QUANTUM GRAPHS - TCC COURSE - JANUARY-MARCH 2012

Homework 1

In the following exercises we consider a particle in a box with δ-potential described by the Schrödinger equation

$$-\frac{d^2 f(x)}{dx^2} + \alpha\delta(x)f(x) = Ef(x)$$

with $\alpha \in \mathbb{R}$ and with Dirichlet boundary conditions $f(-l_1) = 0 = f(f_2)$ at $x = -l_1 < 0$ and $x = l_2 > 0$.

(1) (a) For an irrational lengths ratio ($l_1/l_2 \notin \mathbb{Q}$) show that there cannot be any eigenfunctions with $f(0) \equiv f_0 = 0$.
Conclude that the poles of the secular function

$$\zeta_\alpha(k) = \alpha + k\cot(kl_1) + k\cot(kl_2)$$

cannot belong to the spectrum.
While this is true for attractive ($\alpha < 0$), repulsive ($\alpha > 0$) and vanishing ($\alpha = 0$) δ-potential you may assume a repulsive potential and consider only positive energies $E = k^2 > 0$.

(b) Show that $\zeta_\alpha(k)$ has only single poles if $l_1/l_2 \notin \mathbb{Q}$.
However, if $l_1/l_2 \in \mathbb{Q}$.there are single and double poles of the secular function. Locate them and show that the double poles belong to the spectrum while the single poles do not belong to it. This justifies the need of regularization of the secular function

$$\tilde{\zeta}_\alpha(k) = \zeta_\alpha(k)\sin(kl_1)\sin(kl_2).$$

(2) Consider an attractive δ-potential ($\alpha < 0$). Are there negative energies among the eigenvalues of the Schrödinger operator? Under which conditions and how many?

(3) **Bonus problem** - Assume that the eigenfunction is continuous at $x = 0$ with $f_0 = f(0^+) = f(0^-)$ (otherwise the δ-potential is not well defined) and show that any eigenfunction satisfies the matching condition

$$\alpha f_0 = f'(0^+) - f'(0^-)\ .$$