QUANTUM GRAPHS - TCC COURSE - JANUARY-MARCH 2012

HOMEWORK 5

The constant term of the Trace formula

(1) In the following question we will complete the missing ingredients from the proof done in class for the expression of the constant term, N_0, of the trace formula. Consider a Neumann graph (all vertex conditions are of Neumann type) with a single connected component. Let $\vec{E}$ be the space of directed edges on the graph (this space is of dimension $2E$, where E is the number of edges).

(a) Let $\omega : \vec{E} \to \mathbb{C}$ such that

$$\forall (i,j) \in \vec{E} \ ; \ \ \omega(i,j) = -\omega(j,i)$$

and

$$\forall i \ \sum_{j\sim i} \omega(i,j) = 0,$$

where $j \sim i$ means that the vertex j is adjacent to the vertex i. All such functions $\omega : \vec{E} \to \mathbb{C}$ form a vector space (over $\mathbb{C}$). Prove that the dimension of this space is $\beta := E - V + 1$. You can follow some of the steps we used in class for this purpose.

(b) Let $\vec{a}^{in} \in \mathbb{C}^{2E}$ with entries denoted by $a_j^{(i),in}$ (for $i \sim j$), such that the following is satisified

$$\forall i \sim j \ , \ i \sim k \ ; \ \ a_j^{(i),in} + a_i^{(j),in} = a_k^{(i),in} + a_i^{(k),in}$$

and

$$\forall i \ \sum_{j\sim i} -a_j^{(i),in} + a_i^{(j),in} = 0.$$

Prove that the dimension of the vector space which contains all such solutions $\vec{a}^{in} \in \mathbb{C}^{2E}$ is $\beta + 1$. You can use your answer to section (a).

(c) Note that this means that $\dim\ker(\mathbf{1} - S) = \beta + 1$ and therefore $N_0 = -\frac{\beta+1}{2}$, which gives the following trace formula for the spectral counting function

$$N(k) = \frac{1-\beta}{2} + \frac{\mathcal{L}}{\pi}k + -\lim_{\epsilon\to 0}\frac{1}{\pi}\text{Im}\,\log\zeta(k+i\epsilon).$$

Show that for a Neumann graph with C (disjoint) connected components, this result would be generalized to

$$N(k) = \frac{C-\beta}{2} + \frac{\mathcal{L}}{\pi}k + -\lim_{\epsilon\to 0}\frac{1}{\pi}\text{Im}\,\log\zeta(k+i\epsilon).$$

You can use the generalized definition of β, which is $\beta := E - V + C$ (this value can be obtained by summing over all the β's of the different components).

From quantum graphs to discrete graphs

(2) In this question we'll start revealing the spectral connection between quantum graphs and discrete graphs.

(a) Consider an arbitrary quantum graph with V vertices and E edges. Assume that Neumann conditions are imposed on all vertices and that all edges are of the same length, l. Use the following representation for an eigenfunction with eigenvalue k^2 on the edge (i, j)

$$f_{ij}(x_{ij}) = \frac{f_j \sin(kx_{ij}) + f_i \sin(k(l - x_{ij}))}{\sin(kl)}.$$

and the restrictions dictated by the Neumann conditions to obtain a set of V homogeneous equations for the variables f_i $(i = 1, 2, \ldots V)$.

(b) Denote by $\vec{f}$ the vector whose entries are all the f_i variables. Assume that $sin\,(kl) \neq 0$ and manipulate the linear set of equations you got in (a) to have the following form

$$A\,\vec{f} = \cos(kl)\,\vec{f}.$$

What is the matrix A?

Note that this matrix describes the underlying discrete graph. Note also that (almost all) the eigenvalues of the quantum graph are obtained as $\frac{1}{l}\arccos(\lambda)$, where λ is some eigenvalue of A. This establishes a spectral connection between the discrete and the quantum graph, on which we will elaborate more in the next lecture.