QUANTUM GRAPHS - TCC COURSE - JANUARY-MARCH 2012

Homework 3

Secular equation - first approach

(1) **BONUS** - By writing

$$f_{12}(x_{12}) = A\cos kx_{12} + B\sin kx_{12}$$

and

$$f_{ij}(x_{ij}) = \frac{f_j \sin(kx_{ij}) + f_i \sin(k(l_{ij} - x_{ij}))}{\sin kL_{ij}},$$

for $i < j$ obtain a set of homogeneous equations for the coefficients A, B, f_i $(i = 1, 2, \dots V)$ and derive a secular function that does not have poles at the Dirichlet spectrum of the edge $e = (1, 2)$.
Hint: There is more than one solution to this - one may also reduce the number of equations (and variables) easily to V (the number of vertices) without re-introducing poles.

Secular equation - scattering approach

(2) Show explicitly that continuity of the eigenfunction at a vertex i of degree d_i and the matching condition $\alpha_i f_i = \sum_{j\in V_i} f_{ij}(i, \text{out})$ are equivalent to the vertex scattering matrix

$$\sigma^{(i)}_{jj'}(k) = \frac{2}{d_i + i\frac{\alpha_i}{k}} - \delta_{jj'} = \begin{cases} \frac{2}{d_i+i\frac{\alpha_i}{k}} & j \neq j' \\ \frac{2}{d_i+i\frac{\alpha_i}{k}} - 1 & j = j' \end{cases}, \tag{0.1}$$

where

$$\vec{a}^{(i),\text{out}} = \boldsymbol{\sigma}^{(i)}(k)\vec{a}^{(i),\text{in}} \tag{0.2}$$

and the vectors $\vec{a}^{(i),\text{out}}$, $\vec{a}^{(i),\text{in}}$ are the vectors of coefficients with which the eigenfunction is given in the form

$$f_{ij}(x_j) = a_j^{(i),\text{in}} e^{-ikx_{ij}} + a_j^{(i),\text{out}} e^{ikx_{ij}} \ ,$$

Note that we impose $x_{ij} = 0$ at the vertex i (alternatively, you may assume that $i < j$ for all incident vertices j, and then this follows from our regular convention, $x_{ij} = 0$ at $\min(i, j)$).

(3) **BONUS** - Show that the vertex scattering matrix

$$\boldsymbol{\sigma}^{(i)} = -\mathbf{1}_{d_i} + \frac{2}{v_i + i\frac{\alpha_i}{k}} \mathbb{E}_{d_i}$$

is unitary. Here, $\mathbb{E}_{d_i}$ is the full $d_i \times d_i$ matrix with all matrix elements equal to one. You may use $\mathbb{E}_{d_i}^2 = d_i \mathbb{E}_{v_i}$ and $\mathbb{E}^\dagger = \mathbb{E}$ where $\mathbb{E}^\dagger$ denotes the hermitian conjugate of $\mathbb{E}$.

(4) Consider a star graph which consists of three edges. The central vertex is denoted by 0 and supplied with Neumann vertex conditions. The boundary vertices are denoted by $1, 2, 3$ and are supplied with Neumann, Dirichlet, Dirichlet conditions, correspondingly. These notations and the edge lengths are shown in figure 0.1.

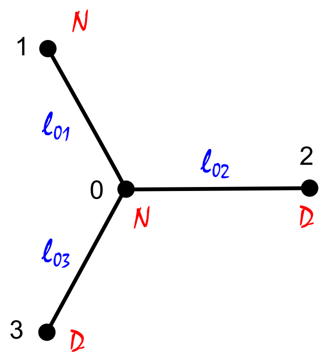

FIGURE 0.1. A star graph.

In this question you will explicitely write the quantum evolution operator, $U(k)$ of the graph above. In order to do so, answer the following sections:

(a) Write explicitely the scattering matrices, $\boldsymbol{\sigma}^{(i)}$, which correspond to each of the vertices $0, 1, 2, 3$. Use equation (0.1) above.

(b) Write the equation (0.2) for each of the vertices. Use the explicit matrices which you have found in the previous section and write the components of the vectors $\vec{a}^{(i),\text{in}}$ and $\vec{a}^{(i),\text{out}}$ with explicit indices in each case (i.e., write $a_0^{(2),in}$, $a_3^{(0),out}$, etc.).

(c) Write (explicitely again) the "big" scattering matrix $S(k)$ to fit the following set of equations

$$\vec{a}^{out} = S(k)\, \vec{a}^{in},$$

where

$$\vec{a}^{in} = \begin{pmatrix} a_0^{(1),in} \\ a_1^{(0),in} \\ a_0^{(2),in} \\ a_2^{(0),in} \\ a_0^{(3),in} \\ a_3^{(0),in} \end{pmatrix} \quad \text{and} \quad \vec{a}^{out} = \begin{pmatrix} a_1^{(0),out} \\ a_0^{(1),out} \\ a_2^{(0),out} \\ a_0^{(2),out} \\ a_3^{(0),out} \\ a_0^{(3),out} \end{pmatrix}.$$

Remember that $S(k)$ merely consists of the different components of the single vertex scattering matrices $\boldsymbol{\sigma}^{(i)}(k)$ and zero elements for edges that are not connected to each other.

(d) Write the matrix $T(k)$such that it fits into the set of equations $\vec{a}^{in} = S(k)\,\vec{a}^{out}$ with $\vec{a}^{in}$ and $\vec{a}^{out}$ as given above.

(e) A few tips to check yourself (no need to calculate, just in order to verify your answer).

(i) The matrix $S(k)$should be k-independent.

(ii) The matrix $S(k)$should be unitary.

(iii) The matrix $T(k)$ should be diagonal.

(f) Congratulations! You have obtained the quantum evolution operator, $U(k) = T(k)\,S(k)$. Once again, you don't need to calculate anything here (I trust that you can multiply matrices, if you really wish), just feel proud for the accomplishment.

(5) Consider a star graph with Neumann vertex conditions at the central vertex $i = 0$ and Dirichlet vertex conditions at the boundary vertices $i = 1, 2, \ldots, E$. Derive the quantum evolution map $U(k)$ and show that the secular function can be reduced to the form

$$\zeta(k) = \det\left(\mathbf{1}_{2E} - U(k)\right) = \det\left(\mathbf{1}_E + \widetilde{T}(2k)\,\boldsymbol{\sigma}^{(0)}\right),$$

where $\sigma^{(0)}$ is the central vertex scattering matrix and $\widetilde{T}(k)$ is a diagonal $E \times E$ matrix, $T(k)_{ee'} = \delta_{ee'} e^{ikl_e}$. You can gain a good intuition for the solution of this question, from your solution of question 4.

(6) **BONUS** - Show that the following secular function is real

$$\tilde{\zeta}(k) = \sqrt{\det\left(S^*(k)T^*(k)\right)}\det\left(\mathbf{1}_{2E} - U(k)\right).$$

Remember that $U(k) = T(k)S(k)$ and use the unitarity of $T(k)$ and $S(k)$.