QUANTUM GRAPHS - TCC COURSE - JANUARY-MARCH 2012

Homework 8

Various Variations

(1) Show that the spectrum of a quantum graph with a non-negative potential ($V \geq 0$) is non-negative if all the vertex conditions are of δ-type with non-negative coupling coefficients (i.e., $\forall\, v \in \mathcal{V}\ ;\ \ \alpha_v \geq 0$).

(2) Prove proposition 2 which was stated in class:
Let $\lambda = \lambda(\alpha)$ be a simple eigenvalue of a graph Γ which satisfies δ-type vertex conditions at a certain vertex v with the coupling coefficient $\alpha \neq \infty$. Then

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\alpha} = |f(v)|^2 .$$

In addition, if we re-parameterize the condition at v as:

$$\zeta \sum_{e \in E_v} \frac{\mathrm{d}f}{\mathrm{d}x_e}(v) = -f(v),$$

now allowing Dirichlet ($\zeta = 0$) and excluding Neumann ($\zeta = \infty$) conditions, the derivative is

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\zeta} = \left| \sum_{e \in E_v} \frac{\mathrm{d}f}{\mathrm{d}x_e}(v) \right|^2 .$$

Hint: for the first part of the proof follow the steps of the proof of proposition 1, which was done in class.