# Can you hear the shape of a graph?

I am a quantum graph - how do I sound?

You can draw a quantum graph by placing any number of points on a page and connecting them with lines. Points on the graph are called vertices and the lines are known as edges. A quantum graph as a physical object would look like a collection of guitar strings, tied together. When you pluck any of the strings the vibration carries along the whole graph. The network of strings can produce an infinite number of notes when plucked in different ways. For each note, the string vibrates at a particular frequency. The collection of all frequencies is known as the **spectrum** of the graph.
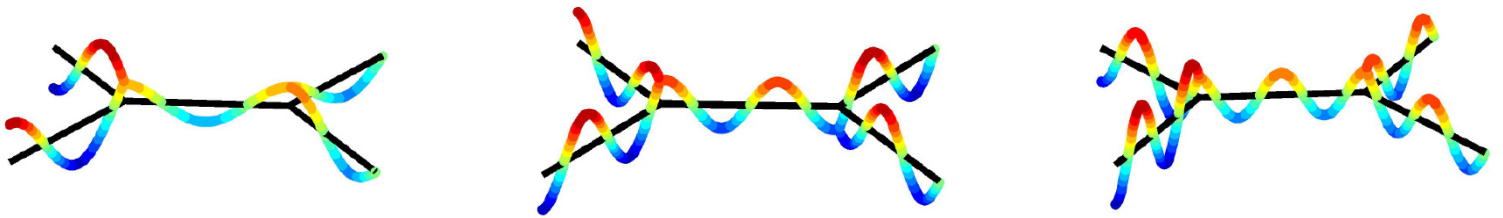

Visualisations of a five-edged quantum graph producing its 8th, 13th and 16th notes

This is my sound - how do I look?

You are awoken from your slumber by hearing your neighbour playing a loud instrument. Can you tell which instrument disturbed your sleep just by hearing the sound it makes? Mathematicians solve problems like this when the instrument is actually a quantum graph. Could you guess what your neighbour's graph looks like just by hearing its sounds? This is an example of what scientists call an **inverse problem**.

The strategy for answering this question involves listening to the pitch of the notes. Long strings have low notes, with a low vibration frequency; we can work out a quantum graph's total length from its sound spectrum. Visualising how the individual strings are connected, **the shape of the graph**, is more complicated.

The shape of a graph is related to its **periodic orbits**. These are journeys along the edges of the graph which start and finish at the same vertex.
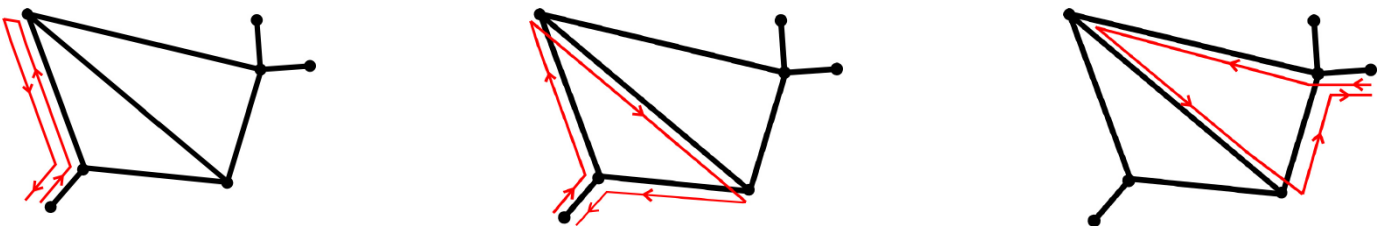
A few periodic orbits of a graph

A special formula shows how the spectrum of sounds determines the lengths of all periodic orbits of a graph [see reference 1 below]. For most graphs, this set of lengths, determined by their spectrum, defines their shape [reference 2]. However, there are pairs of graphs with different shapes but a common spectrum. These are **isospectral** graphs.

Graphs or drums which sound the same

For most of the 20th century, mathematicians and physicists did not know whether isospectral graphs existed. They started investigating this question for another musical instrument – the drum. For Mark Kac, a mathematician, a drum does not have to be circular, but can have any shape. He was curious to find out whether it is possible to build sets of isospectral drums – drums which look different, but sound exactly the same [reference 3]. Since 1966, when Kac started his exploration, this question bothered many mathematicians and physicists. The answer was only found in 1992, when the three mathematicians, Carolyn Gordon, David Webb and Scott Wolpert discovered the first example of isospectral drums [reference 4]. Gordon et al. used a formal theory derived by Toshikazu Sunada to produce their pair of drums [reference 5].

More recently, Ram Band, Ori Parzanchevski and Gilad Ben-Shach developed a new method enabling the construction of isospectral drums and graphs [references 6 and 7].


Gordon and Webb holding their isospectral drums

An application of the isospectral construction

Picture a square-shaped quantum graph, together with its four axes of reflective symmetry [labeled *x*, *y*, *u*, *v*]. Using these symmetries we can create a pair of isospectral graphs, by 'desymmetrising', or cropping, the graph. One is produced if we cut the 'parent' graph along its axes, *x* and *y*, and keep the remaining corner piece. A second graph is created by cutting along axes *u* and *v* and taking the side section that contains a loop. These two new graphs have been produced by this new method for general isospectral construction. We can redraw the pair of graphs with no change in their properties.
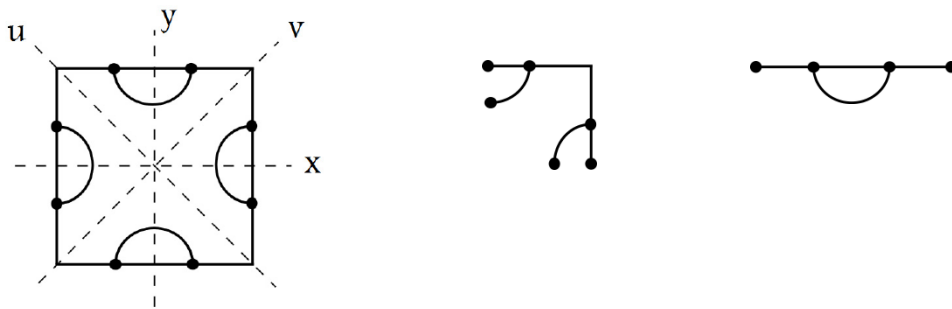
A square-shaped graph and two of its desymmetrisations
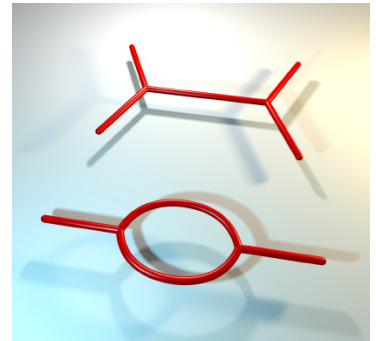
An isospectral pair of graphs

The isospectral construction method begins with a graph with certain symmetry properties. This 'parent' graph can be desymmetrised in a number of ways to generate new graphs. They are not necessarily isospectral and we need to follow a certain rule to obtain isospectrality. The recently developed isospectral theorem supplies us with this rule [references 6 and 7].

Quantum graphs - applications

Quantum graphs, both simple and complex, form a sub-domain within the general research of networks. This research ranges from the human arterial and neural systems to the World Wide Web. We use quantum graphs today to simplify complicated phenomena. They acquired the name "quantum graphs" because one of their earliest applications was to explain molecules – the ubiquitous inhabitants of the quantum mechanical world. They are increasingly useful for explaining other phenomena from "quantum chaos" to understanding the propagation of electromagnetic and acoustic waves used in modern communications and medical technologies [references 8 and 9].

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