $\tilde{f}_r|_{g\tilde{e}^i} = \tilde{f}_{\boldsymbol{b}^{\boldsymbol{i}}_{\#}[\rho_R(g^{-1})]^B_{B^i}\boldsymbol{\alpha}}|_{\tilde{e}^i}$. Linearity now implies

$$\tilde{f}_r\big|_{g\tilde{e}^i} \equiv \tilde{f}_{\boldsymbol{b}^{\boldsymbol{i}}_{\#}[g^{-1}]^B_{B^i}\boldsymbol{\alpha}}\big|_{\tilde{e}^i} \equiv \tilde{\boldsymbol{f}}_{\boldsymbol{b}^{\boldsymbol{i}}_{\#}}\big|_{\tilde{\boldsymbol{e}}^{\boldsymbol{i}}} \cdot [g^{-1}]^B_{B^i} \cdot \boldsymbol{\alpha} \equiv (\boldsymbol{\Psi}\tilde{\boldsymbol{f}})\big|_{\boldsymbol{e}^{\boldsymbol{i}}_{\#}} \cdot [g^{-1}]^B_{B^i} \cdot \boldsymbol{\alpha},$$

where ρ_R is understood, $\tilde{\boldsymbol{f}}_{\boldsymbol{b}^{\boldsymbol{i}}_{\#}}|_{\tilde{\boldsymbol{e}}^{\boldsymbol{i}}} = (\tilde{f}_{b^i_1}|_{\tilde{e}^i}, \ldots, \tilde{f}_{b^i_d}|_{\tilde{e}^i})$, and

$$(\boldsymbol{\Psi}\tilde{\boldsymbol{f}})\big|_{\boldsymbol{e}^{\boldsymbol{i}}_{\#}} = \big((\Psi\tilde{f})\big|_{e^i_1}, \ldots, (\Psi\tilde{f})\big|_{e^i_{d_i}}, 0, \ldots, 0\big),$$

since for $j > d_i$ we have seen that $\tilde{f}_{b^i_j}|_{\tilde{e}^i} \equiv 0$, and we therefore did not include the corresponding e^i_j edge in Γ/R (it is "dead"—see footnote 6). We now see that for any $f \in L_2(\Gamma/R)$ the inverse of Ψ must be given by:

$$(\Psi^{-1}f)_{\mathbf{b}\cdot\boldsymbol{\alpha}}\big|_{g\tilde{e}^i} \equiv \boldsymbol{f}\big|_{\boldsymbol{e}^{\boldsymbol{i}}_{\#}} \cdot [g^{-1}]^B_{B^i} \cdot \boldsymbol{\alpha},$$

(again $[g^{-1}]^B_{B^i}$ stands for $[\rho_R(g^{-1})]^B_{B^i}$), so we need to establish that the r.h.s does not depend on the choice of g. We observe that if g and g' are two possible choices then $g^{-1}g' \in G_{\tilde{e}^i}$, and by the construction of B^i we have $[g^{-1}g']_{B^i} = \left(\begin{array}{c|c} I_{d_i} & 0 \\ \hline 0 & * \end{array}\right)$. As we have agreed that $\boldsymbol{f}|_{\boldsymbol{e}^{\boldsymbol{i}}_{\#}} = (f|_{e^i_1}, \ldots, f|_{e^i_{d_i}}, 0, \ldots, 0)$, we have $\boldsymbol{f}|_{\boldsymbol{e}^{\boldsymbol{i}}_{\#}} \cdot [g^{-1}g']_{B^i} = \boldsymbol{f}|_{\boldsymbol{e}^{\boldsymbol{i}}_{\#}}$ and thus

$$\boldsymbol{f}\big|_{\boldsymbol{e}^{\boldsymbol{i}}_{\#}} \cdot [g'^{-1}]^B_{B^i} = \boldsymbol{f}\big|_{\boldsymbol{e}^{\boldsymbol{i}}_{\#}} \cdot [g^{-1}g']_{B^i} \cdot [g'^{-1}]^B_{B^i} = \boldsymbol{f}\big|_{\boldsymbol{e}^{\boldsymbol{i}}_{\#}} \cdot [g^{-1}]^B_{B^i},$$

so that Ψ^{-1} is well defined. We thus have an isomorphism $\Psi : \mathrm{Hom}_{\mathbb{C}G}(R, L_2(\Gamma)) \cong L_2(\Gamma/R)$, which clearly restricts to $\mathrm{Hom}_{\mathbb{C}G}(R, \bigoplus_{\tilde{e}\in E_\Gamma} H^2(\tilde{e})) \cong \bigoplus_{e\in E_{\Gamma/R}} H^2(e)$, and we are left to handle the boundary conditions.

We now determine matrices A_{v_k} and B_{v_k} for the vertex v_k from the matrices $A_{\tilde{v}_k}$, $B_{\tilde{v}_k}$ of the vertex $\tilde{v}_k$. Assume that the edges entering $\tilde{v}_k$ are $g_1\tilde{e}^{\nu_1}, \ldots, g_n\tilde{e}^{\nu_n}$ (where $n = d_{\tilde{v}_k}$), so that a function f on Γ satisfies the vertex conditions at $\tilde{v}_k$ when

$$A_{\tilde{v}_k} \cdot f\big|_{\tilde{v}_k} + B_{\tilde{v}_k} \cdot f'\big|_{\tilde{v}_k} = 0,$$

where we recall from Sect. 2 that

$$f\big|_{\tilde{v}_k} = \big(f\big|_{g_1\tilde{e}^{\nu_1}}(\tilde{v}_k) \quad \ldots \quad f\big|_{g_n\tilde{e}^{\nu_n}}(\tilde{v}_k)\big)^T,$$

$$f'\big|_{\tilde{v}_k} = \big(f'\big|_{g_1\tilde{e}^{\nu_1}}(\tilde{v}_k) \quad \ldots \quad f'\big|_{g_n\tilde{e}^{\nu_n}}(\tilde{v}_k)\big)^T.$$

$\tilde{f} \in \mathrm{Hom}_{\mathbb{C}G}(R, H^2(\Gamma))$ means that $\tilde{f}_r$ satisfies the conditions at $\tilde{v}_k$ for all $r \in R$, which happens iff the basis functions $\{\tilde{f}_{b_j}\}_{j=1}^d$ satisfy them. Thus, if we define the

$n \times d$ matrix

$$\tilde{\boldsymbol{f}}_{\boldsymbol{b}}|_{\tilde{\boldsymbol{v}}_{\boldsymbol{k}}} = \begin{pmatrix} \tilde{f}_{b_1}|_{g_1 \tilde{e}^{\nu_1}}(\tilde{v}_k) & \cdots & \tilde{f}_{b_d}|_{g_1 \tilde{e}^{\nu_1}}(\tilde{v}_k) \\ \vdots & \ddots & \vdots \\ \tilde{f}_{b_1}|_{g_n \tilde{e}^{\nu_n}}(\tilde{v}_k) & \cdots & \tilde{f}_{b_d}|_{g_n \tilde{e}^{\nu_n}}(\tilde{v}_k) \end{pmatrix}$$

$$= \left(\tilde{f}_{b_1}|_{\tilde{v}_k} \quad \cdots \quad \tilde{f}_{b_d}|_{\tilde{v}_k} \right) = \begin{pmatrix} \tilde{\boldsymbol{f}}_{\boldsymbol{b}_\#}|_{\boldsymbol{g}_1 \tilde{\boldsymbol{e}}^{\boldsymbol{\nu}_1}}(\tilde{\boldsymbol{v}}_{\boldsymbol{k}}) \\ \vdots \\ \tilde{\boldsymbol{f}}_{\boldsymbol{b}_\#}|_{\boldsymbol{g}_n \tilde{\boldsymbol{e}}^{\boldsymbol{\nu}_n}}(\tilde{\boldsymbol{v}}_{\boldsymbol{k}}) \end{pmatrix},$$

and analogously $\tilde{\boldsymbol{f}}'_{\boldsymbol{b}}|_{\tilde{\boldsymbol{v}}_{\boldsymbol{k}}}$, then we need only check that

$$A_{\tilde{v}_k} \cdot \tilde{\boldsymbol{f}}_{\boldsymbol{b}}\big|_{\tilde{\boldsymbol{v}}_{\boldsymbol{k}}} + B_{\tilde{v}_k} \cdot \tilde{\boldsymbol{f}}'_{\boldsymbol{b}}\big|_{\tilde{\boldsymbol{v}}_{\boldsymbol{k}}} = 0_{n \times d}. \tag{4.2}$$

In addition, we note that if the boundary conditions are met by $\tilde{f}$ at $\tilde{v}_k$, then they are also met at any vertex in the orbit $G \cdot \tilde{v}_k$, since G's elements are assumed to preserve boundary conditions (see Sect. 3).

For a $n \times m$ matrix $X = ((x_{ij}))$ we define its row-wise-vectorization to be the $nm \times 1$ matrix

$$\operatorname{rv} X \stackrel{def}{=} \begin{pmatrix} (x_{11}, \ldots, x_{1m})^T \\ \vdots \\ (x_{n1}, \ldots, x_{nm})^T \end{pmatrix} = (x_{11}, x_{12}, \ldots, x_{1m}, x_{21}, \ldots\ldots, x_{nm})^T.$$

Vectorization is linear, and it behaves quite nicely under matrix multiplication. Specifically, $\operatorname{rv}(X \cdot Y \cdot Z) = (X \otimes Z^T) \cdot \operatorname{rv} Y$ whenever $X \cdot Y \cdot Z$ is defined. This allows us to write (4.2) as

$$(A_{\tilde{v}_k} \otimes I_d) \cdot \operatorname{rv} \tilde{\boldsymbol{f}}_{\boldsymbol{b}}\big|_{\tilde{\boldsymbol{v}}_{\boldsymbol{k}}} + (B_{\tilde{v}_k} \otimes I_d) \cdot \operatorname{rv} \tilde{\boldsymbol{f}}'_{\boldsymbol{b}}\big|_{\tilde{\boldsymbol{v}}_{\boldsymbol{k}}} = 0_{nd \times 1}. \tag{4.3}$$

Recalling that $\tilde{\boldsymbol{f}}_{\boldsymbol{b}_\#}|_{\boldsymbol{g}_i \tilde{\boldsymbol{e}}^{\boldsymbol{\nu}_i}} = \boldsymbol{\Psi} \tilde{\boldsymbol{f}}|_{\boldsymbol{e}_\#^{\boldsymbol{\nu}_i}} \cdot [g_i^{-1}]^B_{B^{\nu_i}}$, we have

$$\operatorname{rv} \tilde{\boldsymbol{f}}_{\boldsymbol{b}}\big|_{\tilde{\boldsymbol{v}}_{\boldsymbol{k}}} = \begin{pmatrix} \tilde{\boldsymbol{f}}_{\boldsymbol{b}_\#}|_{\boldsymbol{g}_1 \tilde{\boldsymbol{e}}^{\boldsymbol{\nu}_1}}(\tilde{\boldsymbol{v}}_{\boldsymbol{k}})^T \\ \vdots \\ \tilde{\boldsymbol{f}}_{\boldsymbol{b}_\#}|_{\boldsymbol{g}_n \tilde{\boldsymbol{e}}^{\boldsymbol{\nu}_n}}(\tilde{\boldsymbol{v}}_{\boldsymbol{k}})^T \end{pmatrix} = \begin{pmatrix} ([g_1^{-1}]^B_{B^{\nu_1}})^T \cdot (\boldsymbol{\Psi} \tilde{\boldsymbol{f}}|_{\boldsymbol{e}_\#^{\boldsymbol{\nu}_1}}(\boldsymbol{v}_{\boldsymbol{k}}))^T \\ \vdots \\ ([g_n^{-1}]^B_{B^{\nu_n}})^T \cdot (\boldsymbol{\Psi} \tilde{\boldsymbol{f}}|_{\boldsymbol{e}_\#^{\boldsymbol{\nu}_n}}(\boldsymbol{v}_{\boldsymbol{k}}))^T \end{pmatrix}$$

$$= \operatorname{diag}\big([g_1^{-1}]^B_{B^{\nu_1}}, \ldots, [g_n^{-1}]^B_{B^{\nu_n}}\big)^T \cdot \operatorname{rv} \begin{pmatrix} \boldsymbol{\Psi} \tilde{\boldsymbol{f}}|_{\boldsymbol{e}_\#^{\boldsymbol{\nu}_1}}(\boldsymbol{v}_{\boldsymbol{k}}) \\ \vdots \\ \boldsymbol{\Psi} \tilde{\boldsymbol{f}}|_{\boldsymbol{e}_\#^{\boldsymbol{\nu}_n}}(\boldsymbol{v}_{\boldsymbol{k}}) \end{pmatrix}$$

and likewise for $\operatorname{rv} \tilde{\boldsymbol{f}}'_{\boldsymbol{b}}|_{\tilde{\boldsymbol{v}}_{\boldsymbol{k}}}$. But now, the last vector is almost $\Psi \tilde{f}|_{v_k}$, the vector of values of $\Psi \tilde{f}$ at v_k! Only two changes need to be made: first, if the edges entering v_k

are $e_1^{\mu_1}, \dots, e_{d_{\mu_1}}^{\mu_1}, e_1^{\mu_2}, \dots, e_{d_{\mu_m}}^{\mu_m}$, then by definition $\{\mu_i\}_{i=1}^m = \{\nu_i\}_{i=1}^n$ as sets; however, the μ_i are distinct, whereas in general, repetitions can occur among the ν_i (i.e., two edges in $E_{\tilde{v}_k}$ might belong to the same G-orbit). Second, as in all our expressions there might be "dead" edges, $e_j^{\mu_i}$ with $j > d_{\mu_i}$, which do not really appear in the quotient graph (note, however, that neither of the problems can occur when the action of G is free). We shall deal with these two inconveniences at once: we define the $n \times m$ matrix $(\Theta')_{ij} = \begin{cases} 1 & \nu_i = \mu_j \\ 0 & \text{otherwise} \end{cases}$, and then take Θ to be the $nd \times d_{v_k}$ matrix obtained by removing from $(\Theta' \otimes I_d)$ the columns $\{(i-1)\cdot d + j\}_{\substack{1 \le i \le m \\ d_{\mu_i} < j \le d}}$; these are the columns which would have been multiplied by a "dead" edge in $(\Psi \tilde{f}|_{e_1^{\mu_1}} \dots \Psi \tilde{f}|_{e_d^{\mu_1}} \Psi \tilde{f}|_{e_1^{\mu_2}} \dots \dots \Psi \tilde{f}|_{e_d^{\mu_m}})^T$. We now have

$$\operatorname{rv}\begin{pmatrix} \boldsymbol{\Psi \tilde{f}|_{e_\#^{\nu_1}}(v_k)} \\ \vdots \\ \boldsymbol{\Psi \tilde{f}|_{e_\#^{\nu_n}}(v_k)} \end{pmatrix} = \Theta \cdot \left(\Psi \tilde{f}|_{e_1^{\mu_1}}(v_k) \dots \Psi \tilde{f}|_{e_{d_{\mu_1}}^{\mu_1}}(v_k) \dots \dots \Psi \tilde{f}|_{e_{d_{\mu_m}}^{\mu_m}}(v_k)\right)^T$$

$$= \Theta \cdot \Psi \tilde{f}|_{v_k},$$

and we can thus define

$$A_{v_k} = (A_{\tilde{v}_k} \otimes I_d) \cdot \mathfrak{G} \cdot \Theta, \tag{4.4}$$

$$B_{v_k} = (B_{\tilde{v}_k} \otimes I_d) \cdot \mathfrak{G} \cdot \Theta, \tag{4.5}$$

where $\mathfrak{G} = \operatorname{diag}([g_1^{-1}]_{B^{\nu_1}}^B, \dots, [g_n^{-1}]_{B^{\nu_n}}^B)^T$, and finally rewrite (4.3) as

$$A_{v_k} \cdot \Psi \tilde{f}\big|_{v_k} + B_{v_k} \cdot \Psi \tilde{f}'\big|_{v_k} = 0.$$

These vertex conditions on $\Psi \tilde{f}$ at v_k are equivalent to $\tilde{f}_r$ satisfying the vertex conditions at $\tilde{v}_k$ for all $r \in R$, and therefore also on the entire orbit $G \cdot \tilde{v}_k$. If we repeat this process for each $k = 1 \dots K$, we indeed obtain boundary conditions on Γ/R which are satisfied by $\Psi \tilde{f}$, for $\tilde{f} \in \operatorname{Hom}_{\mathbb{C}G}(R, \bigoplus_{\tilde{e} \in E_\Gamma} H^2(\tilde{e}))$, exactly when $\tilde{f} \in \operatorname{Hom}_{\mathbb{C}G}(R, H^2(\Gamma))$.

If the action of G is free, then Θ is just a permutation matrix (we can even order E_{v_k} so that $\Theta = I$), but in the general case Θ might be non-square (explicitly, it is of size $nd \times d_{v_k}$, where $d_{v_k} = \sum_{i=1}^m d_{\mu_i} \le md \le nd$). When this occurs, the matrices A_{v_k} and B_{v_k} we have obtained are not square matrices, and we therefore obtain a quotient which is only a generalized quantum graph. Nevertheless, as the matrices A_{v_k} and B_{v_k} serve only to represent the system of equations $A_{v_k} \cdot f|_{v_k} + B_{v_k} \cdot f'|_{v_k} = 0$, we can perform elementary row operations on the $nd \times 2d_{v_k}$ matrix $(A_{v_k} \mid B_{v_k})$ without changing the boundary conditions at v_k, and thus perhaps reduce the number of rows of $(A_{v_k} \mid B_{v_k})$. In the case that $\operatorname{rank}(A_{v_k} \mid B_{v_k}) \le d_{v_k}$, we can reduce the matrices A_{v_k} and B_{v_k} to squares ones, and if this holds for all k then we actually have a proper quantum graph. If it further happens that $\mathrm{rank}(A_{v_k} \mid B_{v_k}) = d_{v_k}$ for all k, then the quotient graph is also exact. We now show sufficient conditions for this to happen.

Proposition 6 *If there exist* $\omega \in \mathbb{C}^\times$ *and* $M \in GL_{d_{\tilde{v}_k}}(\mathbb{C})$ *such that* $(A_{\tilde{v}_k} \mid B_{\tilde{v}_k})$ *is row-equivalent to* $(\omega(M - I) \mid M + I)$, *then* $\mathrm{rank}(A_{v_k} \mid B_{v_k}) = d_{v_k}$.

Remark We recall from Sect. 2 that this holds for all k when Γ's Laplacian is self-adjoint. Therefore, in this case Γ/R is exact, as stated in Theorem 5.

Proof Denote $\tilde{v} = \tilde{v}_k$, $v = v_k$, and recall that $E_{\tilde{v}} = \{g_i \tilde{e}^{\nu_i}\}_{i=1}^n$ is the set of edges entering $\tilde{v}$. Assume, by reordering if necessary, that $\nu_i = \mu_i$ for $1 \le i \le m$, i.e., that $\{g_i \tilde{e}^{\nu_i}\}_{i=1}^m$ are representatives for the $G_{\tilde{v}}$-orbits in $E_{\tilde{v}}$. Denote $\tilde{\varepsilon}^i = g_i \tilde{e}^{\nu_i} = g_i \tilde{e}^{\mu_i}$ (where $1 \le i \le m$), and note that $G_{\tilde{\varepsilon}^i}$ is conjugate to $G_{\tilde{e}^{\mu_i}}$. The action of $G_{\tilde{v}}$ on $E_{\tilde{v}}$ gives rise to a representation $\mathbb{C}[E_{\tilde{v}}]$ of $G_{\tilde{v}}$, and the $G_{\tilde{v}}$-set isomorphism $E_{\tilde{v}} = \coprod_{i=1}^m G_{\tilde{v}} \cdot \tilde{\varepsilon}^i \cong \coprod_{i=1}^m {}^{G_{\tilde{v}}}/\!/_{G_{\tilde{\varepsilon}^i}}$ (where ${}^{G}/\!/_{H}$ is the G-set of left H-cosets in G) translates to an isomorphism of $G_{\tilde{v}}$-representations:

$$\mathbb{C}[E_{\tilde{v}}] \cong \bigoplus_{i=1}^m \mathbb{C}[{}^{G_{\tilde{v}}}/\!/_{G_{\tilde{\varepsilon}^i}}] \cong \bigoplus_{i=1}^m \mathrm{Ind}_{G_{\tilde{\varepsilon}^i}}^{G_{\tilde{v}}} \mathbf{1}_{G_{\tilde{\varepsilon}^i}}.$$

Here $\mathbf{1}_G$ denotes the trivial representation of a group G, but we shall also use it to denote its character. We now see that

$$\begin{aligned}
\langle \chi_{\mathbb{C}[E_{\tilde{v}}]}, \chi_R \rangle_{G_{\tilde{v}}} &= \Big\langle \chi_{\bigoplus_{i=1}^m \mathrm{Ind}_{G_{\tilde{\varepsilon}^i}}^{G_{\tilde{v}}} \mathbf{1}_{G_{\tilde{\varepsilon}^i}}}, \chi_R \Big\rangle_{G_{\tilde{v}}} = \sum_{i=1}^m \Big\langle \mathrm{Ind}_{G_{\tilde{\varepsilon}^i}}^{G_{\tilde{v}}} \mathbf{1}_{G_{\tilde{\varepsilon}^i}}, \chi_R \Big\rangle_{G_{\tilde{v}}} \\
&= \sum_{i=1}^m \langle \mathbf{1}_{G_{\tilde{\varepsilon}^i}}, \chi_R \rangle_{G_{\tilde{\varepsilon}^i}} = \sum_{i=1}^m \dim R^{G_{\tilde{\varepsilon}^i}} \\
&= \sum_{i=1}^m \dim R^{G_{\tilde{e}^{\mu_i}}} = \sum_{i=1}^m d_{\mu_i} = d_v.
\end{aligned} \tag{4.6}$$

We return to the matrices $(A_v \mid B_v) \in M_{nd \times 2d_v}(\mathbb{C})$ and $(A_{\tilde{v}} \mid B_{\tilde{v}}) \in M_{n \times 2n}(\mathbb{C})$. For $f \in H^2(\Gamma)$, the action of $G_{\tilde{v}}$ on $E_{\tilde{v}}$ induces a permutation action of $G_{\tilde{v}}$ on the entries of $f|_{\tilde{v}} = (f|_{\tilde{e}}(\tilde{v}))_{\tilde{e} \in E_{\tilde{v}}}$, and exactly the same action is induced on the entries of $f'|_{\tilde{v}}$. Thus, the space $\mathbb{C}^{2n}$ of possible values and derivatives at $\tilde{v}$ has naturally the structure of the $G_{\tilde{v}}$-representation $\mathbb{C}[E_{\tilde{v}}] \oplus \mathbb{C}[E_{\tilde{v}}]$. Furthermore, as by assumption G preserves the boundary conditions, $\ker(A_{\tilde{v}} \mid B_{\tilde{v}}) \subseteq \mathbb{C}^{2n}$ is a sub-$G_{\tilde{v}}$-representation of $\mathbb{C}^{2n} \cong \mathbb{C}[E_{\tilde{v}}] \oplus \mathbb{C}[E_{\tilde{v}}]$. We observe that the encoding and decoding processes are "rigid", in the sense that for $x \in [0, l_{\tilde{e}^i}]$ it suffices to know $\{\tilde{f}_r|_{\tilde{e}^i}(x)\}_{r \in R}$ to determine $\{\Psi \tilde{f}|_{e_j^i}(x)\}_{j=1..d_i}$, and vice versa. Likewise, $\{\tilde{f}_r|_{\tilde{v}}\}_{r \in R}$ and $\Psi \tilde{f}|_v$ determine one another, and the same goes for the corresponding derivatives. This means that in the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbb{C}G}(R, H^2(\Gamma)) & \longrightarrow & \mathrm{Hom}_{\mathbb{C}G_{\tilde{v}}}(R, \ker(A_{\tilde{v}} \mid B_{\tilde{v}})) \\
\Psi \Big\downarrow \quad \tilde{f} \longmapsto (r \mapsto (\tilde{f}_r|_{\tilde{v}}, \tilde{f}_r'|_{\tilde{v}})) & & \\
\qquad \quad \Downarrow \qquad\qquad\qquad \Downarrow & & \Big\downarrow \psi \\
\qquad \Psi\tilde{f} \longmapsto (\Psi\tilde{f}|_v, (\Psi\tilde{f})'|_v) & & \\
H^2(\Gamma/R) & \longrightarrow & \ker(A_v \mid B_v)
\end{array}$$

the map ψ, which is this "local" encoding, is in fact an isomorphism. This gives us

$$\mathrm{null}(A_v \mid B_v) = \langle \chi_R, \chi_{\ker(A_{\tilde{v}}|B_{\tilde{v}})} \rangle_{G_{\tilde{v}}}, \tag{4.7}$$

so that by (4.6)

$$\mathrm{rank}(A_v \mid B_v) = \langle 2\chi_{\mathbb{C}[E_{\tilde{v}}]} - \chi_{\ker(A_{\tilde{v}}|B_{\tilde{v}})}, \chi_R \rangle_{G_{\tilde{v}}}.$$

We therefore have

$$\mathrm{rank}(A_v \mid B_v) = d_v \quad \Leftrightarrow \quad \langle \chi_{\mathbb{C}[E_{\tilde{v}}]} - \chi_{\ker(A_{\tilde{v}}|B_{\tilde{v}})}, \chi_R \rangle_{G_{\tilde{v}}} = 0,$$

and the last equality holds for all representations R of G if and only if $\mathrm{Ind}_{G_{\tilde{v}}}^G \mathbb{C}[E_{\tilde{v}}] \cong \mathrm{Ind}_{G_{\tilde{v}}}^G \ker(A_{\tilde{v}} \mid B_{\tilde{v}})$. In particular, this happens if $\mathbb{C}[E_{\tilde{v}}]$ and $\ker(A_{\tilde{v}} \mid B_{\tilde{v}})$ are isomorphic $G_{\tilde{v}}$-representations, which we now show to follow from our assumptions.

Observe that $\xi : \mathbb{C}[E_{\tilde{v}}] \oplus \mathbb{C}[E_{\tilde{v}}] \to \mathbb{C}[E_{\tilde{v}}]$, defined by $\xi(\underline{a}, \underline{b}) = \omega\underline{a} - \underline{b}$ is a homomorphism of $G_{\tilde{v}}$-representations, and recall that $\ker(A_{\tilde{v}} \mid B_{\tilde{v}})$ is naturally embedded in $\mathbb{C}[E_{\tilde{v}}] \oplus \mathbb{C}[E_{\tilde{v}}]$. When restricting ξ to $\ker(A_{\tilde{v}} \mid B_{\tilde{v}})$ we obtain the desired isomorphism onto $\mathbb{C}[E_{\tilde{v}}]$, since $\dim \ker(A_{\tilde{v}} \mid B_{\tilde{v}}) = \mathrm{null}(\omega(M - I) \mid (M + I)) = d_{\tilde{v}} = \dim \mathbb{C}[E_{\tilde{v}}]$, and

$$(\underline{a}, \underline{b}) \in \ker\left(\xi\Big|_{\ker(A_{\tilde{v}}|B_{\tilde{v}})}\right) \;\Rightarrow\; \left\{\begin{array}{c} \omega(M - I)\underline{a} + (M + I)\underline{b} = 0 \\ \omega\underline{a} - \underline{b} = 0 \end{array}\right\} \;\Rightarrow\; (\underline{a}, \underline{b}) = 0.$$

□

4.3 Remarks

4.3.1

If G acts on Γ and R is a representation of $H \le G$, we can consider the composition of isomorphisms

$$H^2(\Gamma/R) \xrightarrow{\Psi^{-1}} \mathrm{Hom}_{\mathbb{C}H}(R, H^2(\Gamma)) \xrightarrow{\mathcal{F}} \mathrm{Hom}_{\mathbb{C}G}(\mathrm{Ind}_H^G R, H^2(\Gamma)) \xrightarrow{\Upsilon} H^2(\Gamma/\mathrm{Ind}_H^G R)$$

where Ψ and Υ are the "encoding maps" defined during the constructions of Γ/R and $\Gamma/\mathrm{Ind}_H^G R$, respectively, and $\mathcal{F}$ is the Frobenius isomorphism.[7] We obtain what is known as a *transplantation* (see [23, 24]) between Γ/R and $\Gamma/\mathrm{Ind}_H^G R$, an operator which constructs functions on one graph as linear combinations of segments of functions on the second graph. Note that the case of two quotients obtained from a single representation using different bases for the construction is covered as well, by taking $H = G$, and $\mathcal{F} = id$. This is developed in more details in [15].

4.3.2

If Γ' and Γ'' are two Γ/R-graphs, with corresponding encoding maps Ψ and Υ, then the commutative diagram

$$\begin{array}{ccccc} H^2(\Gamma') & \xleftarrow[\Psi|_{\mathrm{Hom}_{\mathbb{C}G}(R,H^2(\Gamma))}]{\cong} & \mathrm{Hom}_{\mathbb{C}G}(R, H^2(\Gamma)) & \xrightarrow[\Upsilon|_{\mathrm{Hom}_{\mathbb{C}G}(R,H^2(\Gamma))}]{\cong} & H^2(\Gamma'') \\ \Delta_{\Gamma'} \big\downarrow & & \big\downarrow \Delta & & \big\downarrow \Delta_{\Gamma''} \\ L_2(\Gamma') & \xleftarrow[\Psi]{\cong} & \mathrm{Hom}_{\mathbb{C}G}(R, L_2(\Gamma)) & \xrightarrow[\Upsilon]{\cong} & L_2(\Gamma'') \end{array}$$

gives eigenvalue-isospectrality of Γ' and Γ''. However, as $\Upsilon \circ \Psi^{-1}$ need not be unitary, in general $\Delta_{\Gamma'}$ and $\Delta_{\Gamma''}$ are not unitarily equivalent operators. Thus, their general spectra might differ, and it can also happen that one of them is a self-adjoint operator and the other is not.

Assume, however, that these graphs are constructed by the method presented above, with respect to the bases $B^i = (b^i_j)_{j=1}^d$ for Γ' and $\mathfrak{B}^i = (\mathfrak{b}^i_j)_{j=1}^d$ for Γ'' ($1 \le i \le I$). Recall that $(b^i_j)_{j=1}^{d_i}$ and $(\mathfrak{b}^i_j)_{j=1}^{d_i}$ are each a basis for $R^{G_{\tilde{e}^i}}$. Let $P_i \in GL_{d_i}(\mathbb{C})$ be the corresponding change of basis matrix, i.e., $(b^i_1 \ldots b^i_{d_i}) \cdot P_i = (\mathfrak{b}^i_1 \ldots \mathfrak{b}^i_{d_i})$. We have for any $\tilde{f} \in \mathrm{Hom}_{\mathbb{C}G}(R, L_2(\Gamma))$

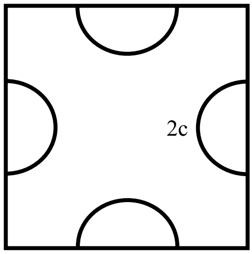
$$\boldsymbol{\Psi} \tilde{\boldsymbol{f}}\big|_{e^i_\#} \cdot P_i \equiv \tilde{\boldsymbol{f}}_{b^i_\#}\big|_{\tilde{e}^i} \cdot P_i \equiv \tilde{\boldsymbol{f}}_{\mathfrak{b}^i_\#}\big|_{\tilde{e}^i} \equiv \boldsymbol{\Upsilon} \tilde{\boldsymbol{f}}\big|_{e^i_\#},$$

so that $\Upsilon \circ \Psi^{-1}$ is given by $\mathrm{diag}(P_1, \ldots, P_I)$, where we have identified both $L_2(\Gamma')$ and $L_2(\Gamma'')$ with $\bigoplus_{i=1}^I (\bigoplus_{j=1}^{d_i} L_2([0, l_{\tilde{e}^i}]))$. Thus, the transplantation map $\Upsilon \circ \Psi^{-1}$ is unitary if and only if $P_i \in U(d_i)$ for all i, i.e., the $R^{G_{\tilde{e}^i}}$-bases we have chosen for the two quotients are unitarily equivalent themselves. When this is so, $\Delta_{\Gamma'}$ and $\Delta_{\Gamma''}$ are indeed unitarily equivalent operators.

By analyzing the transplantations arising from different representatives for the action of G on Γ, or from inducing R to some supergroup in $\mathrm{Aut}\,\Gamma$, one can arrive at similar criteria for unitary-equivalence of the corresponding Laplacians. In particular, in [15] is presented an extremely simple transplantation between Γ/R and $\Gamma/\mathrm{Ind}_H^G R$, which is unitary.

[7] Taking the induction to be the scalar extension $\mathrm{Ind}_H^G R = \mathbb{C}G \otimes_{\mathbb{C}H} R$, $\mathcal{F} : \mathrm{Hom}_{\mathbb{C}H}(R, R') \to \mathrm{Hom}_{\mathbb{C}G}(\mathrm{Ind}_H^G R, R')$ is defined by linearly extending $(\mathcal{F}\tilde{f})_{g \otimes r} = g \cdot \tilde{f}_r$.

Fig. 1 A graph that obeys the dihedral symmetry of the square. The lengths of some edges are marked



4.3.3

It is natural to ask, for a graph Γ whose Laplacian is self-adjoint, whether the Laplacian on Γ/R is self-adjoint. This turns out to depend on both the action of G and the choices of bases in the construction, and it is addressed for some special cases in [15].

4.3.4

Another interesting question is the following: for a quantum graph Γ acted upon by G, when does an irreducible representation S of G appear in some eigenspace of Γ?[8] It is known that every quantum graph with edges whose Laplacian is self-adjoint has a nonempty spectrum (see for example [22]). Therefore, if Γ/S's Laplacian is self-adjoint then S appears in some eigenspace of Γ iff Γ/S has edges, and by the construction method this happens iff for at least one edge e in Γ the representation $\mathrm{Res}^G_{G_e} S$ has a nonempty trivial component, i.e., $\langle \chi_S, \mathbf{1} \rangle_{G_e} \neq 0$. In particular, if Γ's Laplacian is self adjoint, and G acts freely on Γ, then a self-adjoint quotient can always be obtained [15], and each stabilizer has only the trivial irreducible representation. Thus, every irreducible representation of G appears in some eigenspace of Γ.

5 Examples of Isospectral Quantum Graphs

We now demonstrate several applications of the theory presented above which yield isospectral graphs. All the examples below are direct consequences of the theorem or the corollary presented in Sect. 3.

Let Γ be the graph given in Fig. 1. The lengths of the edges are determined by the parameters a, b, c and it has Neumann boundary conditions at all vertices. $G = D_4$, the dihedral group of the square, is a symmetry group of Γ. Denote by τ the reflection of Γ along the horizontal axis and by σ the rotation of Γ counterclockwise by $\pi/2$. Then we can describe G and some of its subgroups $H_1, H_2, H_3 \leq G$ by:

$$G = \{e, \sigma, \sigma^2, \sigma^3, \tau, \tau\sigma, \tau\sigma^2, \tau\sigma^3\},$$
$$H_1 = \{e, \tau, \tau\sigma^2, \sigma^2\},$$

[8] This question, in the context of compact Lie groups acting on Riemannian manifolds, is addressed in [21].

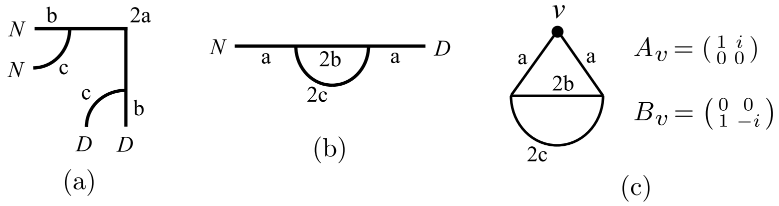


Fig. 2 The three isospectral graphs $\Gamma/R_1, \Gamma/R_2, \Gamma/R_3$. Neumann boundary conditions are assumed if nothing else is specified. D stands for Dirichlet boundary conditions and N for Neumann

$$H_2 = \{e,\ \tau\sigma,\ \tau\sigma^3,\ \sigma^2\},$$
$$H_3 = \{e,\ \sigma,\ \sigma^2,\ \sigma^3\}.$$

Consider the following one-dimensional representations of H_1, H_2 and H_3 respectively:

$$R_1 : \{e \mapsto (1),\ \tau \mapsto (-1),\ \tau\sigma^2 \mapsto (1),\ \sigma^2 \mapsto (-1)\}, \tag{5.1}$$

$$R_2 : \{e \mapsto (1),\ \tau\sigma \mapsto (1),\ \tau\sigma^3 \mapsto (-1),\ \sigma^2 \mapsto (-1)\}, \tag{5.2}$$

$$R_3 : \{e \mapsto (1),\ \sigma \mapsto (i),\ \sigma^2 \mapsto (-1),\ \sigma^3 \mapsto (-i)\}. \tag{5.3}$$

These representations fulfill the condition in Corollary 4: $\mathrm{Ind}_{H_1}^G R_1 \cong \mathrm{Ind}_{H_2}^G R_2 \cong \mathrm{Ind}_{H_3}^G R_3$ and thus we obtain that Γ/R_1, Γ/R_2 and Γ/R_3 are isospectral (Fig. 2).

We now explain the process of building the graph Γ/R_1. First we give an intuition which suffices to obtain the quotient in this case, and afterwards we strictly implement the method that is described in Sect. 4.2. Going back to (3.4), we observe that its upper-right corner is in our example

$$\mathrm{Hom}_{\mathbb{C}H_1}(R_1, H^2(\Gamma)) \cong H^2(\Gamma)^{R_1}, \tag{5.4}$$

where $H^2(\Gamma)^{R_1}$ is the R_1-isotypic component of $H^2(\Gamma)$ (considered as a $\mathbb{C}H_1$-module); the isomorphism is due to the fact that R_1 is one-dimensional, and can be realized by $\tilde{f} \mapsto \tilde{f}_b$, where b is a fixed nonzero vector in R_1. In order to construct Γ/R_1 and have the upper isomorphism $H^2(\Gamma/R_1) \cong \mathrm{Hom}_{\mathbb{C}H_1}(R_1, H^2(\Gamma))$ of (3.4), we now study the properties of $\tilde{f} \in H^2(\Gamma)^{R_1}$. We know (see (5.1)) that $\tau\tilde{f} = -\tilde{f}$, which means that $\tilde{f}$ is an anti-symmetric function with respect to the horizontal reflection. We deduce that $\tilde{f}$ vanishes on the fixed points of τ (marked with diamonds in Fig. 3(a)).

In a similar manner, we see that $\tilde{f}$ is symmetric with respect to the vertical reflection since $\tau\sigma^2\tilde{f} = \tilde{f}$, and therefore the derivative of $\tilde{f}$ must vanish at the corresponding points (the squares in Fig. 3(a)). Furthermore, it is enough to know the values of $\tilde{f}$ restricted to the first quadrant (the bold subgraph in Fig. 3(a)) in order to deduce $\tilde{f}$ on the whole graph, using the known action of the reflections, which follows from $\tilde{f} \in H^2(\Gamma)^{R_1}$:

$$\tau\tilde{f} = -\tilde{f}, \qquad \tau\sigma^2\tilde{f} = \tilde{f}. \tag{5.5}$$

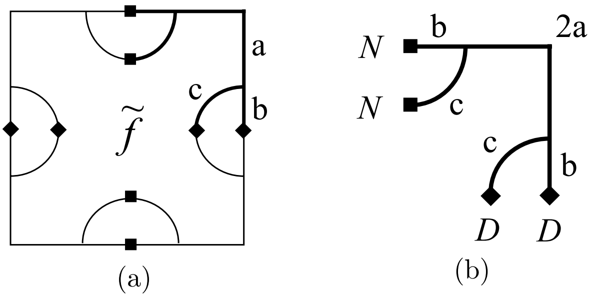


Fig. 3 (**a**) The information we have on $\tilde{f} \in H^2(\Gamma)^{R_1}$. *Diamonds* mark the vertices on which the function vanishes and *squares* the vertices with zero derivative. (**b**) The quotient graph Γ/R_1 which encodes this information. D stands for Dirichlet boundary conditions and N for Neumann

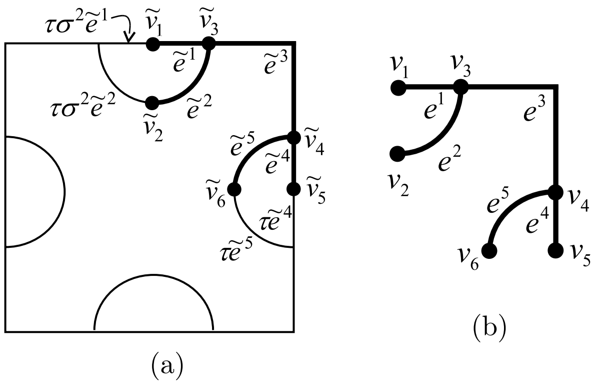


Fig. 4 (**a**) The graph Γ with the representatives of E/H_1, V/H_1 marked in bold (the corners are not vertices). (**b**) The resulting quotient graph Γ/R_1

Our encoding is now complete and the quotient Γ/R_1 is the subgraph which lies in the first quadrant, with the boundary conditions of Dirichlet and Neumann in the appropriate locations as was found for $\tilde{f}$ (Fig. 3(b)). The encoding is described by the map $\Psi : \mathrm{Hom}_{\mathbb{C}H_1}(R_1, H^2(\Gamma)) \to H^2(\Gamma/R_1)$ which is just the restriction map of functions in $H^2(\Gamma)^{R_1} \cong \mathrm{Hom}_{\mathbb{C}H_1}(R_1, H^2(\Gamma))$ to the mentioned subgraph. An important observation is that given $f \in H^2(\Gamma/R_1)$ it is possible to construct a function $\tilde{f} \in H^2(\Gamma)^{R_1}$ (using (5.5)), whose restriction to the first quadrant subgraph is f. As mentioned above, such $\tilde{f}$ is unique, and it follows that Ψ is invertible and thus is an isomorphism. It is easy to see (by removing all boundary conditions, and admitting all L_2-functions) that this isomorphism extends to $L_2(\Gamma/R_1) \cong \mathrm{Hom}_{\mathbb{C}H_1}(R_1, L_2(\Gamma))$, which is the lower edge of (3.4). This ends the intuitive approach and we now proceed to the rigorous derivation.

First, we add "dummy" vertices to the graph Γ so that no vertex is carried by the action of H_1 to one of its neighbors, and choose representatives $\{\tilde{e}^i\}_{i=1}^5$ for the orbits E/H_1, and $\{\tilde{v}_k\}_{k=1}^6$ for the orbits V/H_1. These representatives are marked in Fig. 4(a) by bold lines and points. The dummy vertices amongst the representatives are $\tilde{v}_1, \tilde{v}_2, \tilde{v}_5, \tilde{v}_6$. R_1 is one-dimensional, and $d_i = 1$ for all i since the stabilizers of all edges are trivial. Therefore, the quotient graph is formed by taking one copy of each of the representative edges (Fig. 4(b)). Now, let us determine the boundary conditions using (4.4), (4.5). For all vertices we have $d = 1$ and therefore $A_{\tilde{v}_k} \otimes I_d = A_{\tilde{v}_k}$ and

$B_{\tilde{v}_k} \otimes I_d = B_{\tilde{v}_k}$. Consider the vertex $v_k = v_3$ for which

$$n = 3, \quad m = 3, \quad d_{v_3} = 3,$$

$$g_1 = e, \quad \nu_1 = \mu_1 = 1, \quad g_2 = e, \quad \nu_2 = \mu_2 = 2, \quad g_3 = e, \quad \nu_3 = \mu_3 = 3,$$

$$\mathfrak{G} = I_3, \quad \Theta = (\Theta' \otimes I_d) = \Theta' = I_3.$$

Plugging all this into (4.4), (4.5) and using the boundary conditions on $\tilde{v}_3$ which are given by $A_{\tilde{v}_3} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$, $B_{\tilde{v}_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ gives Neumann boundary conditions for v_3 as well: $A_{v_3} = A_{\tilde{v}_3}$, $B_{v_3} = B_{\tilde{v}_3}$. Exactly the same treatment can be done for the vertex v_4 and the same boundary conditions are obtained. The case is different for the vertex v_5:

$$n = 2, \qquad m = 1, \qquad d_{v_5} = 1,$$

$$g_1 = e, \quad \nu_1 = \mu_1 = 4, \qquad g_2 = \tau, \qquad \nu_2 = \mu_1 = 4$$

$$\mathfrak{G} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \Theta = (\Theta' \otimes I_d) = \Theta' = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The boundary conditions on $\tilde{v}_5$ are of Neumann type as well: $A_{\tilde{v}_5} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, $B_{\tilde{v}_5} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$. This time we obtain

$$A_{v_5} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix},$$

$$B_{v_5} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

A_{v_5} and B_{v_5} are then reduced to square one-dimensional matrices as expected, by removing the second row in both of them. We remain with $A_{v_5} = (2)$, $B_{v_5} = (0)$ which means Dirichlet boundary conditions on the vertex v_5. The same boundary conditions are obtained for v_6. Similar derivation for vertices v_1, v_2 gives Neumann boundary conditions for each one of them. The rigorous construction thus gives us the same quotient graph that was obtained by the intuitive method (Figs. 2(a), 3(b)).

The quotient Γ/R_2 can be constructed in a similar manner, and is shown in Fig. 2(b). We proceed to demonstrate the construction method for the quotient Γ/R_3.[9] We first add the corners of the square as dummy vertices to Γ ($\tilde{v}_1$ in Fig. 5(a) is one of them). We are not obliged to do so, but it yields a quotient with simpler boundary conditions. The choice of representatives for the edges and the vertices is shown in Fig. 5(a) and the resulting quotient in Fig. 5(b).

[9]This result was obtained with G. Ben-Shach.

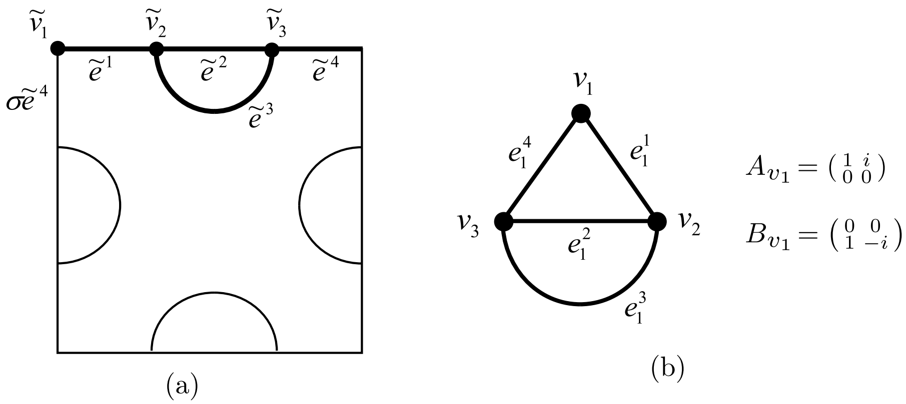


Fig. 5 (**a**) The graph Γ with the representatives of E/H_3, V/H_3 marked in bold. (**b**) The resulting quotient Γ/R_3. v_2, v_3 possess Neumann boundary conditions

The vertices v_2 and v_3 have Neumann boundary conditions exactly as their predecessors, $\tilde{v}_2$ and $\tilde{v}_3$. For v_1 we obtain more interesting boundary conditions:

$$\begin{aligned}
&A_{\tilde{v}_1} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \qquad B_{\tilde{v}_1} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \\
&n = 2, \quad m = 2, \qquad d_{v_1} = 2, \\
&g_1 = e, \quad \nu_1 = \mu_1 = 1, \qquad g_2 = \sigma, \quad \nu_2 = \mu_2 = 4, \\
&\mathfrak{G} = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}, \qquad \Theta = (\Theta' \otimes I_d) = \Theta' = I_2,
\end{aligned}$$

which gives

$$A_{v_1} = \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}, \qquad B_{v_1} = \begin{pmatrix} 0 & 0 \\ 1 & -i \end{pmatrix}. \tag{5.6}$$

Non-formally speaking, the vertex v_1 "applies a factor of i" to the functions that cross it. The resulting graph is the one that was shown in Fig. 2(c).

In order to exhaust this example, we observe that $\mathrm{Ind}_{H_1}^G R_1 \cong \mathrm{Ind}_{H_2}^G R_2 \cong \mathrm{Ind}_{H_3}^G R_3$ is the two-dimensional irreducible representation of D_4, which we denote by R. By Theorem 3, the isospectral family of the three graphs given in Fig. 2 can be extended by adding any Γ/R-graph. We therefore construct now such a graph. Let us use the intuitive approach first. Recall that (5.4) was the key for the intuitive construction of Γ/R_1. Analogously to (5.4), we make the observation that encoding $\mathrm{Hom}_{\mathbb{C}G}(R, H^2(\Gamma))$, the upper-right corner of (3.4), is similar in nature to encoding $H^2(\Gamma)^R$, the R-isotypic component of $H^2(\Gamma)$, due to the simplicity of R as a $\mathbb{C}G$-module. This can be understood as follows: making a choice of a basis $\{b_1, b_2\}$ for R, and given a function $\tilde{f} \in \mathrm{Hom}_{\mathbb{C}G}(R, H^2(\Gamma))$, we have that $\tilde{f}_{b_1}, \tilde{f}_{b_2} \in H^2(\Gamma)^R$ and furthermore $\{\tilde{f}_{b_1}, \tilde{f}_{b_2}\}$ spans over $\mathbb{C}$ a $\mathbb{C}G$-module isomorphic to R. In order to exhibit the general behavior we avoid sparse matrices, and pick a basis $\{b_1, b_2\}$ for

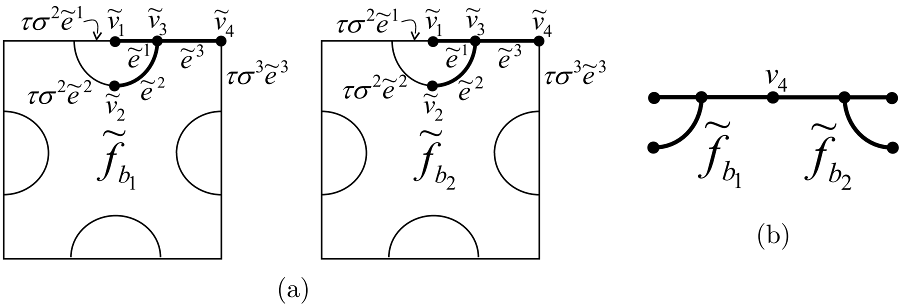


Fig. 6 (**a**) Two copies of the graph Γ with the representatives of E/D_4, V/D_4 marked in bold. These two copies are merely a visualization of the "basis functions" $\tilde{f}_{b_1}, \tilde{f}_{b_2}$ on Γ. (**b**) The first stage in the formation of Γ/R is the gluing of both copies in the vertex v_4, with the boundary conditions given in (5.14)

which the matrix representation of R is

$$\left\{\tau\sigma^2 \mapsto \tfrac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}, \tau\sigma^3 \mapsto \tfrac{1}{2}\begin{pmatrix} \sqrt{3} & -1 \\ -1 & -\sqrt{3} \end{pmatrix}\right\}. \tag{5.7}$$

It is enough to consider only the matrices of these two elements for the construction of the quotient.

Examine the properties of $\tilde{f}_{b_1}, \tilde{f}_{b_2}$ that follow from the above matrix representation (Fig. 6(a)). Since $\tilde{f} \in \mathrm{Hom}_{\mathbb{C}G}(R, H^2(\Gamma))$ we have $\tau\sigma^3 \tilde{f}_{b_1} = \tilde{f}_{(\tau\sigma^3)^{-1}b_1} = \tilde{f}_{\tau\sigma^3 b_1}$, and thus the first column of the matrix representing $\tau\sigma^3$ tells us that

$$\tau\sigma^3 \tilde{f}_{b_1} = \frac{\sqrt{3}}{2}\tilde{f}_{b_1} - \frac{1}{2}\tilde{f}_{b_2}, \tag{5.8}$$

$$\tau\sigma^3 \tilde{f}'_{b_1} = \frac{\sqrt{3}}{2}\tilde{f}'_{b_1} - \frac{1}{2}\tilde{f}'_{b_2} \tag{5.9}$$

and enables us to relate the values and the derivatives of $\tilde{f}_{b_1}, \tilde{f}_{b_2}$ on the vertex $\tilde{v}_4$. Since $\tilde{v}_4$ is a fixed point under the action of $\tau\sigma^3$ and there are Neumann boundary conditions on it, we have that

$$\left(\tau\sigma^3 \tilde{f}_{b_1}\right)\big|_{\tilde{e}_3}(\tilde{v}_4) = \tilde{f}_{b_1}\big|_{\tilde{e}_3}(\tilde{v}_4), \tag{5.10}$$

$$\left(\tau\sigma^3 \tilde{f}'_{b_1}\right)\big|_{\tilde{e}_3}(\tilde{v}_4) = -\tilde{f}'_{b_1}\big|_{\tilde{e}_3}(\tilde{v}_4). \tag{5.11}$$

Evaluating (5.8) on v_4 and combining this with (5.10) gives

$$\left(1 - \frac{\sqrt{3}}{2}\right)\tilde{f}_{b_1}\big|_{\tilde{e}_3}(\tilde{v}_4) + \frac{1}{2}\tilde{f}_{b_2}\big|_{\tilde{e}_3}(\tilde{v}_4) = 0. \tag{5.12}$$

Similarly, from (5.9) and (5.11) we obtain

$$\left(-1 - \frac{\sqrt{3}}{2}\right)\tilde{f}'_{b_1}\big|_{\tilde{e}_3}(\tilde{v}_4) + \frac{1}{2}\tilde{f}'_{b_2}\big|_{\tilde{e}_3}(\tilde{v}_4) = 0. \tag{5.13}$$

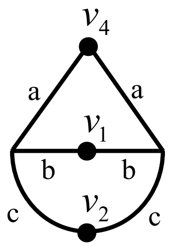


Fig. 7 The quotient graph Γ/R which is isospectral to the graphs in Fig. 2. The boundary conditions are as described in (5.14), (5.15)

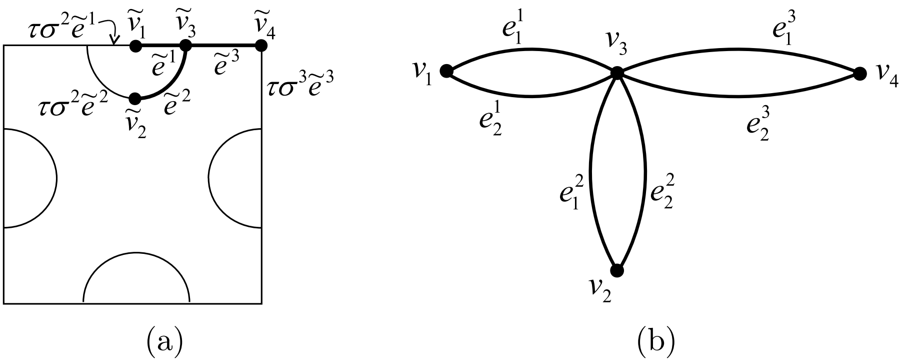


Fig. 8 (**a**) The graph Γ with the representatives of E/D_4, V/D_4 marked in bold. (**b**) The resulting quotient Γ/R

We may therefore think of two copies of the graphs. Each of the basis functions $\tilde{f}_{b_1}$, $\tilde{f}_{b_2}$ resides on one of the copies, and the relations between the values and the derivatives of the functions allow us to take a subgraph out of each copy (marked in bold in Fig. 6(a)) and glue both of them together with the appropriate boundary conditions. The first stage in this gluing process, visualized in Fig. 6(b), is to identify the vertex $\tilde{v}_4$ in the two copies and turn it into the vertex v_4 of the quotient with the boundary conditions that were derived in (5.12), (5.13):

$$A_{v_4} = \begin{pmatrix} 1-\sqrt{3}/2 & 1/2 \\ 0 & 0 \end{pmatrix}, \qquad B_{v_4} = \begin{pmatrix} 0 & 0 \\ -1-\sqrt{3}/2 & 1/2 \end{pmatrix}. \tag{5.14}$$

After treating similarly vertices $\tilde{v}_1, \tilde{v}_2$ we get the quotient Γ/R (Fig. 7) whose remaining boundary conditions are given by:

$$A_{v_1} = A_{v_2} = \begin{pmatrix} 3/2 & \sqrt{3}/2 \\ 0 & 0 \end{pmatrix}, \qquad B_{v_1} = B_{v_2} = \begin{pmatrix} 0 & 0 \\ -1/2 & \sqrt{3}/2 \end{pmatrix}. \tag{5.15}$$

We now use the rigorous approach for the same quotient, Γ/R. The representatives of the orbits E/G are $\{\tilde{e}^i\}_{i=1}^3$ and the representatives of V/G are $\{\tilde{v}_k\}_{k=1}^4$ (Fig. 8(a)). This time the representation is not one-dimensional ($d=2$) so there are additional details to consider. First, note that we have two copies of each representative of E/G in the quotient and both of the copies "survive" since all edges have trivial stabilizers (Fig. 8(b)). This last observation ensures that we can take $B^i = B$ for all i (i.e., the same basis for all edges). We again take B to be the basis for which the matrix representation of R is (5.7).

We treat the boundary conditions at the vertices one by one:

- v_4 has the following data:

$$n=2,\quad m=1,\quad d_{v_4}=2,\quad g_1=e,\quad \nu_1=\mu_1=3,\quad g_2=\tau\sigma^3,\quad \nu_2=\mu_1=3,$$

$$\Theta=(\Theta'\otimes I_d)=\begin{pmatrix}1\\1\end{pmatrix}\otimes I_2=\begin{pmatrix}1&0\\0&1\\1&0\\0&1\end{pmatrix}.$$

$A_{\tilde{v}_4}$, $B_{\tilde{v}_4}$ are the regular Neumann matrices and we therefore obtain

$$A_{v_4}=\begin{pmatrix}1&0&-1&0\\0&1&0&-1\\0&0&0&0\\0&0&0&0\end{pmatrix}\cdot\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\frac{\sqrt{3}}{2}&-\frac{1}{2}\\0&0&-\frac{1}{2}&-\frac{\sqrt{3}}{2}\end{pmatrix}\cdot\begin{pmatrix}1&0\\0&1\\1&0\\0&1\end{pmatrix}=\begin{pmatrix}1-\frac{\sqrt{3}}{2}&\frac{1}{2}\\\frac{1}{2}&1+\frac{\sqrt{3}}{2}\\0&0\\0&0\end{pmatrix},$$

$$B_{v_4}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\1&0&1&0\\0&1&0&1\end{pmatrix}\cdot\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\frac{\sqrt{3}}{2}&-\frac{1}{2}\\0&0&-\frac{1}{2}&-\frac{\sqrt{3}}{2}\end{pmatrix}\cdot\begin{pmatrix}1&0\\0&1\\1&0\\0&1\end{pmatrix}=\begin{pmatrix}0&0\\0&0\\1+\frac{\sqrt{3}}{2}&-\frac{1}{2}\\-\frac{1}{2}&1-\frac{\sqrt{3}}{2}\end{pmatrix}.$$

Noting that both A_{v_4} and B_{v_4} are of rank one, we see that they express the same boundary conditions as given in (5.14).

- v_1 obviously has the same boundary conditions as v_2. We examine v_1:

$$n=2,\quad m=1,\quad d_{v_1}=2,\quad g_1=e,\quad \nu_1=\mu_1=1,\quad g_2=\tau\sigma^2,\quad \nu_2=\mu_1=1,$$

$$\Theta=(\Theta'\otimes I_d)=\begin{pmatrix}1\\1\end{pmatrix}\otimes I_2=\begin{pmatrix}1&0\\0&1\\1&0\\0&1\end{pmatrix}.$$

Again, $A_{\tilde{v}_1}$ and $B_{\tilde{v}_1}$ are the regular Neumann matrices and we get:

$$A_{v_1}=\begin{pmatrix}1&0&-1&0\\0&1&0&-1\\0&0&0&0\\0&0&0&0\end{pmatrix}\cdot\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-\frac{1}{2}&-\frac{\sqrt{3}}{2}\\0&0&-\frac{\sqrt{3}}{2}&\frac{1}{2}\end{pmatrix}\cdot\begin{pmatrix}1&0\\0&1\\1&0\\0&1\end{pmatrix}=\begin{pmatrix}\frac{3}{2}&\frac{\sqrt{3}}{2}\\\frac{\sqrt{3}}{2}&\frac{1}{2}\\0&0\\0&0\end{pmatrix},$$

$$B_{v_1}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\1&0&1&0\\0&1&0&1\end{pmatrix}\cdot\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-\frac{1}{2}&-\frac{\sqrt{3}}{2}\\0&0&-\frac{\sqrt{3}}{2}&\frac{1}{2}\end{pmatrix}\cdot\begin{pmatrix}1&0\\0&1\\1&0\\0&1\end{pmatrix}=\begin{pmatrix}0&0\\0&0\\\frac{1}{2}&-\frac{\sqrt{3}}{2}\\-\frac{\sqrt{3}}{2}&\frac{3}{2}\end{pmatrix}$$

which are matrices of rank one and again we may reduce these matrices into two dimensional ones which are exactly those given in (5.15).

- The case of v_3 is a bit more interesting:

$$n=3,\quad m=3,\quad d_{v_3}=6,$$

$$g_1=e,\quad \nu_1=\mu_1=1,\quad g_2=e,\quad \nu_2=\mu_2=2,\quad g_3=e,\quad \nu_3=\mu_3=3,$$

$$\Theta=(\Theta'\otimes I_d)=I_3\otimes I_2=I_6.$$

As $A_{\tilde{v}_3}$ and $B_{\tilde{v}_3}$ are Neumann matrices, we have

$$A_{v_3}=\begin{pmatrix}1&0&-1&0&0&0\\0&1&0&-1&0&0\\0&0&1&0&-1&0\\0&0&0&1&0&-1\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\cdot I_6\cdot I_6=\begin{pmatrix}1&0&-1&0&0&0\\0&1&0&-1&0&0\\0&0&1&0&-1&0\\0&0&0&1&0&-1\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}$$

$$B_{v_3}=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\1&0&1&0&1&0\\0&1&0&1&0&1\end{pmatrix}\cdot I_6\cdot I_6=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\1&0&1&0&1&0\\0&1&0&1&0&1\end{pmatrix},$$

and we see that the above boundary conditions separate the edges into two sets, $\{e_1^1,e_1^2,e_1^3\}$ and $\{e_2^1,e_2^2,e_2^3\}$, each dominated by a regular Neumann condition. This enables us to split the vertex v_3 into two distinct vertices of degree 3, each connected to a different set of edges and possessing Neumann boundary conditions. We remark that this would happen for any choice of basis for R, as here $g_1=g_2=g_3=e$.

Note that the resulting quotient is the same as was obtained previously (Fig. 7).

Finally, we repeat the construction for an arbitrary choice of basis which yields an orthogonal matrix representation for R. We can parametrize such a representation in the following way:

$$\left\{\begin{aligned}&\tau\sigma^2\mapsto\begin{pmatrix}\cos^2\theta-\sin^2\theta & -2\cos\theta\sin\theta\\ -2\cos\theta\sin\theta & -\cos^2\theta+\sin^2\theta\end{pmatrix},\\ &\tau\sigma^3\mapsto\begin{pmatrix}2\cos\theta\sin\theta & \cos^2\theta-\sin^2\theta\\ \cos^2\theta-\sin^2\theta & -2\cos\theta\sin\theta\end{pmatrix}\end{aligned}\right\}.$$

For example, the basis we chose in (5.7) is obtained by $\theta=\pi/3$. As remarked, v_3 always splits into two vertices with Neumann conditions, so that Fig. 7 can describe the quotient with respect to any basis. For the parametrization above, we obtain the following boundary conditions:

$$A_{v_1} = A_{v_2} = \begin{pmatrix} 2\sin^2\theta & \sin 2\theta \\ \sin 2\theta & 2-2\sin^2\theta \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A_{v_4} = \begin{pmatrix} 1-\sin 2\theta & 2\sin^2\theta - 1 \\ 2\sin^2\theta - 1 & 1+\sin 2\theta \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$B_{v_1} = B_{v_2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 2-2\sin^2\theta & -\sin 2\theta \\ -\sin 2\theta & 2\sin^2\theta \end{pmatrix}, \qquad B_{v_4} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1+\sin 2\theta & 1-2\sin^2\theta \\ 1-2\sin^2\theta & 1-\sin 2\theta \end{pmatrix}.$$

All of these matrices are of rank one, and can therefore be reduced to square ones by deleting the appropriate rows.[10] We thus get a continuous family of isospectral graphs. Examine two members of this family: $\theta = 0$ and $\theta = 3\pi/4$. The boundary conditions for the case $\theta = 0$ are:

$$A_{v_1} = A_{v_2} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \qquad A_{v_4} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix},$$
$$B_{v_1} = B_{v_2} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B_{v_4} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

When applying this to Fig. 7, we notice that the vertices v_1, v_2 do not stay vertices of degree two, but rather, each of them splits into two vertices of degree one, one with Dirichlet boundary condition, and the other with Neumann. The vertex v_4, however, stays connected and obtains Neumann boundary conditions. Observe that the resulting quotient is the one that we have already obtained as Γ/R_1 (Fig. 2(a)). In a similar manner, the quotient Γ/R_2 (Fig. 2(b)) is obtained from the choice $\theta = 3\pi/4$. We conclude by pointing out that the graph described in Fig. 7 is a good prototype for the mentioned isospectral family, yet it might also be misleading, since there are members of the family whose boundary conditions tear apart the edges connected to some of the vertices and thus change the connectivity of the graph (boundary conditions with this property are called "decoupling" in [12, 13]). One should also pay attention to the fact that we have treated only orthogonal representations of D_4. These are not the most general ones, and we may extend the isospectral family presented above by considering the broader case of all matrix representations of R. In particular, the quotient Γ/R_3 (Fig. 2(c)) is obtained from the unitary representation

$$\left\{ \sigma \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \tau \mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right\}.$$

[10] However, there is no a priori reduction which is valid for all θ!

6 Isospectral Manifolds and Complexes

We now take Γ to be a Riemannian manifold, possibly with a boundary, at which differential boundary conditions are imposed. If Γ admits an action of a finite group G, we can retrace the definitions and results of Sect. 3: the eigenspaces $\Phi_\Gamma(\lambda) = \ker(\lambda I - \Delta)$ are again representations of G, where Δ is the Laplace-Beltrami operator (or any differential operator, for that matter), and again we define Γ's R-spectrum by $\sigma_\Gamma^R : \lambda \mapsto \langle \chi_R, \chi_{\Phi_\Gamma(\lambda)} \rangle_G$.

In order to simplify the presentation, we limit our attention to $\bigoplus_{\lambda\in\mathbb{C}} \Phi_\Gamma(\lambda)$, the space spanned by the Laplacian's eigenfunctions, which we denote by $\mathcal{H}(\Gamma)$. For a representation R of G, we define a Γ/R-manifold to be any Riemannian manifold Γ' such that there is an isomorphism

$$\mathcal{H}(\Gamma') \cong \operatorname{Hom}_{\mathbb{C}G}(R, \mathcal{H}(\Gamma)) \tag{6.1}$$

intertwining the Laplacian (which is again defined on $\operatorname{Hom}_{\mathbb{C}G}(R, \mathcal{H}(\Gamma))$ by $(\Delta f)(r) = \Delta(f(r))$). Note that (6.1) is simply what one gets from (3.4) upon restricting one's attention to $\mathcal{H}$. As before, the eigenvalue spectrum of Γ/R is well-defined and equals the R-spectrum of Γ, as from

$$\operatorname{Hom}_{\mathbb{C}G}(R, \Phi_\Gamma(\lambda)) = \ker_{\operatorname{Hom}_{\mathbb{C}G}(R,\mathcal{H}(\Gamma))}(\lambda I - \Delta) \cong \ker_{\mathcal{H}(\Gamma/R)}(\lambda I - \Delta)$$

we obtain

$$\sigma_{\Gamma/R}(\lambda) = \dim \operatorname{Hom}_{\mathbb{C}G}(R, \Phi_\Gamma(\lambda)) = \langle \chi_R, \chi_{\Phi_\Gamma(\lambda)} \rangle_G = \sigma_\Gamma^R(\lambda).$$

The algebra underlying the rest of Sect. 3 remains valid in the new settings, and gives us that $\Gamma/\mathbb{C}G$ is isospectral to Γ, and that for a representation R of $H \le G$, Γ/R and $\Gamma/\operatorname{Ind}_H^G R$ are isospectral, from which follows that for representations R_1, R_2 of $H_1, H_2 \le G$ satisfying $\operatorname{Ind}_{H_1}^G R_1 \cong \operatorname{Ind}_{H_2}^G R_2$, Γ/R_1 and Γ/R_2 are isospectral.

The reader might thus wonder why have we focused on quantum graphs, until now. The main reason is that under fairly moderate assumptions (e.g., self-adjoint Laplacian or a free action) one can actually produce a quotient graph for every representation, as is demonstrated in Sect. 4. Graphs are one-dimensional manifolds with singularities (at the vertices), and it is these singularities that we exploit, by endowing them with the appropriate boundary conditions, to encapsulate the restrictions arising from a choice of a representation. In higher dimensions, manifolds with a boundary, carrying Neumann, Dirichlet, or a more complicated boundary condition, are a generalization of this idea, and one goal of this section is to demonstrate that some known examples of isospectral manifolds can be understood by our theory. That is, we show that for some known isospectral pairs the manifolds and their boundary conditions are such that they are quotients (in the sense of (6.1)) of a common covering manifold by two representations with isomorphic inductions in some supergroup of symmetries.

It turns out, however, that in order to form a quotient by a general representation we need more singularities than just boundaries (at least via our construction). A graph is a one-dimensional manifold when all of its vertices are of degree two, and a manifold with boundary when all vertices are of degree at most two. Unfortunately,

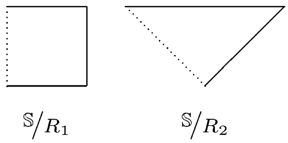


Fig. 9 The two isospectral domains presented in [26], obtained as quotients of the square $\mathbb{S}$ (Fig. 10) by the representations (5.1), (5.2). *Solid lines* indicate Dirichlet boundary conditions and *dotted* ones Neumann

even if a graph has one of these properties, its quotient by a multidimensional representation (as constructed in Sect. 4) need not have either, since the degrees of the vertices are multiplied, in general, by the dimension of the representation (they can decrease due to nontrivial stabilizers and decoupling).

Carrying over the construction method of Sect. 4 to general Riemannian manifolds (e.g., by replacing graphs with higher dimensional simplicial structures) yields objects we might call "quantum-complexes". In general, these consist of several Riemannian manifolds of the same dimension "glued" along their boundaries by homogeneous boundary conditions (so in dimension one, we obtain the notion of quantum graphs). When a boundary condition involves the boundaries of more than two manifolds, the result is no longer a manifold.[11] We remark that this gives Corollary 4 an additional significance, as it shows that working with representations of lower dimension is not only a computational convenience, but also leads to quotients with simpler singularities, or none at all.

6.1 Isospectral Drums

In [25, 26], Jakobson et al., and Levitin et al., respectively, obtain several examples of isospectral domains with mixed Dirichlet-Neumann boundary conditions, all of which can be interpreted as quotients with respect to representations sharing a common induction. As a basic demonstration of the generalization of our theory to higher dimensions, we reconstruct an isospectral pair consisting of a square and a triangle with mixed boundary conditions (Fig. 1 in [26], 9 here).

This example rests upon our acquaintance D_4, so that we can reuse the definitions and results of Sect. 5. In place of the graph in Fig. 1, we now consider the full square $\mathbb{S}$, with Dirichlet boundary conditions, and with $G = D_4$ acting as one would expect (Fig. 10).

The domains in Fig. 9 are quotients of the square $\mathbb{S}$ (Fig. 10) by the representations R_1 and R_2 of $H_1, H_2 \leq G$, which are defined in (5.1), (5.2). Since $\mathrm{Ind}_{H_1}^{G} R_1 \cong \mathrm{Ind}_{H_2}^{G} R_2$, the two domains are isospectral.

We demonstrate the construction of $\mathbb{S}/R_1$. As before, $\mathrm{Hom}_{\mathbb{C}H_1}(R_1, \mathcal{H}(\mathbb{S})) \cong \mathcal{H}(\mathbb{S})^{R_1}$ (since R_1 is one-dimensional), hence $\mathcal{H}(\mathbb{S}/R_1)$ should encode the R_1-isotypic component of $\mathcal{H}(\mathbb{S})$. $\mathbb{T}$, the first quadrant of $\mathbb{S}$ (Fig. 11(a)), is a fundamental domain for the action of H_1, so that given $f \in \mathcal{H}(\mathbb{T})$ it is possible to construct at

[11] But even though this is in general the case, by choosing an appropriate action, representation and bases, it is possible to obtain manifolds even when taking a quotient by a multidimensional representation, due to the mentioned phenomena of stabilizers and decoupling.

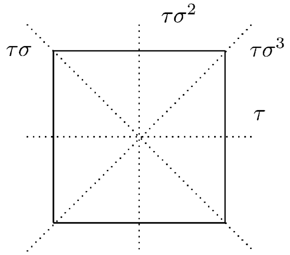


Fig. 10 The square $\mathbb{S}$, and the axes of reflection elements in D_4

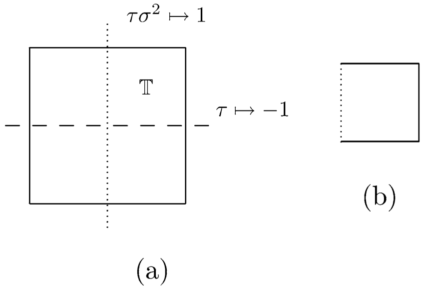


Fig. 11 (**a**) The fundamental domain $\mathbb{T}$ for $\mathbb{S}/H_1$; every $\tilde{f} \in \mathcal{H}(\mathbb{S})^{R_1}$ vanishes along the *dashed line* and has zero normal derivative at the *dotted line*. (**b**) The quotient planar domain $\mathbb{S}/R_1$ which encodes this information. The *solid lines* represent Dirichlet boundary conditions and the *dotted* one Neumann

most one function in $\mathcal{H}(\mathbb{S})^{R_1}$ whose restriction to $\mathbb{T}$ is f. Thus, the restriction map $\Psi : \mathcal{H}(\mathbb{S})^{R_1} \to \mathcal{H}(\mathbb{T})$ is injective. In order for it to be surjective, we must impose suitable boundary conditions on $\mathbb{T}$. From (5.1) we obtain information on $\tilde{f} \in \mathcal{H}(\mathbb{S})^{R_1}$. Since such $\tilde{f}$ is anti-symmetric with respect to the action of τ, it must vanish at the horizontal axis of reflection, and therefore every $f \in \operatorname{im}\Psi$ vanishes at the lower edge of $\mathbb{T}$. Similarly, every $\tilde{f} \in \mathcal{H}(\mathbb{S})^{R_1}$ is symmetric with respect to $\tau\sigma^2$, so that its normal derivative at the vertical axis of reflection is zero, and thus all functions in $\operatorname{im}\Psi$ have vanishing normal derivatives at the left edge of $\mathbb{T}$. This information, summarized in Fig. 11(a), suggests the domain presented in Fig. 11(b) as the quotient $\mathbb{S}/R_1$: a square identical to $\mathbb{T}$, three of whose edges have Dirichlet boundary condition and one Neumann.

Once these boundary conditions are imposed on $\mathbb{S}/R_1$, Ψ is indeed onto: for $f \in \mathcal{H}(\mathbb{S}/R_1)$ which obeys them, we define a function $\tilde{f}$ on $\mathbb{S}$ by $\tilde{f}|_{\mathbb{T}} = f$, $\tau\tilde{f} = -\tilde{f}$, $\tau\sigma^2\tilde{f} = \tilde{f}$, $\sigma^2\tilde{f} = -\tilde{f}$. While $\tilde{f}$ is well defined on the vertical $\tau\sigma^2$-axis even if f does not obey any boundary conditions, it is the requisition that f vanish on the lower edge of $\mathbb{S}/R_1$ that guarantees that $\tilde{f}$ is well defined on the horizontal τ-axis. In a similar manner, while at the τ-axis the two one-sided normal derivatives of $\tilde{f}$ agree a priori, it is the Neumann condition at the left edge of $\mathbb{S}/R_1$ which ensures this at the $\tau\sigma^2$-axis. The boundary conditions thus assure that $\tilde{f}$ is well defined and continuously differentiable, and being piecewise smooth and a sum of Laplacian eigenfunctions, it is smooth, and therefore in $\mathcal{H}(\mathbb{S})$, so that $f = \Psi\tilde{f} \in \operatorname{im}\Psi$. As Ψ and its inverse are linear and commute with the Laplacian, we have established $\operatorname{Hom}_{\mathbb{C}H_1}(R_1, \mathcal{H}(\mathbb{S})) \cong \mathcal{H}(\mathbb{S})^{R_1} \cong \mathcal{H}(\mathbb{S}/R_1)$, as the definition of a $\mathbb{S}/R_1$-domain in (6.1) calls for.

Analogously, from the properties of $\tilde{f} \in \mathcal{H}(\mathbb{S})^{R_2}$ we can deduce the corresponding quotient $\mathbb{S}/R_2$. This process is summarized in the two parts of Fig. 12.

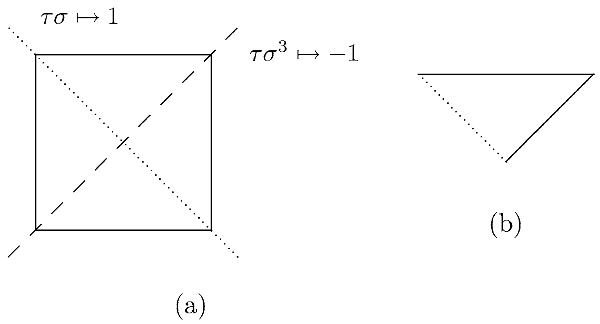


Fig. 12 (**a**) The information we have on $\tilde{f} \in \mathcal{H}(\mathbb{S})^{R_2}$: it vanishes along the *dashed line* and has zero normal derivative at the *dotted line*. (**b**) The quotient planar domain $\mathbb{S}/R_2$ which encodes this information. The *solid lines* represent Dirichlet boundary conditions and the *dotted* one Neumann

We remark that the various constructions demonstrated in Sect. 5 can be applied analogously to $\mathbb{S}$, enriching the isospectral pair in Fig. 9. For example, $\mathbb{S}/R_3$ would be an orbifold with a line that applies a factor of i to functions crossing it. The other isospectral families in [25, 26] can be obtained from various representations of the general dihedral groups D_n, and of the product $D_4 \times D_4$. The interested reader will find some of these constructions in [15].

6.2 The Gordon-Webb-Wolpert Drums

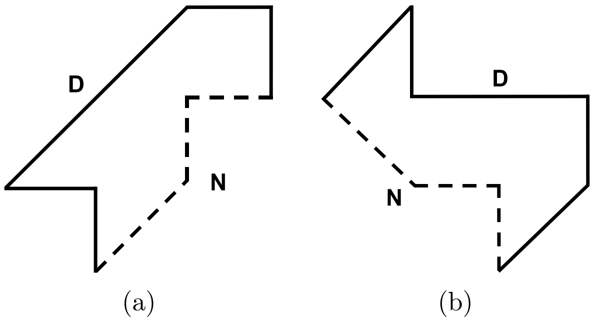
In a similar fashion, we can apply our method to the Gordon-Webb-Wolpert construction [3, 4], obtaining their isospectral planar domains with new boundary conditions. We follow the exposition of Buser et al. [5], who obtain the mentioned drums as follows: they consider G_0, a group of motions of the hyperbolic plane $\mathbb{H}$ ($*444$ in Conway's orbifold notation), and an epimorphism $\pi : G_0 \twoheadrightarrow G = PSL_3(2)$. In G they exhibit two subgroups A and B, each isomorphic to S_4, that satisfy the Sunada condition [1] with respect to G. The quotients of $\mathbb{H}$ by $\pi^{-1}(A)$ and $\pi^{-1}(B)$ are isometric domains. Both are composed of seven copies of a hyperbolic triangle (which is a fundamental domain for the action of G_0), assembled in different configurations (which are determined by the coset structure of the pre-images). Finally, by replacing the fundamental hyperbolic triangle with a suitable Euclidean one, the non-isometric isospectral drums of Gordon et al. are obtained.

An elegant formulation of the Sunada condition for H_1 and H_2 in G is that the inductions of the trivial representations $\mathbf{1}_{H_1}$ and $\mathbf{1}_{H_2}$ to G are isomorphic, i.e.

$$\mathrm{Ind}_{H_1}^{G}\mathbf{1}_{H_1} \cong \mathrm{Ind}_{H_2}^{G}\mathbf{1}_{H_2}. \tag{6.2}$$

In fact, the connection between A and B is stronger than this (reflecting a line-point duality in the Fano plane): it turns out that for every representation R of S_4, $\mathrm{Ind}_A^G R \cong \mathrm{Ind}_B^G R$. For each such R, we can thus construct an isospectral pair by taking the quotient of $\mathbb{H}$ by the pullbacks of R to $\pi^{-1}(A)$ and $\pi^{-1}(B)$. Taking $R = \mathbf{1}_{S_4}$ will produce once again the planar drums of Gordon et al. In fact, we shall see in Sect. 6.3 that taking quotient (in our sense) by the trivial representation of a group is equal to taking quotient (in the classical sense) by the group. Taking R to be the sign representation of S_4, and again replacing the fundamental hyperbolic triangles with Euclidean ones, we obtain the same drums but with different boundary conditions (Fig. 13).

Fig. 13 The isospectral drums of Gordon et al. with new boundary conditions



We conclude this example by pointing out that in [5] a wide variety of isospectral pairs is presented, using various symmetry groups of $\mathbb{H}$. All these examples can be exploited to construct other isospectral pairs, as isomorphic inductions may be found either from Sunada triples or by taking appropriate sums of irreducible representations.

6.3 The Sunada Method

We recall the classical theorem of Sunada [1]:

> If G acts freely on a Riemannian manifold Γ, and $H_1, H_2 \leq G$ satisfy (6.2), then Γ/H_1 and Γ/H_2 are isospectral manifolds.

Sunada's theorem follows from our theory, once we show that for a finite group G acting freely on a manifold Γ, the quotient manifold Γ/G is a $\Gamma/\mathbf{1}_G$-manifold, that is,

$$\mathcal{H}(\Gamma/G) \cong \mathrm{Hom}_{\mathbb{C}G}(\mathbf{1}_G, \mathcal{H}(\Gamma)). \tag{6.3}$$

This follows from the observation that $\mathrm{Hom}_{\mathbb{C}G}(\mathbf{1}_G, L_2(\Gamma))$ corresponds naturally to $L_2(\Gamma)^{\mathbf{1}_G} = L_2(\Gamma)^G$, the trivial component of $L_2(\Gamma)$, and this is the space of L_2-functions on Γ which are stable under all elements of G. But these are exactly the functions which factor through Γ/G, hence $L_2(\Gamma/G) \cong \mathrm{Hom}_{\mathbb{C}G}(\mathbf{1}_G, L_2(\Gamma))$, and in particular (6.3) follows.

Remark We can view the preceding argument as yet another proof for Sunada's theorem, but this would be presumptuous. In fact, Pesce [27] uses Frobenius Reciprocity in exactly the same manner to reprove Sunada's theorem. A survey of different proofs for Sunada's theorem, among them Pesce's, can be found in [10].

7 Summary and Open Questions

The main construction presented in this paper is that of objects denoted Γ/R, where R is a complex representation of a finite group acting on a geometric object Γ. For such Γ and R there can be, in general, many objects so denoted, and they are all isospectral to one another. Furthermore, these objects are defined so that whenever

$\mathrm{Hom}_{\mathbb{C}H_1}(R_1,\_) \cong \mathrm{Hom}_{\mathbb{C}H_2}(R_2,\_)$, where each R_i is a representation of a group H_i acting on Γ, there is also isospectrality between Γ/R_1 and Γ/R_2. The consequences of this are explored in Sect. 3, and in particular we find two convenient means for the construction of isospectral objects:

Starting with a group G: take subgroups $H_1, H_2 \leq G$ and corresponding representations R_1, R_2 sharing a common induction in G.[12] For any object Γ on which G acts by symmetries, Γ/R_1 and Γ/R_2 are isospectral.

Starting with an object Γ: find a group G acting on Γ and construct $\Gamma/\mathbb{C}G$ (by some choice of representatives and bases, as explained in Sect. 4.2). Any quotient thus obtained is isospectral to Γ itself, by the analogue of Proposition 2 for arbitrary geometric objects.

It is natural to ask to what extent the various methods for obtaining isospectral objects overlap. For example, in Sect. 5, three isospectral graphs (Fig. 2) are obtained from representations with isomorphic inductions. However, at the end of the same section it is demonstrated that all of them (together with others) could have also been obtained as Γ/R for a single R, their induction, by different choices of bases. Can one expect that given a basis for R, there is always a basis for $\mathrm{Ind}_H^G R$ with respect to which Γ/R and $\Gamma/\mathrm{Ind}_H^G R$ are isometric?

Even when limiting to the basic quotient construction, questions arise. For R and Γ as above, we have a family of isospectral objects Γ/R, varying as one moves between different choices of bases in the construction, as explained in Sect. 4.2 and demonstrated in the last part of Sect. 5. This family has the topology of a manifold, being parametrized by the action of a general linear group on the space of possible bases. Surveying this continuum of quotient objects, one might ask where along it do changes occur in the shape of the objects (in contrast with only boundary condition changes), in the number of connected components, etc. One can look for certain types of objects in this continuum, such as manifolds, billiards, objects with real boundary conditions, or ones with a self-adjoint Laplacian. Such questions seem to lead to a deeper research in differential and algebraic geometry, investigating the critical points at which changes occur or the algebraic varieties at which certain conditions are fulfilled. Except for the basic demonstration of these phenomena in Sect. 5, we have not treated these questions.

We list some more questions that seem interesting, and which we have not regarded:

- Γ is naturally a $\Gamma/\mathbb{C}G$-graph. Does it occur by our construction? It seems that the answer is yes, by taking G as a basis for $\mathbb{C}G$, but we have not shown this.
- Given two isospectral objects, can it be decided algorithmically whether they are representation-quotients of a common object? Can it be done given a transplantation between the two?

[12] This resembles the Sunada condition, but is dramatically easier to achieve, since we are free to take any representations of the subgroups (instead of only the trivial ones). A systematic approach would be to take all irreducible characters of subgroups of G, induce them to G, find linear dependencies, and sum the corresponding representations accordingly. Also, any H_1 and R_1 are usable with $H_2 = G$, by taking $R_2 = \mathrm{Ind}_{H_1}^G R_1$.

- What are the necessary and sufficient conditions for the quotients constructed in Sect. 4.2 to be proper quotient graphs (in contrast with generalized ones)? Exact quantum graphs? Graphs with a self-adjoint Laplacian?
- Can the isomorphism (3.4) be understood as natural, in a suitable category? This can be interpreted both as (contravariant) functoriality in R, or as functoriality in Γ, which would require a definition of quantum graph morphisms.
- Can the theory presented in this paper be applied to discrete graphs? To representations of Lie groups acting on Riemannian manifolds?
- It is clear that $\mathcal{H}(\Gamma \amalg \Gamma') = \mathcal{H}(\Gamma) \oplus \mathcal{H}(\Gamma')$, so that $\sigma_{\Gamma \amalg \Gamma'} \equiv \sigma_\Gamma + \sigma_{\Gamma'}$, and given bases for R and R', their union is a basis for $R \oplus R'$ with respect to which $\Gamma/R \oplus R'$ is isometric to $\Gamma/R \amalg \Gamma/R'$ (see [15], 9.3 for an example). Is there an operation $\otimes$ on graphs, or general geometric object, which gives $\mathcal{H}(\Gamma \otimes \Gamma') = \mathcal{H}(\Gamma) \otimes \mathcal{H}(\Gamma')$, so that $\sigma_{\Gamma \otimes \Gamma'} \equiv \sigma_\Gamma \cdot \sigma_{\Gamma'}$? What about convolution: $\sigma_{\Gamma \star \Gamma'} \equiv \sigma_\Gamma \star \sigma_{\Gamma'}$?
- A classical conjecture, originally aimed at Riemannian manifolds[13] [28]: for $G = \mathrm{Aut}\Gamma$, and $R = \bigoplus_{i=1}^r S_i$, where $\{S_i\}_{i=1}^r$ are the irreducible representations of G, is $\sigma_\Gamma^R \le 1$?

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References

1. Sunada, T.: Riemannian coverings and isospectral manifolds. Ann. Math. **121**, 169–186 (1985)
2. Kac, M.: Can one hear the shape of a drum? Am. Math. Mon. **73**, 1–23 (1966)
3. Gordon, C., Webb, D., Wolpert, S.: One cannot hear the shape of a drum. Bull. Am. Math. Soc. **27**, 134–138 (1992)
4. Gordon, C., Webb, D., Wolpert, S.: Isospectral plane domains and surfaces via Riemannian orbifolds. Invent. Math. **110**, 1–22 (1992)
5. Buser, P., Conway, J., Doyle, P., Semmler, K.-D.: Some planar isospectral domains. Int. Math. Res. Not. **9**, 391–400 (1994)
6. von Below, J.: Can one hear the shape of a network. In: Partial Differential Equations on Multistructures. Lecture Notes in Pure and Applied Mathematics, vol. 219, pp. 19–36. Dekker, New York (2000)
7. Milnor, J.: Eigenvalues of the Laplace operator on certain manifolds. Proc. Natl. Acad. Sci. USA **51**, 542 (1964)
8. Gordon, C., Perry, P., Schueth, D.: Isospectral and isoscattering manifolds: A survey of techniques and examples. Contemp. Math. **387**, 157–179 (2005)
9. Brooks, R.: Constructing isospectral manifolds. Am. Math. Mon. **95**, 823–839 (1988)
10. Brooks, R.: The Sunada method. Contemp. Math. **231**, 25–35 (1999)

[13] This question, in the context of quantum graphs, was suggested to us by L. Friedlander.

11. Gnutzmann, S., Smilansky, U.: Quantum graphs: Applications to quantum chaos and universal spectral statistics. Adv. Phys. **55**, 527–625 (2006)
12. Kuchment, P.: Quantum graphs: I. Some basic structures. Waves Random Media **14**, S107 (2004)
13. Kuchment, P.: Quantum graphs: An introduction and a brief survey. In: Proc. Symp. Pure Math., pp. 291–314. AMS, Providence (2008)
14. Kostrykin, V., Schrader, R.: Neumann's rule for quantum wires. J. Phys. A **32**, 595–630 (1999)
15. Band, R., Parzanchevski, O., Ben-Shach, G.: The isospectral fruits of representation theory: Quantum graphs and drums. J. Phys. A **42**, 175202 (2009)
16. Gutkin, B., Smilansky, U.: Can one hear the shape of a graph? J. Phys. A **31**, 6061–6068 (2001)
17. Roth, J.P.: Le spectre du Laplacien sur un graphe. In: Proceedings of the Colloque J. Deny Orsay 1983, Lect. Not. Math., vol. 1096, pp. 521–539 (1984)
18. Oren, I.: Private communication (2008)
19. Shapira, T., Smilansky, U.: Quantum graphs which sound the same. In: Khanna, F., Matrasulov, D. (eds.) Nonlinear Dynamics and Fundamental Interactions. NATO Science Series II: Mathematics, Physics and Chemistry, vol. 213, pp. 17–29 (2005)
    The updated reference for item 19 (Tashkent) is: Shapira, T., Smilansky U.: Quantum graphs which sound the same. In:
20. Band, R., Shapira, T., Smilansky, U.: Nodal domains on isospectral quantum graphs: The resolution of isospectrality? J. Phys. A, Math. Gen. **39**, 13999–14014 (2006)
21. Brüning, J., Heintze, E.: Représentations des groupes d'isométries dans les sous-espaces propres du laplacien. C. R. Acad. Sci. Paris **286**, 921–923 (1978)
22. Bolte, J., Endres, S.: The trace formula for quantum graphs with general self-adjoint boundary conditions. Ann. Henri Poincaré **10**(1), 189–223 (2009)
23. Buser, P.: Isospectral Riemann surfaces. Ann. Inst. Fourier **36**, 167–192 (1986)
24. Berard, P.: Transplantation et isospectralité I. Math. Ann. **292**, 547–559 (1992)
25. Jakobson, D., Levitin, M., Nadirashvili, N., Polterovich, I.: Spectral problems with mixed Dirichlet-Neumann boundary conditions: Isospectrality and beyond. J. Comput. Appl. Math. **194**, 141–155 (2004)
26. Levitin, M., Parnovski, L., Polterovich, I.: Isospectral domains with mixed boundary conditions. J. Phys. A, Math. Gen. **39**, 2073–2082 (2005)
27. Pesce, H.: Variétés isospectrales et représentations des groupes. Contemp. Math. **173**, 231–240 (1994)
28. Wigner, E.P.: Group Theory and its Applications to the Quantum Mechanics of Atomic Spectra. Academic Press, New York (1959)