QUANTUM GRAPHS - TCC COURSE - JANUARY-MARCH 2012

Homework 2

(1) Show that a vertex of degree two with Neumann vertex conditions ($\alpha = 0$) is superfluous (i.e. one can erase the vertex and join the two incident edges to a one single edge such that the lengths add up).

(2) Consider a star graph which consists of one central vertex which is connected to all other $V - 1$ vertices.
We will enumerate the vertices as $i = 0, 1, \ldots, E$ where $E = V - 1$ is the total number of edges. The central vertex is privileged to have the index $i = 0$. The lengths of the edges are $\{l_{01}, \ldots, l_{0E}\}$. We will assume Dirichlet boundary conditions at the vertices $i = 1, \ldots, E$, and a repulsive vertex potential whose strength is $\alpha_0 \geq 0$ at the central vertex. Namely, the vertex conditions at the boundary vertices are

$$\forall\, 1 \leq i \leq E \; ; \; f_{0i}\left(l_{0i}\right) = 0,$$

and the vertex conditions at the central vertex are

$$\forall\, 1 \leq i < j \leq E \; ; \quad f_{0i}\left(0\right) = f_{0j}\left(0\right) \equiv f_0$$

and

$$\sum_{i=1}^{E} f'_{0i}\left(0\right) = \alpha_0 f_0,$$

where we follow the convention introduced in class, $x_{0i} = 0$ at the central vertex and $x_{0i} = l_{0i}$ at the vertex i.

(a) Find the secular function for this graph.
You can verify that you have the correct answer by setting $E = 2$ and comparing to the secular function we have obtained in the previous lesson for the single edge with a delta potential.

(b) What is the weakest assumption you need to assume in order to have a one to one correspondence between the zeors of the secular function and the graph's eigenvalues?

(c) Show that under the assumption you got in the previous section, the graph's spectrum is non degenerate. Namely, that each eigenvalue appears with multiplicity one.

(d) Under the same assumption as in (b) and (c), show the following interlacing properties:

  (i) The Dirichlet spectrum ($\alpha_0 \to \infty$) of the star graph interlaces with the spectrum of the same star graph but with a Neumann vertex condition at the central point ($\alpha_0 = 0$).

  (ii) For any positive value of α_0 the n-the eigenvalue of the star graph is bounded from below by the n-th eigenvalue for $\alpha_0 = 0$ and from above by the n-th eigenvalue of the Dirichlet spectrum.

(e) **BONUS** - Assume now that $\alpha = 0$ (Neumann vertex conditions at the centre). Consider all star graphs with any number of edges E and any edge lengths $\{l_{01}, \ldots, l_{0E}\}$, such that the total length of the edges, $L = \sum_{i=1}^{E} l_{0i}$ is fixed.

  (i) What is the supremum of the first eigenvalue among all choices of values for E and $\{l_{01}, \ldots, l_{0E}\}$ (under the constraint above)? Is this supremum attained (i.e., is it a maximum)?

  (ii) What is the infimum? Is it attained?

(f) **BONUS** - Choose any value for the number of edges, E, and any values for the edge lengths $\{l_{01}, \ldots, l_{0E}\}$. Plot the secular function for your choice and find its thirteen first zeros (numerically).