QUANTUM GRAPHS - TCC COURSE - JANUARY-MARCH 2012

Homework 7

Quantum to Classical correspondence for Quantum Graphs

In this homework we study the classical dynamics of a quantum graph. This will help us to understand in what sense the classical dynamics that corresponds to a quantum graph is 'chaotic'. As you remember, the quantum evolution map,$U(k)$, contains amplitudes for scattering processes to go from one directed edge to another. We define a corresponding classical map, M, by replacing the amplitudes $U(k)_{\alpha\alpha'}$ by

$$M_{\alpha\alpha'} = |U(k)_{\alpha\alpha'}|^2 = |S_{\alpha\alpha'}|^2 .$$

M is a matrix of dimensions $2E \times 2E$, which contains the probabilities for the scattering events.

(1) Prove that the matrix M is a *bi-stochastic (doubly stochastic) matrix*. Namely, prove that

$$\sum_{\alpha=1}^{2B} M_{\alpha\alpha'} = \sum_{\alpha'=1}^{2B} M_{\alpha\alpha'} = 1.$$

The stochastic property allows to define a (classical) Markov process on the graph:
Let $P_\alpha(n)$ be the probability find a particle on the directed edge α, at some (discrete) time n. We can then define the probabilities to find the particle on the directed edge α, at time $n+1$, by

$$P_\alpha(n+1) = \sum_{\alpha'} M_{\alpha\alpha'} P_{\alpha'}(n) .$$

(2) Show that if $P(n)$ satisfies $\sum_\alpha P_\alpha(n) = 1$, then $P(n+1)$ satisfies the same property. This means that a probability vector stays a probability vector after acting on it with M.

Let $P^{\mathrm{inv}} = \frac{1}{2E}$ be the equi-distributed probability vector on the graph. It is easy to show that $M\, P^{\mathrm{inv}} = P^{\mathrm{inv}}$ for any quantum graph.

The classical dynamics which corresponds to a quantum graph is chaotic in the following sense.

The Markov process on the graph is called *ergodic* if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} P(m) = P^{\text{inv}}$$

for every initial probability vector $P(0)$.
Every connected graph is ergodic.

Most graphs are also *mixing*, i.e.,

$$\lim_{n\to\infty} P(n) = P^{\text{inv}}$$

for every initial probability vector $P(0)$.

In the following questions we will characterize ergodicity and mixing in terms of M's eigenvalues.

(3) Prove that all the eigenvalues of M are either on the unit circle or inside it. Namely, if we denote the set of eigenvalues by $\{\lambda_i\}$, then $\forall i\,;\ |\lambda_i| \leq 1$.

(4) We know that M has at least one eigenvalue which equals 1 (the corresponding eigenvector is P^{inv}). Let us denote this eigenvalue by λ_1 ($\lambda_1 = 1$). Prove that if $\min_{2\leq i\leq 2E}\left(1 - |\lambda_i|\right) > 0$ then the graph is mixing.
hint - it might be useful to prove the convergence property using the vectors L_1-norm.

(5) Using the notation above ($\lambda_1 = 1$), prove that if $\min_{2\leq i\leq 2E}\left(|1 - \lambda_i|\right) > 0$ then the graph is ergodic.
hint - the same hint as in the previous question.

Note that from the conditions above you can deduce that mixing is a stronger notion than ergodicity (namely, that every mixing system is also ergodic).

The quantity $\Delta := \min_{2\leq i\leq 2E}\left(|1 - \lambda_i|\right)$ is called the spectral gap and it determines the convergence rate of $\lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} P(m)$ (the greater the gap, the quicker is the convergence).

Similarly, $\tilde{\Delta} := \min_{2\leq i\leq 2E}\left(1 - |\lambda_i|\right)$ determines the convergence rate of $\lim_{n\to\infty} P(n)$.