QUANTUM GRAPHS - TCC COURSE - JANUARY-MARCH 2012

HOMEWORK 4

The Trace Formula

(1) Consider an $M \times M$ unitary matrix U with unimodular eigenvalues $e^{i\theta_\ell}$ for $\ell = 1, \dots, M$. One may extend the spectrum of eigenphases θ_ℓ periodically beyond the interval $0 \le \theta < 2\pi$. The extended spectrum then consists of the numbers

$$\theta_{\ell,n} = \theta_\ell + n2\pi \qquad n \in \mathbb{Z} \ .$$

Assume that $\theta_\ell \neq 0$ and show that one may write the corresponding spectral counting function

$$N(\theta) = \sum_{n=0}^{\infty} \sum_{\ell=1}^{M} \vartheta(\theta - \theta_\ell - n2\pi)$$

as the following trace formula

$$N(\theta) = \frac{M\theta}{2\pi} - \frac{1}{\pi}\mathrm{Im} \log \det (1 - U) + \frac{1}{\pi}\mathrm{Im} \log \det (1 - e^{-i\theta}U).$$

Compare this to the trace formula of a quantum graph. Show that the above formula is equivalent to the one obtained in the lecture for quantum graphs if all the edge lengths of the graph are the same.
Hint: You can either perform a similar calculation to the one done in class, or alternatively (and more interesting):

- Write the spectral counting function as

$$N(\theta) = \sum_{n=-\infty}^{\infty} \sum_{\ell=1}^{M} \vartheta(\theta - \theta_\ell - n2\pi)\vartheta(\theta_\ell + n2\pi).$$

- Use the Poisson summation formula

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{\nu=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\pi i \nu x} f(x) dx \ .$$

- While using the formula above, evaluate separately the $\nu = 0$ term from the other terms.

Periodic Orbits

Recall that we have obtained in class the following

$$\text{tr } U(k)^n = \sum_{\alpha_1,\ldots,\alpha_n=1}^{2E} e^{ikl_{\alpha_1}} S_{\alpha_1\alpha_n} e^{ikl_{\alpha_n}} S_{\alpha_n\alpha_{n-1}} \ldots e^{ikl_{\alpha_3}} S_{\alpha_3\alpha_2} e^{ikl_{\alpha_2}} S_{\alpha_2\alpha_1} \ ,$$

where $\{\alpha_1, \ldots, \alpha_n\}$ is a set of indices which denote some n directed edges of the graph.

In the sum above, each term is of the form $A_\gamma e^{ik\ell_\gamma}$

$$\begin{aligned} A_\gamma =& S_{\alpha_1\alpha_n} S_{\alpha_n\alpha_{n-1}} \ldots S_{\alpha_2\alpha_1} \\ \ell_\gamma =& l_{\alpha_n} + l_{\alpha_{n-1}} + \ldots l_1 \end{aligned}$$

Here $\gamma = (\alpha_1, \ldots, \alpha_n)$ is a fixed set of summaton indices which corresponds to a sequence of directed edges.

Note that $A_\gamma \neq 0$ only if the edge α_{j+1} follows the edge α_j in the graph, for $j = 1, \ldots, n$ (in this context $\alpha_{n+1} \equiv \alpha_1$) – i.e. only if γ is a closed trajectory on the graph. For such a trajectory, l_γ is its total length, i.e., the sum of its edge lengths.

You can see a few examples of such closed trajectories in figure 0.1.

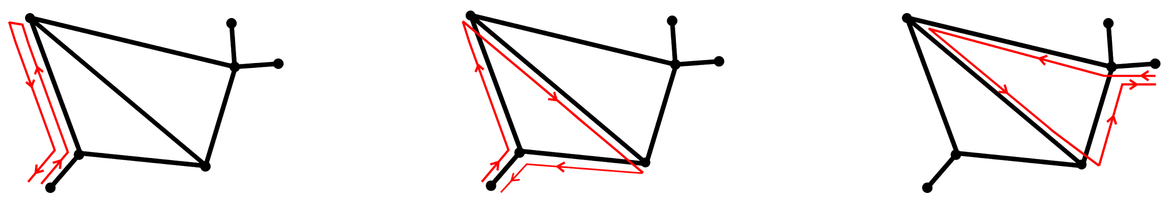

FIGURE 0.1. Three examples of closed trajectories on a graph.

We now bring a few observations and definitions related to periodic orbits.

- Note that any cyclic permutation, γ', of the indices in the trajectory γ gives a different closed trajectory with the same contribution $A_{\gamma'} = A_\gamma$ and $\ell_{\gamma'} = \ell_\gamma$.

- The equivalence class $\overline{\gamma} = \overline{\alpha_1 \ldots \alpha_n}$ that contains all cyclic permutations of a given closed trajectory $\gamma = (\alpha_1, \ldots, \alpha_n)$ is called a ***periodic orbit*** with ***period*** n on the graph.

- The periodic orbit $\overline{\gamma} = \overline{\alpha_1 \ldots \alpha_n}$ is a ***primitive periodic orbit*** of primitive period n if the sequence of indices $\alpha_1, \ldots, \alpha_n$ is not a repetition of a shorter sequence.
  All the closed trajectories in figure 0.1 represent primitive periodic orbits.

- If $\overline{\gamma}$ is a periodic orbit with period n, then there exists a unique primitive periodic orbit $\overline{\gamma}_p$ with ***primitive period*** n_p such that $\overline{\gamma}$ is a repetition of $\overline{\gamma}_p$ and $n = rn_p$. Here $r \geq 1$ is the integer ***repetition number*** of $\overline{\gamma}$ and n_p the ***primitive period*** of $\overline{\gamma}$.

- If $r = 1$ then $\overline{\gamma} = \overline{\gamma}_p$ and $\overline{\gamma}$ is primitive.

- We write $\overline{\gamma} = \overline{\gamma}_p^r$ for the r-th repetition of $\overline{\gamma}_p$.

Further reading is available from the review paper "Quantum graphs: Applications to quantum chaos and universal spectral statistics" by S. Gnutzmann and U. Smilansky (see link in the course website).

And now, some questions about periodic orbits:

(2) Consider the tetrahedron graph, i.e. the complete Neumann graph with $V = 4$ vertices. Choose two periodic orbits on the graph, such that one of them is primitive and the second is some repetition of the first. For each of those periodic orbits, $\overline{\gamma}$, evaluate
  - (a) The period n of the orbit.
  - (b) The length ℓ_γ of the orbit (expressed in terms of the graph edge lengths).
  - (c) The coefficient A_γ which corresponds to the orbit in the periodic orbits expansion (write the explicit number).
  - (d) For the non-primitive orbit, what is the repetition number, r, and the primitive period n_p?

(3) The number of periodic orbits up to a given period can be counted using the $2E \times 2E$ edge adjacency matrix $\mathcal{B}$ whose indices correspond to the directed edges where

$$\mathcal{B}_{\alpha\alpha.} = \begin{cases} 1 & \text{if } \alpha \text{follows } \alpha', \\ 0 & \text{else.} \end{cases}$$

  - (a) Show that $\frac{1}{n}\text{tr}\mathcal{B}^n = \sum_{\gamma:n_\gamma=n} \frac{1}{r_\gamma}$ where the sum is over all periodic orbits of period n. Conclude that if n is a prime number then $\frac{1}{n}\text{tr}\mathcal{B}^n$ is the number of periodic orbits of period n.
  - (b) Derive an expression for the number of periodic orbits of period n in terms of traces of powers of $\mathcal{B}$ for (*i.*) $n = p^j$ where p is a prime number and $j \geq 2$ an integer, and (*ii.*) $n = p_1 p_2$ where p_1 and p_2 are prime numbers. Make an educated guess for the general expression when n has the prime number decomposition $n = \prod_m p_m^{j_m}$ where $j_m \geq 0$ is the multiplicity of the m-the prime.

(4) This question demonstrates that given some lengths of periodic orbits of an unknown graph, one can reconstruct the graph.
  - (a) There is a graph whose total length (sum of all edge lengths) is $1\frac{5}{6}$. Some of the graph's periodic orbits are of the following lengths: $\frac{2}{3}$, $1\frac{1}{3}$, 2, $2\frac{2}{3}$, 3, $3\frac{1}{3}$, $3\frac{2}{3}$, 4, $4\frac{1}{3}$, $4\frac{2}{3}$, 5. These are actually the lengths of all periodic orbits whose length is not greater than 5. What is the corresponding graph? Draw the graph and indicate the edge lengths on the drawing.
  - (b) There is a graph whose total length is $5\frac{13}{15}$. The lengths of its shortest periodic orbits (all of whose lengths are no greater than 5) are 2, $2\frac{1}{3}$, $2\frac{2}{5}$, $2\frac{2}{3}$, 4, $4\frac{1}{3}$, $4\frac{2}{5}$, $4\frac{2}{3}$, $4\frac{2}{3}$, $4\frac{11}{15}$, 5. When a certain length appears more than once, this means that there are few different periodic orbits of the same length. What is the graph this time?

Hint: how does the shortest periodic orbit of any graph look like?